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Local A-packets for $U_{E/F}(4)$ and a conjecture of Hiraga on the Zelevinskii duality *

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Contents

1 Introduction

In this note we calculate the candidates of the non-trivial A-packets [1] (see also [7]) for the quasisplit unitary group in four variables $U_{E/F}(4)$.

As is well-known, A-packets and the Arthur conjecture were introduced in order to suitably generalize the *strong multiplicity one theorem* to general reductive groups. In other words, to recover the multiplicity of each irreducible automorphic representations from the Hecke algebra action. We assume this expectation, and use this to define A-packets. This global postulate combined with some local part of the Arthur conjecture allows us to determine completely the candidates of such packets of $U_{E/F}(4)$.

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Of course our former result on the irreducible non-supercuspidal representations [9], [10] is the base of this work. But the main construction depends on the detailed study of the local and global theta correspondences. We hope that our approach will yield the global multiplicity formula for these A-packets in some near future.

As an application we verify a conjecture of Hiraga on the effect of the Zelevinskii duality to L and A-packets. At the time of the conference, we announced that there exists a counter example. But this is false, and that case forms the most interesting example ever known. We thank T. Ikeda for the discussion on this point, and of course, for the organization of a pleasant symposium.

2 CAP parameters for $U_{E/F}(4)$

We first determine the set of A-parameters which should correspond to the non-tempered A-packets. Although our primary concern is local A-packets, we need a global setting.

Let K be a quadratic extension of an algebraic number field k. Write σ for the generator of the Galois group $\mathrm{Gal}(K/k)$. The adele ring of k is denoted by $\mathbb{A}$ while $\mathbb{A}_K$ denotes that of K.

Let G be the connected reductive group over k such that

$$G(R) = \{g \in GL_4(R \otimes_k K) \mid gI_4{}^t\sigma(g) = I_4\}, \tag{2.1}$$

for any k-algebra R. We have written

$$I_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

The L-group ${}^LG = \widehat{G} \rtimes_{\rho_G} W_k$ is given by

$$\widehat{G} = GL_4(\mathbb{C}), \quad \rho_G(w)g = \begin{cases} g & \text{if } w \in W_K \\ \mathrm{Ad}(I_4)^t g^{-1} & \text{otherwise.} \end{cases}$$

Write $\mathcal{L}_k$ for the hypothetical Langlands group of k. An *A-parameter* is a continuous homomorphism $\phi : \mathcal{L}_k \times SL_2(\mathbb{C}) \to {}^LG$ such that

- the restriction $\phi|_{\mathcal{L}_k}$ is a tempered Langlands parameter;
- $\phi|_{SL_2(\mathbb{C})}$ is analytic.

We usually do not distinguish a parameter and its equivalence class, i.e. its $\widehat{G}$-orbit. Write $\Psi(G)$ for the set of equivalence classes of A-parameters. We shall be concerned with the parameters which (conjecturally) parameterize automorphic representations occurring discretely in the automorphic spectrum and have some non-tempered local components. More precisely, we say that an A-parameter ϕ is of *CAP type* (cuspidal but associated to parabolic) if

- ϕ is elliptic, that is, the connected centralizer $\mathrm{Cent}(\phi, \widehat{G})^0$ is contained in $Z(\widehat{G})$, and
- $\phi|_{SL_2(\mathbb{C})}$ is not trivial.

We consider only parameters of this type.

By virtue of Rogawski's detailed study of automorphic representations on $U(3)$ [16], we can classify the CAP-parameters for G.

Proposition 2.1. *The following list gives the complete representatives of equivalence classes of A-parameters of CAP type for G. We conventionally write η, μ for characters of $K^\times\backslash\mathbb{A}_K^\times$ satisfying $\eta|_{\mathbb{A}^\times} = \mathbf{1}$, $\mu|_{\mathbb{A}^\times} = \omega_{E/F}$. $\omega_{E/F}$ is the quadratic character of $k^\times\backslash\mathbb{A}^\times$ associated to K/k by the class field theory. Also T denotes an elliptic L-packet of the quasisplit unitary group G_1 of two variables. Such L-packets and the associated Langlands parameters*

$$\varphi_T : \mathcal{L}_k \ni w \longmapsto \varphi_T^0(w) \rtimes_{\rho_{G_1}} \mathrm{p}_{W_k}(w) \in \widehat{G}_1 \rtimes_{\rho_{G_1}} W_k$$

are described in [16]. Here p_{W_k} is the conjectural morphism $\mathcal{L}_k \to W_k$. We fix $w_\sigma \in W_k \setminus W_K$.

