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DENSITY THEOREMS RELATED TO PREHOMOGENEOUS VECTOR SPACES

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In this survey we discuss old and new density theorems which can be obtained by the zeta function theory of prehomogeneous vector spaces.

1. Density theorems

In this section we state all results assuming that the ground field is $\mathbb{Q}$ for simplicity, even though they can be generalized to statements with a finite number of local conditions over an arbitrary number field.

We start with new results. If k is a number field then let Δ_k, h_k and R_k be the absolute discriminant (which is an integer), the class number and the regulator, respectively.

We fix two prime numbers $q_1 \neq q_2$. Let Q_{q_1,q_2} be the set of quartic extensions $F/\mathbb{Q}$ such that $F\otimes\mathbb{Q}_{q_1}$ is a field and that $F\otimes\mathbb{Q}_{q_2}$ is a direct sum of $\mathbb{Q}_{q_2}$ and a cubic extension of $\mathbb{Q}_{q_2}$. Note that if $F\in Q_{q_1,q_2}$ then the Galois group of the Galois closure of k over $\mathbb{Q}$ is either $\mathfrak{S}_4$ or A_4. Also each isomorphism class appears four times in Q_{q_1,q_2}.

Define

$$E'_p = \begin{cases} 1+p^{-2}-p^{-3}-p^{-4} & p\neq q_1,q_2, \\ (1-q_1^{-1})(\frac{1}{4}+q_1^{-3}+\frac{1}{2}q_1^{-2}) & p=q_1, \\ (1-q_2^{-1})(\frac{1}{3}+q_2^{-2}) & p=q_2. \end{cases}$$

The following theorem is our first result.

Theorem 1.1. *We have*

$$\lim_{X\to\infty} X^{-1} \sum_{\substack{F\in Q_{q_1,q_2} \\ |\Delta_F|\le X}} 1 = \frac{37}{48}\prod_p E'_p.$$

Also in the above limit one can ignore $F\in Q_{q_1,q_2}$ such that the Galois group of the Galois closure of F over $\mathbb{Q}$ is A_4.

The proof of the above theorem shall be published in the future.

Let Q be the set of quartic extensions $k/\mathbb{Q}$ such that the Galois group of the Galois closure of k over $\mathbb{Q}$ is $\mathfrak{S}_4$ or A_4. Then we also make the following conjecture.

Conjecture 1.2.

$$\lim_{X\to\infty} X^{-1} \sum_{\substack{F\in Q \\ |\Delta_F|\le X}} 1 = \frac{37}{48}\prod_p (1+p^{-2}-p^{-3}-p^{-4}).$$

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Also in the above limit one can ignore $F \in Q$ such that the Galois group of the Galois closure of F over $\mathbb{Q}$ is A_4.

Our next result is regarding biquadratic extensions with one square root fixed. Let $\widetilde{k} = \mathbb{Q}(\sqrt{d_0})$ where $d_0 \neq 1$ is a square free integer. Suppose $|\Delta_{\mathbb{Q}(\sqrt{d_0})}| = \prod_p p^{\widetilde{\delta}_p(d_0)}$ is the prime decomposition. Note that $\widetilde{\delta}_p(d_0) > 0$ if and only if p is ramified in $\mathbb{Q}(\sqrt{d_0})$. Moreover, if $p \neq 2$ is ramified in $\mathbb{Q}(\sqrt{d_0})$ then $\widetilde{\delta}_p(d_0) = 1$, and if $p = 2$ then $\widetilde{\delta}_p(d_0) = 2$ when $d_0 \equiv 3\ (4)$ and $\widetilde{\delta}_p(d_0) = 3$ when d_0 is an even number. Note that if $d_0 \equiv 1, 5\ (8)$ then the prime 2 is split or inert in $\mathbb{Q}(\sqrt{d_0})$, respectively.

For any prime number p, we put

$$E'_p(d_0) = \begin{cases} 1 - 3p^{-3} + 2p^{-4} + p^{-5} - 2p^{-6} & \text{if } p \text{ is split in } \widetilde{k}, \\ (1+p^{-2})(1-p^{-2}-p^{-3}+p^{-4}) & \text{if } p \text{ is inert in } \widetilde{k}, \\ (1-p^{-1})(1+p^{-2}-p^{-3}+p^{-2\widetilde{\delta}_p(d_0)-2\lfloor\widetilde{\delta}_p(d_0)/2\rfloor-1}) & \text{if } p \text{ is ramified in } \widetilde{k} \end{cases}$$

where $\lfloor\widetilde{\delta}_p(d_0)/2\rfloor$ is the largest integer less than or equal to $\widetilde{\delta}_p(d_0)/2$.

We define

$$c_+(d_0) = \begin{cases} 16 & d_0 > 0, \\ 8\pi & d_0 < 0, \end{cases} \quad c_-(d_0) = \begin{cases} 4\pi^2 & d_0 > 0, \\ 8\pi & d_0 < 0, \end{cases}$$

$$M(d_0) = |\Delta_{\mathbb{Q}(\sqrt{d_0})}|^{\frac{1}{2}} \zeta_{\mathbb{Q}(\sqrt{d_0})}(2) \prod_p E'_p(d_0).$$

The following theorem was proved by A. Kable and the author in [28], [11], [12].

Theorem 1.3. *With either choice of sign we have*

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:\mathbb{Q}]=2, \\ 0<\pm\Delta_F\leq X}} h_{F(\sqrt{d_0})} R_{F(\sqrt{d_0})} = c_\pm(d_0)^{-1} h_{\mathbb{Q}(\sqrt{d_0})} R_{\mathbb{Q}(\sqrt{d_0})} M(d_0).$$

Suppose $a_n \geq 0$ is a non-negative real number for $n = 1, 2, \cdots$. Consider two statements as follows.

Theorem A *There exist constants a, b, c such that*

$$\lim_{X\to\infty} (X^a (\log X)^b)^{-1} \sum_{1\leq n\leq X} a_n = c.$$

Theorem B *Theorem A holds and the constants a, b, c can be determined.*

Theorem A is called the existence theorem of the density, and Theorem B the precise form of the density theorem. We generally refer to theorems of the above form as *density theorems*.

Of course the value of a density theorem depends on how interesting the number a_n is. If a_n is the number of an algebraic object then the corresponding density theorem asserts that the algebraic object in question is distributed regularly in some sense.

Probably the most famous density theorem is the prime number theorem. However, it is purely of multiplicative nature. Density theorems which we consider are of both additive and multiplicative nature and density theorems like the prime number theorem are not in the scope of our consideration.

We now state known density theorems related to the theory of prehomogeneous vector spaces. We first describe Gauss' conjecture which played a historical role in the development of the theory of automorphic forms.

Let $h(D)$ be the number of $\mathrm{SL}(2)_{\mathbb{Z}}$-equivalence classes of primitive integral forms of discriminant D. Note that if $h(D) \neq 0$ then $D \equiv 0, 1 \mod 4$. It is known that if D is the discriminant of a quadratic field F, then $h(D)$ is the narrow class number of F. One can also interpret $h(D)$ for general D by the order of F of discriminant D. Let ε_D be the smallest unit with norm 1 of $\mathbb{Q}(\sqrt{D})$ which may be written as

$$\varepsilon_D = \frac{1}{2}(t + u\sqrt{D})$$

where $t, u \in \mathbb{Z}$.

The following theorem was called Gauss' conjecture. The imaginary case was proved by Lipschitz in 1865 [13] and the real case was proved by Siegel in 1944 [22]. There are also subsequent works on the error term estimate such as Mertens [14], Vinogradov [24], Shintani [21], Chamizo–Iwaniec [1]. Let

$$c_{q,1+} = \frac{4\pi^2}{21\zeta(3)}, \; c_{q,1-} = \frac{4\pi}{21\zeta(3)},$$
$$c_{q,2+} = \frac{\pi^2}{18\zeta(3)}, \; c_{q,2-} = \frac{\pi}{18\zeta(3)}.$$

Theorem 1.4. *With either choice of sign we have*

$$\lim_{X\to\infty} X^{-3/2} \sum_{0<\pm D\leq X} h(4D)\log\varepsilon_{4D} = c_{q,1\pm},$$
$$\lim_{X\to\infty} X^{-3/2} \sum_{0<\pm D\leq X} h(D)\log\varepsilon_{D} = c_{q,2\pm}.$$

Gauss considered binary quadratic forms $ax^2 + 2bxy + cy^2$ with $a, b, c \in \mathbb{Z}$ and so the first statement in the above theorem is equivalent to what Gauss conjectured.

Note that if $D = m^2 D_0$, there is a simple relation between $h(D)$ and $h(D_0)$ (resp. $h(D)\log\varepsilon_D$ and $h(D_0)\log\varepsilon_{D_0}$) if $D < 0$ (resp. $D > 0$). So in Theorem 1.4, essentially the same object is counted infinitely many times. This ambiguity was first removed by Goldfeld–Hoffstein [8] as follows.

Let

$$c_{q,3+} = \frac{\pi^2}{36}\prod_p(1 - p^{-2} - p^{-3} + p^{-4}), \; c_{q,3-} = \frac{\pi}{36}\prod_p(1 - p^{-2} - p^{-3} + p^{-4}).$$

Theorem 1.5. (Goldfeld–Hoffstein, 1985) *With either choice of sign we have*

$$\lim_{X\to\infty} X^{-3/2} \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<\pm\Delta_F\leq X}} h_F R_F = c_{q,3\pm}$$

The above theorem was first proved using Eisenstein series of half integral weight. Datskovsky [2] later gave a proof based on the zeta function for the space of binary quadratic forms.

Next we consider cubic fields. The following theorem was proved by Davenport–Heilbronn [5], [6] using the "fundamental domain" method.

Theorem 1.6. **(Davenport–Heilbronn, 1971)**

$$\lim_{X\to\infty} X^{-1} \sum_{\substack{[F:\mathbb{Q}]=3 \\ |\Delta_F|\le X}} 1 = \frac{1}{\zeta(3)}.$$

Note that in the above theorem, each isomorphism class of a non-normal cubic field is counted three times.

We discuss the notion of prehomogeneous vector spaces and explain how the above density theorems are related to certain prehomogeneous vector spaces for the rest of this note.

2. Prehomogeneous vector spaces

The notion of prehomogeneous vector spaces was introduced by M. Sato in early 1960's. We first recall the definition of prehomogeneous vector spaces. Let k be a field.

Definition 2.1. Let G be a connected reductive group, V a representation of G, and χ a non-trivial primitive character of G, all defined over k. Then (G, V, χ) is called a *prehomogeneous vector space* if it satisfies the following properties.