(1) If $\phi|_{SL_2(\mathbb{C})} \simeq \mathrm{Sym}^3$, then $\phi = \phi_\eta$: $\phi_\eta|_{\mathcal{L}_K} = \eta\mathbf{1}_4 \times \mathrm{p}_{W_K}$, $\phi_\eta(w_\sigma) = \mathbf{1}_4 \rtimes w_\sigma$.

(2) If $\phi|_{SL_2(\mathbb{C})} \simeq \mathrm{Sym}^2 \oplus \mathbf{1}_{SL_2}$, then $\phi = \phi_{\mu,\eta}$:

$$\phi_{\mu,\eta}|_{\mathcal{L}_K} = \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \mu & \\ & & & \mu\eta \end{pmatrix} \times \mathrm{p}_{W_K}, \quad \phi_{\mu,\eta}(w_\sigma) = \begin{pmatrix} & & 1 & \\ & & & 1 \\ & & & 1 \\ 1 & & & \end{pmatrix} \rtimes w_\sigma.$$

(3) When $\phi|_{SL_2(\mathbb{C})} \simeq \mathrm{St}^{\oplus 2}$, we have the following two possibilities.

(a) $\phi = \phi_{T,\mu}$, where T is a stable L-packet of G_1:

$$\phi_{T,\mu}|_{\mathcal{L}_K} = \left(\begin{array}{c|c} \mu\varphi_T^0 & \\ \hline & \mu\varphi_T^0 \end{array}\right) \times \mathrm{p}_{W_K}, \quad \phi_{T,\mu}(w_\sigma) = \left(\begin{array}{c|c} \varphi_T^0(w_\sigma) & \\ \hline & -\varphi_T^0(w_\sigma) \end{array}\right) \rtimes w_\sigma,$$

$$\phi_{T,\mu}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\begin{array}{c|c} a\mathbf{1}_2 & b\mathbf{1}_2 \\ \hline c\mathbf{1}_2 & d\mathbf{1}_2 \end{array}\right) \times 1.$$

(b) $\phi = \phi_{\boldsymbol{\eta}}$ where $\boldsymbol{\eta} = (\eta_1, \eta_2)$ is such that $\eta_1 \neq \eta_2$:

$$\phi_{\boldsymbol{\eta}}|_{\mathcal{L}_K} = \mathrm{diag}(\eta_1, \eta_2, \eta_2, \eta_1) \times \mathrm{p}_{W_K}, \quad \phi_{\boldsymbol{\eta}}(w_\sigma) = \mathbf{1}_4 \rtimes w_\sigma,$$

$$\phi_{\boldsymbol{\eta}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d \end{pmatrix} \times 1.$$

(4) If $\phi|_{SL_2(\mathbb{C})} \simeq \mathrm{St} \oplus \mathbf{1}^2_{SL_2}$, $\phi = \phi_{T,\eta}$:

$$\phi_{T,\eta}|_{\mathcal{L}_K} = \left(\begin{array}{c|c|c} \eta & & \\ \hline & \varphi^0_T & \\ \hline & & \eta \end{array}\right) \times \mathrm{p}_{W_K}, \quad \phi_{T,\eta}(w_\sigma) = \left(\begin{array}{c|c|c} 1 & & \\ \hline & \varphi^0_T(w_\sigma) & \\ \hline & & 1 \end{array}\right) \rtimes w_\sigma.$$

$\phi|_{SL_2(\mathbb{C})}$ *is omitted when it is obvious. Also we identify each quasi-character* χ *of the idele class group of* K *with the composite*

$$\chi : \mathcal{L}_K \xrightarrow{\mathrm{p}_{W_K}} W_K \longrightarrow W_K^{\mathrm{ab}} \xrightarrow{\textit{reciprocity}} \mathbb{A}_K^\times / K^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

Note that both symplectic and orthogonal representations of $\mathcal{L}_K$ appear according to the action of w_σ. This is an interesting feature of the unitary groups.

3 S-groups and base point representations

3.1 Local assumptions

Let v_0 be a place of k. We abbreviate $k_{v_0} = F$, $K_{v_0} := K \otimes_k k_{v_0} = E$ and identify the generator of $\mathrm{Aut}_F(E)$ with σ. In what follows, we shall be interested only in the case when F is non-archimedean and E is a quadratic extension of F (inert case). Then the Langlands group $\mathcal{L}_F$ of F is the direct product $W_F \times SU(2)$, where W_F is the Weil group of F. Using this, local A-parameters are defined similarly as in the global case. Write $\Psi(G_F)$ for the set of equivalence classes of A-parameters for $G_F = G \otimes_k F$. We often write $\Gamma = \mathrm{Gal}(\overline{F}/F)$, $\overline{F}$ being an algebraic closure of F containing E.