(1) There exists a Zariski open orbit.
(2) There exists a non-constant polynomial $\Delta(x) \in k[V]$ such that $\Delta(gx) = \chi(g)^a\Delta(x)$ for a positive integer a.

In (1) of the above definition, if $U \subset V$ is an open set, it is a single G-orbit if there exists $x \in U_{\bar{k}}$ such that $U_{\bar{k}} = G_{\bar{k}}x$. We are mainly interested in irreducible prehomogeneous vector spaces. If the representation is irreducible then χ turns out to be unique and so we shall write (G, V) instead of (G, V, χ) from now on. Any polynomial $\Delta(x)$ which satisfies the condition (2) of the above definition is called a *relative invariant polynomial.* Let $V^{\mathrm{ss}} = \{x \in V \mid \Delta(x) \neq 0\}$, which is called the set of semi-stable points. Irreducible prehomogeneous vector spaces were classified by Sato–Kimura in [18].

We now assume that k is a number field and discuss the zeta functions of prehomogeneous vector spaces. The set of all places, infinite places, and finite places are denoted by $\mathfrak{M}, \mathfrak{M}_\infty, \mathfrak{M}_{\mathrm{f}}$ respectively. If $v \in \mathfrak{M}$ then k_v denotes the completion of k at v. We denote the spaces of Schwartz–Bruhat functions on $V_{\mathbb{A}}, V_{k_v}$ by $\mathscr{S}(V_{\mathbb{A}}), \mathscr{S}(V_{k_v})$ respectively.

For any group over k, we denote the group of rational characters by $X^*(G)$. Let $\widetilde{T} = \mathrm{Ker}(G \to \mathrm{GL}(V))$. We put $\widetilde{G} = G/\widetilde{T}$. We assume that $\widetilde{T}$ is a split torus (usually it is possible to choose the representation so that this condition is satisfied). It follows that the Galois cohomology set $\mathrm{H}^1(k, \widetilde{T})$ is trivial and so for any field $k \subset K$, we have $\widetilde{G}_K = G_K/\widetilde{T}_K$. Therefore, $\widetilde{G}_{\mathbb{A}} = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ also. We define

$$L_0 = \{x \in V_k^{\mathrm{ss}} \mid X^*(Z(G_x^\circ)/\widetilde{T}) = \{1\}\}.$$

We choose a relative invariant polynomial $P(x)$ so that the degree of $P(x)$ is the smallest. Let χ_0 be the character such that $P(gx) = \chi_0(g)P(x)$.

Definition 2.2. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ and a complex variable s, we define

$$Z(\Phi, s) = \int_{\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k} |\chi_0(g)|^s \sum_{x\in L_0} \Phi(gx)d\widetilde{g}$$

where $d\widetilde{g}$ is a Haar measure on $\widetilde{G}_{\mathbb{A}}$.

The integral $Z(\Phi, s)$ is called the *global zeta function*. The convergence of the above integral in some right half plane was considered by Weil [25], Igusa [9], M. Sato–Shintani [19], F. Sato [17], Yukie [29], [30], Ying [27], and H. Saito [16]. H. Saito [16] proved the convergence of $Z(\Phi, s)$ in some right half plane for all regular prehomogeneous vector spaces including reducible representations along the line of [17]. However, the range of the convergence is not optimum in [16] nor any explicit estimate of the incomplete theta series $\sum_{x\in L_0} \Phi(gx)$ is given unlike [25], [9], [19], [20], [29], [30]. Such an estimate is needed in order to carry out the global theory of zeta functions. Also if the estimate is optimum then there may be applications to certain arithmetic questions. So even though the problem of convergence is settled in some sense, more work in this direction is anticipated.

For $x \in L_0$, let $d\widetilde{g}'_x$, $d\widetilde{g}''_x$ be invariant measures on $G_{\mathbb{A}}/G^\circ_{x\,\mathbb{A}}$, $G^\circ_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ such that $d\widetilde{g} = d\widetilde{g}'_x d\widetilde{g}''_x$. We choose $d\widetilde{g}''_x$ to be the unnormalized Tamagawa measure on $G^\circ_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$. For example, if $k = \mathbb{Q}$, $F/\mathbb{Q}$ is a quadratic extension, and $G^\circ_x/\widetilde{T} \cong \mathrm{R}_{F/\mathbb{Q}}(\mathrm{GL}(1))/\mathrm{GL}(1)$ ($\mathrm{R}_{F/\mathbb{Q}}(\mathrm{GL}(1))$ is the restriction of scalar) then the volume of $G^\circ_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}}G^\circ_{x\,k}$ is $2h_k$ if F is real and $2\pi h_k R_k$ if F is imaginary and $F \neq \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

Definition 2.3. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ and a complex variable s, we define

$$Z_x(\Phi, s) = \int_{G_{\mathbb{A}}/G^\circ_{x\,\mathbb{A}}} |\chi_0(\widetilde{g}'_x)|^s \Phi(\widetilde{g}'_x x) d\widetilde{g}'_x$$

The integral $Z_x(\Phi, s)$ is called the *orbital zeta function*. Let $o(x) = [G_{x\,k} : G^\circ_{x\,k}]$. By the obvious modification of the integral,

$$Z(\Phi, s) = \sum_{x\in G_k\backslash L_0} o(x)^{-1}\mathrm{vol}(G^\circ_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}}G^\circ_{x\,k}) Z_x(\Phi, s). \tag{2.4}$$

The relation (2.4) suggests that the zeta function theory may yield the density of the unnormalized Tamagawa number of $G^\circ_x/\widetilde{T}$. So in order to determine the interpretation of the problem, one has to describe the orbit space $G_k\backslash L_0$ and determine the stabilizer G°_x for all $x \in L_0$. We call this problem the problem of rational orbit decomposition. We shall discuss rational orbit decompositions of prehomogeneous vector spaces which are related to density theorems in section 1 in the next section.

3. Rational orbit decompositions of prehomogeneous vector spaces

We consider the following prehomogeneous vector spaces

(1) $G = \mathrm{GL}(3) \times \mathrm{GL}(2)$, $V = \mathrm{Sym}^2\mathrm{Aff}^3 \otimes \mathrm{Aff}^2$,
(2) $G = \mathrm{R}_{k(\sqrt{d_0})/k}(\mathrm{GL}(2)) \times \mathrm{GL}(2)$, $V = W \otimes \mathrm{Aff}^2$ where $\mathrm{R}_{k(\sqrt{d_0})/k}(\mathrm{GL}(2))$ is the restriction of scalar and W is the space of binary Hermitian forms,
(3) $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$, $V = \mathrm{Sym}^2\mathrm{Aff}^2$,
(4) $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$, $V = \mathrm{Sym}^3\mathrm{Aff}^2$.

These prehomogeneous vector spaces correspond to Theorems 1.1, 1.3–1.6.

Definition 3.1. We define $\mathfrak{Ex}_i$ to be the set of Galois extensions of k which are splitting fields of degree i equations.

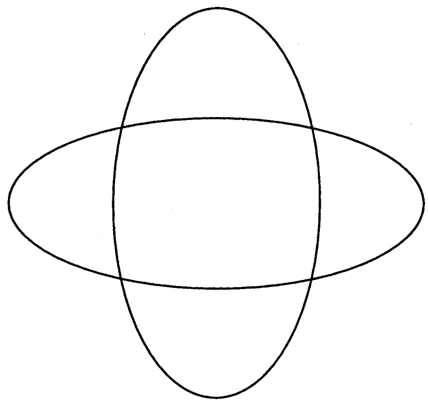
Let $\mathrm{H}^1(k, \mathfrak{S}_i)$ be the first Galois cohomology set where the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts on $\mathfrak{S}_i$ trivially. Then $\mathrm{H}^1(k, \mathfrak{S}_i)$ corresponds bijectively with conjugacy classes of homomorphisms ϕ from $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ to $\mathfrak{S}_i$. By Galois theory, $\mathrm{Ker}(\phi)$ determines a field F which belongs to $\mathfrak{Ex}_i$ and so it determines a map $\mathrm{H}^1(k, \mathfrak{S}_i) \to \mathfrak{Ex}_i$.

Rational orbit decompositions of the cases (1)–(4) are given as follows.

(1) $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{H}^1(k, \mathfrak{S}_4)$, and $G_x^\circ \cong \mathrm{GL}(1)$ for all $x \in V_k^{\mathrm{ss}}$.
(2) $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{H}^1(k, \mathfrak{S}_2) \cong \mathfrak{Ex}_2$, and if $x \in V_k^{\mathrm{ss}}$ corresponds to a quadratic extension F/k and $F \neq k(\sqrt{d_0})$, then $G_x^\circ \cong \mathrm{R}_{F(\sqrt{d_0})/k}(\mathrm{GL}(1))$. If x corresponds to k, then $x \notin L_0$.
(3) $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{H}^1(k, \mathfrak{S}_2) \cong \mathfrak{Ex}_2$, and if $x \in V_k^{\mathrm{ss}}$ corresponds to a quadratic extension F/k, then $G_x^\circ \cong \mathrm{R}_{F/k}(\mathrm{GL}(1))$. If x corresponds to k, then $x \notin L_0$.
(4) $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{H}^1(k, \mathfrak{S}_3) \cong \mathfrak{Ex}_3$, and $G_x^\circ \cong \mathrm{GL}(1)$ for all $x \in V_k^{\mathrm{ss}}$,

The cases (3), (4) are very classical. The case (3) goes back to the work of Gauss [7]. The cases (1), (2) are proved in [26], [10] respectively.

In the case (1), a point $x \in V_k^{\mathrm{ss}}$ is a pair $x = (Q_1, Q_2)$ of ternary quadratic forms. Then one can consider the intersection of two conics determined by Q_1, Q_2 as follows.



Given a quartic equation

$$t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4 = 0,$$

if we substitute $y = t^2$, we get

$$\begin{cases} y = t^2, \\ y^2 + a_1 t y + a_2 t^2 + a_3 t + a_4 = 0. \end{cases}$$

The homogeneous form of the above equation is a pair of ternary quadratic forms. This consideration goes back more than 900 years to the work of a medieval Persian mathematician-poet Omar Khayyam (see [23]).

In the cases (1), (4), the stabilizer G_x° does not depend on x and so the weighting factor is 1. In the cases (2), (3), the weighting factor is more or less $h_F R_F$ of biquadratic fields or quadratic fields. These are the reasons why the zeta function theory for these cases yield the density theorems in section 1

4. The filtering process

Let $a_n \geq 0$ for $n = 1, 2, \cdots$. A general approach to prove density theorems is to consider the generating function, i.e.,

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

The following theorem is a fundamental tool to prove density theorems.