For a p-adic group H, we write $\Pi(H)$ for the set of isomorphism classes of irreducible admissible representations of H. $\Pi_{\mathrm{unit}}(H) \supset \Pi_{\mathrm{temp}}(H) \supset \Pi_{\mathrm{disc}}(H) \supset \Pi_0(H)$ denote the subset of unitarizable, tempered, square-integrable and supercuspidal elements in $\Pi(H)$, respectively. For an F-parabolic subgroup $P = MU \subset G$ and a smooth representation τ of M, we write

$$I_P^G(\tau) := \mathrm{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}]$$

for the parabolically induced representation of $G(F)$ from τ. If moreover $\tau = \tau_0 \otimes e^\lambda$ with $\tau_0 \in \Pi_{\mathrm{temp}}(M(F))$ and a regular exponent $\lambda \in \mathfrak{a}_M^*$, we write $J_P^G(\tau)$ for the Langlands subquotient of $I_P^G(\tau)$.

Fix a non-trivial character $\psi := \bigotimes_v \psi_v : \mathbb{A}/k \to \mathbb{C}^1$. Write $\psi_F := \psi_{v_0}$. This combined with the standard splitting $\mathbf{spl}_G = (\mathbf{B}, \mathbf{T}, \{X_\alpha\}_{\alpha \in \Delta})$ of the group G_F yields a character $\psi_{\mathbf{U}}$ of the unipotent radical $\mathbf{U}(F)$ of $\mathbf{B}(F)$ such that

$$\psi_{\mathbf{U}}(\exp tX_\alpha) = \psi(t), \quad \forall t \in F.$$

This is non-degenerate in the sense that $\mathrm{Stab}(\psi_{\mathbf{U}}, \mathbf{T}(F)) = Z(G)(F)$. Recall that $\pi \in \Pi(G(F))$ is $\psi_{\mathbf{U}}$*-generic* if there is a non-zero linear functional $\Lambda_{\psi_{\mathbf{U}}} : V_\pi \to \mathbb{C}$ on a realization V_π of π satisfying

$$\Lambda_{\psi_{\mathbf{U}}}(\pi(u)\xi) = \psi_{\mathbf{U}}(u)\Lambda_{\psi_{\mathbf{U}}}(\xi), \quad \forall u \in \mathbf{U}(F),\ \xi \in V_\pi.$$

We need the following local assertion of the Arthur conjecture.

Conjecture 3.1 ([1] Conj. 6.1). *(A) For each $\phi \in \Psi(G_F)$ there exists a finite subset $\Pi_\phi \subset \Pi_{\mathrm{unit}}(G(F))$ called the A*-packet *associated to ϕ.*
(B) Set $S_\phi := \mathrm{Cent}(\phi, \widehat{G})$, $\mathbb{S}_\phi := \pi_0(S_\phi / Z(\widehat{G})^\Gamma)$. There exist a function $\delta : S_\phi \times \Pi_\phi \to \mathbb{C}$ and a normalization function $\rho : S_\phi \to \mathbb{C}$ such that

(1) $\rho(s_\phi) \in \{\pm 1\}$, where s_ϕ is the image of $1 \times -\mathbf{1}_2 \in \mathcal{L}_F \times SL_2(\mathbb{C})$ under ϕ.

(2) The normalized function

$$S_\phi \times \Pi_\phi \ni (s, \pi) \longmapsto \langle s, \pi \rangle := \frac{1}{\rho(s)} \delta(s, \pi) \in \mathbb{C}$$

reduces to a class function on $\mathbb{S}_\phi$.

(3) Writing $\mathbf{s}_\phi$ for the image of s_ϕ in $\mathbb{S}_\phi$, we have

$$\langle \mathbf{s}_\phi \mathbf{s}, \pi \rangle = e_\phi(\mathbf{s}_\phi, \pi) \langle \mathbf{s}, \pi \rangle, \quad \forall \mathbf{s} \in \mathbb{S}_\phi.$$

Here $e_\phi(\bullet, \pi)$ is a $\{\pm 1\}$-valued character on $\mathbb{S}_\phi$.

(C) Identifying the norm $|\,|_F$ of $F^\times$ with the composite

$$|\,|_F : \mathcal{L}_F \xrightarrow{\mathrm{pr}_{W_F}} W_F \longrightarrow W_F^{\mathrm{ab}} \xrightarrow{\textit{reciprocity}} F^\times \xrightarrow{|\,|_F} \mathbb{R}_+^\times$$

as in the global case, we write

$$\varphi_\phi : \mathcal{L}_F \ni w \longmapsto \phi\left(w, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2} \end{pmatrix}\right) \in {}^L G_F.$$

φ_ϕ is a Langlands parameter which corresponds to a non-tempered L-packet Π_{φ_ϕ}. Moreover