Theorem 4.1. (Tauberian Theorem) *Suppose $f(s)$ is holomorphic in* $\mathrm{Re}(s) \geq a$ *except for a pole of order $b+1$ at $s = a$ with leading term $c(s-a)^{-(b+1)}$. Then*

$$\lim_{X\to\infty} (X^a (\log X)^b)^{-1} \sum_{1\leq n\leq x} a_n = \frac{c}{ab!}.$$

For the proof of this theorem, see Theorem I [15, p. 464].

We explain our approach by mainly considering the prehomogeneous vector space (4). The meromorphic continuation and the functional equation of the global zeta function can be proved using the theory of b-functions. The b-function is explicitly computed and so we know the location of the poles of the global zeta function. The reader may think that Theorem A for this case may follow from Theorem 4.1 and the knowledge of the location of the poles. However, that is not the case and in fact Theorem A and Theorem B are proved simultaneously.

If the global zeta function were a product of gamma factors and the Dirichlet series

$$\sum_{[k:\mathbb{Q}]=3} |\Delta_k|^{-3}, \tag{4.2}$$

then Theorem A would have followed from the meromorphic continuation as long as all poles are real. However, the global zeta function is not in this form and we explain how discriminants appear by modifying the relation (2.4).

Since $G_x^\circ = \widetilde{T}$ for all $x \in V_k^{\mathrm{ss}}$, $d\widetilde{g} = d\widetilde{g}'_x$. Since the group is a product of $\mathrm{GL}(n)$'s for this case (in fact for all the cases (1)–(4)), we choose the standard measure $d\widetilde{g}_v$ on $\widetilde{G}_{k_v}$. There exists a constant c_G such that $d\widetilde{g} = c_G \prod_v d\widetilde{g}_v$. Let $\mathcal{O}_v \subset k_v$ be the integer ring and $|\ |_v$ the absolute value on k_v. If F/k is a finite extension of number fields then we denote the relative discriminant by $\Delta_{F/k}$. It is an ideal in k and we denote its ideal norm by $\mathcal{N}(\Delta_{F/k})$. Relative discriminants and their norms are similarly defined for k_v also.

We choose representatives $w_{v,1}, \cdots, w_{v,N_v}$ of $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$ so that they satisfy the following condition.

Condition 4.3. (1) If $v \in \mathfrak{M}_{\mathrm{f}}$ then $w_{v,1}, \cdots, w_{v,N_v} \in V_{\mathcal{O}_v}$, and if $v \in \mathfrak{M}_\infty$ then $|P(w_{v,i})|_v = 1$ ($P(x)$ is the relative invariant polynomial of the smallest degree).

(2) If $w_{v,i}$ corresponds to the Galois closure of a field F/k_v, then $|\Delta(w_{v,i})|_v = \mathcal{N}(\Delta_{F/k_v})^{-1}$.

(3) If $y \in G_{k_v} w_{v,i} \cap V_{\mathcal{O}_v}$ then $|\Delta(w_{v,i})|_v \geq |\Delta(y)|_v$.

In the cases (2)–(4), representatives which satisfy Condition 4.3 exist. In the case (1), $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$ corresponds bijectively with isomorphism classes of fields F/k_v of degree up to four and pairs (F_1, F_2) of quadratic extensions of k_v and Condition 4.3(2) has to be replaced by $|\Delta(w_{v,i})|_v = \mathcal{N}(\Delta_{F_1/k_v})^{-1}\mathcal{N}(\Delta_{F_2/k_v})^{-1}$ if x corresponds to a pair (F_1, F_2) of quadratic extensions. We call representatives which satisfy Condition 4.3 "good" representatives.

Let $x \in V_{k_v}^{\mathrm{ss}}$.

Definition 4.4. For $\Phi \in \mathscr{S}(V_{k_v})$ and a complex variable s, we define

$$Z_{x,v}(\Phi, s) = \int_{G_{k_v}/G^\circ_{x\,k_v}} |\chi_0(\widetilde{g}_v)|_v^s \Phi(\widetilde{g}_v x) d\widetilde{g}_v$$

The above integral is called the *local orbital integral.*

If $x \in V_{k_v}^{\mathrm{ss}}$, $\Phi \in \mathscr{S}(V_{k_v})$, and $x \in G_{k_v} w_{v,i}$, then we define

$$\Xi_{x,v}(\Phi, s) = Z_{w_{v,i},v}(\Phi, s).$$

We call $\Xi_{x,v}(\Phi, s)$ the *standard local zeta function.* If $x \in V_k^{\mathrm{ss}}$ and $\Phi = \otimes \Phi_v$, then we put

$$\Xi_x(\Phi, s) = \prod_v \Xi_{x,v}(\Phi_v, s).$$

By the condition (3), if $x \in G_{k_v} w_{v,i}$ then

$$Z_{x,v}(\Phi, s) = |P(w_{v,i})|_v |P(x)|_v^{-1} \Xi_{x,v}(\Phi, s).$$

Suppose $x \in V_k^{\mathrm{ss}}$ corresponds to the Galois closure of a field $F(x)/k$ of degree up to three. Since $\prod_v |P(x)|_v = 1$, we have

$$Z_x(\Phi, s) = c_G \mathcal{N}(\Delta_{F(x)/k})^{-s} \Xi_x(\Phi, s).$$

Therefore, by (2.4), we get

$$Z(\Phi, s) = c_G \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} o(x)^{-1} \mathcal{N}(\Delta_{F(x)/k})^{-s} \Xi_x(\Phi, s).$$

By a similar consideration, if we choose the definitions of $d\widetilde{g}'_x$, $d\widetilde{g}''_x$, their local versions $d\widetilde{g}'_{x,v}$, $d\widetilde{g}''_{x,v}$, and $\Xi_x(\Phi, s)$, $\Xi_{x,v}(\Phi, s)$, then it is possible to prove a similar formula

$$Z(\Phi, s) = c_G \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} o(x)^{-1} \operatorname{wt}(x) D(x)^{-s} \Xi_x(\Phi, s) \tag{4.5}$$

where $\operatorname{wt}(x) = \operatorname{vol}(G^\circ_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}} G^\circ_{x\,k})$ and

$$D(x) = \mathcal{N}(\Delta_{F(x)/k}) \text{ or } \mathcal{N}(\Delta_{F_1(x)/k}) \mathcal{N}(\Delta_{F_2(x)/k}) \tag{4.6}$$

depending on whether x corresponds to a field $F(x)$ or a pair $(F_1(x),\ F_2(x))$ of fields (the second case happens only in the case (1)).

For prehomogeneous vector spaces (1)–(4), there is a map from the orbit space $G_k \backslash V_k^{\mathrm{ss}}$ to the set of field extensions and it is natural to consider the discriminants of the corresponding fields. However, the interpretation of the orbit space $G_k \backslash V_k^{\mathrm{ss}}$ is not known for all the cases. Therefore, expressing the global zeta function in terms of an intrinsic invariant such as the discriminant has yet to be done systematically.

It is obvious from (4.5) that the global zeta function is not in the form (4.2). In some sense we have to approximate the Dirichlet series (4.2) by (4.5). This process is called the *filtering process.* The filtering process was developed by Datskovsky and Wright in [3], [4] (it was used intrinsically in the original work of Davenport–Heilbronn [5], [6]). The reader can see Wright's note §0.5 [29] also.

Generally speaking it is more difficult to count objects which are scarce. For example if we directly apply the Tauberian theorem to the Riemann zeta function, we simply get the trivial result

$$\lim_{X\to\infty} X^{-1} \sum_{1\le n\le X} 1 = 1$$

and we do not obtain the prime number theorem.

Let $\mathcal{O}_k$ be the integer ring of k. In our case the orbit space $G_k\backslash V_k^{\mathrm{ss}}$ parametrizes interesting algebraic objects, but there is an ambiguity in the integral equivalence classes $G_{\mathcal{O}_k}\backslash V_{\mathcal{O}_k}^{\mathrm{ss}}$. The situation of Gauss' conjecture corresponds to $G_{\mathcal{O}_k}\backslash V_{\mathcal{O}_k}^{\mathrm{ss}}$ and the situation of Goldfeld–Hoffstein theorem corresponds to $G_k\backslash V_k^{\mathrm{ss}}$. As we pointed out earlier, if we count integral equivalence classes then we are counting essentially the same object infinitely many times. To count $G_{\mathcal{O}_k}\backslash V_{\mathcal{O}_k}^{\mathrm{ss}}$, one can use the Tauberian theorem and Theorem A follows from the meromorphic continuation of the global zeta function. However, $G_k\backslash V_k^{\mathrm{ss}}$ is more scarce and removing the ambiguity is what the filtering process does. Intuitively speaking, we start with $G_{\mathcal{O}_k}\backslash V_{\mathcal{O}_k}^{\mathrm{ss}}$ and consider smaller and smaller sets by changing the test function Φ. Then we take the limit of density theorems at each step in some sense. Of course such an argument has to be justified but the reader can probably understand that at each step one has to know Theorem B rather than Theorem A. By the time we prove that we can take this "limit of limits", we end up with proving Theorem B and so Theorem A and Theorem B are proved simultaneously. For this reason it is absolutely necessary to describe the principal parts of the global zeta function at its poles by invariant distributions of the test function Φ.

The principal parts of the global zeta function for the cases (3), (4) were computed by Shintani in [21], [20] respectively. The author computed the principal parts of the global zeta function for the cases (1), (2) in [29], [28] respectively. Finding the principal parts of the global zeta function is a very difficult problem and still more than ten meaningful cases have yet to be handled.

Let $s=\kappa$ be the rightmost pole of $Z(\Phi,s)$. In the case (1) it is possible to choose the test function Φ so that the intersection of V_k^{ss} and the support of Φ is precisely the family Q_{q_1,q_2} and that $s=\kappa$ is a simple pole with residue

$$\widehat{\Phi}(0) = \mathfrak{V}\int_{V_{\mathbb{A}}} \Phi(x)dx \tag{4.7}$$

where $\mathfrak{V}$ is a constant. In the cases (2), (3) $s=\kappa$ is a simple pole and the residue is in the form (4.7) also. In the case (4) it is possible to carry out more global theory so that if we consider orbits which correspond to cubic extension instead of $G_k\backslash V_k^{\mathrm{ss}}$, then $s=\kappa$ is a simple pole and the residue is in the form (4.7) again.

So instead of the global zeta function, we consider

$$Z_I(\Phi,s) = \int_{\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k} |\chi_0(g)|^s \sum_{x\in I} \Phi(gx)d\widetilde{g}$$

where $I\subset L_0$ is a G_k-invariant subset and assume the following condition.

Condition 4.8. (1) The function $Z_I(\Phi,s)$ can be continued meromorphically to $\mathrm{Re}(s)\ge\kappa$ with a simple pole at $s=\kappa$ with residue (4.7).

(2) The function $Z_I(\Phi,s)$ has the expansion (4.5) with V_k^{ss} replaced by I.

(3) The set I corresponds bijectively with isomorphism classes of fields of degree up to $4, 2, 2, 3$ for the cases (1)–(4) respectively.

We now describe the filtering process. We fix a finite set $S \supset \mathfrak{M}_\infty$ of places of k. For each finite subset $T \supset S$ of $\mathfrak{M}$, we consider T-tuples $\omega_T = (\omega_v)_{v\in T}$ where each ω_v is one of the good representatives. If $x \in V_k^{\mathrm{ss}}$ and $x \in G_{k_v}\omega_v$ then we write $x \approx \omega_v$ and if $x \approx \omega_v$ for all $v \in T$ then we write $x \approx \omega_T$. Suppose that we have Dirichlet series $L_i(s) = \sum_{m=1}^{\infty} \ell_{i,m} m^{-s}$ for $i = 1, 2$. If $\ell_{1,m} \leq \ell_{2,m}$ for all $m \geq 1$ then we shall write $L_1(s) \preccurlyeq L_2(s)$.

For later purposes, it is convenient to make the following definition.

Definition 4.9. For any $v \in \mathfrak{M}_{\mathrm{f}}$, $\Phi_{v,0}$ is the characteristic function of $V_{\mathcal{O}_v}$.

Let $\Xi_{x,v}(s) = \Xi_{x,v}(\Phi_{v,0}, s)$ and $\Xi_{x,T}(s) = \prod_{v\notin T} \Xi_{x,v}(s)$. Let q_v be the order of the set $\mathcal{O}_v/\mathfrak{p}_v$ where $\mathfrak{p}_v$ is the maximal ideal. From the integral defining $\Xi_{x,v}(s)$ it follows that for $v \notin S$ this function may be expressed as $\Xi_{x,v}(s) = \sum_{n=-\infty}^{\infty} a_{x,v,n} q_v^{-ns}$ for certain numerical coefficients $a_{x,v,n}$. The following is the conditions necessary to apply the filtering process.

Condition 4.10. (1) For all $v \notin S$ and all $x \in V_{k_v}^{\mathrm{ss}}$ we have $a_{x,v,n} = 0$ for $n < 0$, $a_{x,v,0} = 1$ and $a_{x,v,n} \geq 0$ for all n.

(2) There exists a Dirichlet series $L_v(s) = \sum_{n=0}^{\infty} \ell_{v,n} q_v^{-ns}$ for all $v \notin S$ such that for all $x \in V_{k_v}^{\mathrm{ss}}$, $\Xi_{x,v}(s) \preccurlyeq L_v(s)$.

(3) There exists $\epsilon > 0$ which does not depend on v such that the series defining $L_v(s)$ converges to a holomorphic function in the region $\mathrm{Re}(s) > \kappa - \epsilon$ and the product $\prod_{v\notin S} L_v(s)$ converges absolutely and locally uniformly in the region $\mathrm{Re}(s) > \kappa - \epsilon$.

If $\omega_T = (\omega_v)_{v\in T}$ is a T-tuple where each ω_v is one of the good representatives, then we denote the S-tuple $(\omega_v)_{v\in S}$ by $\omega_T|_S$. We put

$$
\begin{aligned}
\xi_{\omega_T}(s) &= \sum_{\substack{x\in G_k\backslash I \\ x\approx\omega_T}} o(x)^{-1}\,\mathrm{wt}(x)\mathcal{N}(\Delta_{F(x)/k})^{-s}\Xi_{x,T}(s), \\
\xi_{\omega_S,T}(s) &= \sum_{\substack{x\in G_k\backslash I \\ x\approx\omega_S}} o(x)^{-1}\,\mathrm{wt}(x)\mathcal{N}(\Delta_{k(x)/k})^{-s}\Xi_{x,T}(s) \qquad (4.11)\\
&= \sum_{\omega_T|_S=\omega_S} \xi_{\omega_T}(s).
\end{aligned}
$$

For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ and $\Phi_v \in \mathcal{S}(V_{k_v})$ we put

$$
\Sigma(\Phi) = \int_{V_{\mathbb{A}}} \Phi(x)dx, \quad \Sigma_v(\Phi_v) = \int_{V_{k_v}} \Phi_v(x_v)dx_v.
$$

If $\Phi = \otimes_v \Phi_v$ then there exists a constant $c(\Sigma)$ such that $\Sigma(\Phi) = c(\Sigma)\prod_v \Sigma_v(\Phi_v)$. If $x \in V_{k_v}^{\mathrm{ss}}$, by the invariance properties of distributions, there exists a constant $r_{x,v} > 0$ such that if the support of Φ_v is contained in $G_{k_v}x$ then

$$
\Sigma_v(\Phi_v) = r_{x,v}\Xi_v(\Phi_v, \kappa).
$$

It is easy to choose such Φ_v so that $\Sigma_v(\Phi_v) \neq 0$. Since $\Sigma_v(\Phi_{v,0}) = 1$, we get the following proposition.

Proposition 4.12. *The Dirichlet series $\xi_{\omega_S,T}(s)$ has a meromorphic continuation to* $\mathrm{Re}(s) \geq \kappa$ *with a simple pole at $s = \kappa$ with residue*

$$\mathfrak{V}c(\Sigma)c_G^{-1}\left(\prod_{v\in S} r_{\omega_v,v}\right)\left(\prod_{v\in T\setminus S}\sum_x r_{x,v}\right)$$

where the sum is over the complete set $\{x\}$ of good representatives.

We put

$$E_v = \sum_x r_{x,v}. \tag{4.13}$$

Suppose $\prod_v E_v$ converges absolutely to a positive number. If T approaches to $\mathfrak{M}$ then $\Xi_{x,T}(s)$ approaches to 1. So if we are allowed to take the limit $T \to \mathfrak{M}$, we get the density of G_k-orbits. The following proposition is proved in Theorem 4.1 [4, pp. 129,130] and Proposition (0.5.4) [29, pp. 17,18] (which is also due to Wright).

Proposition 4.14. *Suppose Conditions 4.8, 4.10 are satisfied. Then*

$$\lim_{X\to\infty} X^{-\kappa} \sum_{\substack{x\in G_k\setminus I, x\approx\omega_S \\ \mathcal{N}(\Delta_{F(x)/k})\leq X}} o(x)^{-1}\,\mathrm{wt}(x) = \mathfrak{V}c(\Sigma)c_G^{-1}\kappa^{-1}\left(\prod_{v\in S} r_{\omega_v,v}\right)\prod_{v\notin S} E_v$$

if $\prod_v E_v$ converges absolutely to a positive number.

In the case (1), S has to contain two finite places. However, in the cases (2)–(4), there is no restriction on S. So taking the sum over all ω_S, we get the following corollary.

Corollary 4.15. *Suppose Conditions 4.8, 4.10 are satisfied. Then*

$$\lim_{X\to\infty} X^{-\kappa} \sum_{\substack{x\in G_k\setminus I \\ \mathcal{N}(\Delta_{F(x)/k})\leq X}} o(x)^{-1}\,\mathrm{wt}(x) = \mathfrak{V}c(\Sigma)c_G^{-1}\kappa^{-1}\prod_{v\in\mathfrak{M}} E_v$$

if $\prod_v E_v$ converges absolutely to a positive number.

Note that if one can carry out more global theory and prove Condition 4.8(1) for the family Q, then the above corollary holds and so Conjecture 1.2 follows.

Finally we discuss how to compute $r_{x,v}$. In the cases (1)–(4), it turns out that

$$r_{x,v} = \mathrm{vol}(G_{\mathcal{O}_v}x)\mathrm{vol}(G^\circ_{x\,k_v} \cap G_{\mathcal{O}_v}).$$

We have to say a few words about the definitions of measures to compute two volumes appearing in the above formula. Since $G_{\mathcal{O}_v}x \subset V_{\mathcal{O}_v}$, we choose the standard measure on V_{k_v}, i.e., the measure such that $\mathrm{vol}(V_{\mathcal{O}_v}) = 1$. The definition of the measure on $G^\circ_{x\,k_v}$ is more difficult. For the cases (1), (4), this group does not depend on x and so there is no problem defining a measure on it. For the cases (2), (3), one has to use the identification $G^\circ_x \cong \mathrm{R}_{F(x)/k}(\mathrm{GL}(1))$. There is a standard measure on the idele group of $F(x)$. However, the above identification is not unique. The idea to define a measure on $G^\circ_{x\,k_v}$ is to fix 1-cocycles which represent $\mathrm{H}^1(k, \mathfrak{S}_2)$ and consider the above identifications which are compatible with these 1-cocycles. Then it turns out that the choice of the measure does not depend on such identifications both globally and locally.

The choice of the measure on $G^\circ_{x\,k_v}$ is discussed in section 5 [11]. Datskovsky's choice of such a measure in [2, p. 218, l.13] is wrong because it does not satisfy the functorial property (i.e., the property that if $x = gy$ then the measure on $G^\circ_{x\,k_v}$ is induced by the measure on $G^\circ_{y\,k_v}$ by conjugation) which he implicitly uses in [2, p. 230]. However, the final answer in that paper is correct due to the fact that his measure on $G^\circ_{x\,k_v}$ coincides with the correct measure for good representatives and $\mathrm{vol}(G^\circ_{x\,k_v} \cap G_{\mathcal{O}_v})$ happens to be 1 for all good representatives in the case (3). The volume $\mathrm{vol}(G^\circ_{x\,k_v} \cap G_{\mathcal{O}_v})$ is not computed in [2], but it can be verified to be 1. In the case (2), there are orbits such that $\mathrm{vol}(G^\circ_{x\,k_v} \cap G_{\mathcal{O}_v})$ is not 1.

Datskovsky and Wright did not exactly compute $r_{x,v}$ in the above manner in [3], [4], but it is possible to do so and it is easier. This method can intrinsically be seen in the original work of Davenport–Heilbronn [6] and the computation is only a few pages long (for the case (4)).

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