(1) There exists an F-parabolic subgroup $P = MU \subset G$ containing **B** *such that $\varphi_\phi = e^\lambda \otimes \varphi^M$ for some regular exponent $\lambda \in \mathfrak{a}_M^*$ and a tempered (i.e. bounded) Langlands parameter $\varphi^M : \mathcal{L}_F \to {}^L M_F$. If we set $S_{\varphi^M} := \mathrm{Cent}(\varphi^M, \widehat{M})$ and $\mathbb{S}_{\varphi^M} := \pi_0(S_{\varphi_M} / Z(\widehat{M})^\Gamma)$, there should be an injective map*

$$\Pi_{\varphi^M} \ni \tau \longmapsto \langle \bullet, \tau \rangle \in \Pi(\mathbb{S}_{\varphi^M}).$$

Here $\Pi(\mathbb{S}_{\varphi^M})$ is the set of isomorphism classes of irreducible representations of $\mathbb{S}_{\varphi^M}$, whose elements are identified with their characters.

(2) Π_{φ^M} contains a unique ψ_{U^M}-generic element τ_1 (the generic packet conjecture).

(3) From definition, we have $\Pi_{\varphi_\phi} = \{J_P^G(e^\lambda \otimes \tau) \,|\, \tau \in \Pi_{\varphi^M}\}$, and $S_{\varphi_\phi} = S_{\varphi^M}$ since λ is regular. If we set

$$\mathbb{S}_{\varphi_\phi} = \mathbb{S}_{\varphi^M} \ni \mathbf{s} \longmapsto \langle \mathbf{s}, J_P^G(e^\lambda \otimes \tau) \rangle := \langle \mathbf{s}, \tau \rangle \in \mathbb{C}^\times,$$

then the following diagram commutes:

$$\begin{array}{ccc} \Pi_{\varphi_\phi} \ni \pi \longmapsto \langle \bullet, \pi \rangle \in \Pi(\mathbb{S}_{\varphi_\phi}) \\ {\scriptstyle inclusion}\big\downarrow \qquad\qquad\qquad \big\downarrow{\scriptstyle inclusion} \\ \Pi_\phi \ni \pi \longmapsto \frac{\langle \bullet, \pi \rangle}{\langle \bullet, \pi_1 \rangle} \in \Pi(\mathbb{S}_\phi) \end{array}$$

Here we have written $\pi_1 := J_P^G(e^\lambda \otimes \tau_1) \in \Pi_\phi$. *We call this the* base-point representation *in* Π_ϕ. *Its dependence on* ψ *is obvious. Also note that* $\mathbb{S}_{\varphi_\phi}$ *is a quotient of* $\mathbb{S}_\phi$. *Finally it follows from this diagram that* $|\delta(s_\phi, \pi_1)| = 1$.

Recall the conjectural homomorphism $\iota_{v_0} : \mathcal{L}_F \to \mathcal{L}_k$. This allows us to speak of the local component

$$\phi_F : \mathcal{L}_F \times SL_2(\mathbb{C}) \xrightarrow{\iota_{v_0} \times \mathrm{id}} \mathcal{L}_k \times SL_2(\mathbb{C}) \xrightarrow{\phi} {}^L G$$

of the A-parameters given in Prop. 2.1. Note that the image of ϕ_F is in fact contained in the image of $\mathrm{id}_{\widehat{G}} \rtimes \iota_{v_0} : \widehat{G} \rtimes_{\rho_G} W_F \to \widehat{G} \rtimes_{\rho_G} W_k$, and we can view ϕ_F as a local parameter.

In the rest of this section, we describe the base point representations in the local packets Π_{ϕ_F} and the S-groups S_{ϕ_F}, $\mathbb{S}_{\phi_F}$ associated to the relevant local parameters ϕ_F.

3.2 Representations of $G(F)$

Next we review some results from [9]. We need some more notation to describe them.

Write $\omega_{E/F}$ for the quadratic character of $F^\times$ associated to E/F by the local classfield theory. As in the global setting, we reserve η and μ to denote characters of $E^\times$ such that $\eta|_{F^\times} = \mathbf{1}_{F^\times}$ and $\mu|_{F^\times} = \omega_{E/F}$, respectively. η defines a character $\eta_u : U(1,F) \ni x\sigma(x)^{-1} \mapsto \eta(x) \in \mathbb{C}^1$ of $U(1)_{E/F}(F)$. For any unitary group $U(V)$ of a hermitian space $(V, (\ ,\ ))$ over E, this defines a 1-dimensional representation $\eta^{U(V)} : G \overset{\det}{\to} U(1)_{E/F}(F) \overset{\eta_u}{\to} \mathbb{C}^1$. Here det denotes the determinant morphism $\det : GL_E(V) \to \mathbb{G}_{m,E}$.

Let G_1 be the quasisplit unitary group in two variables defined by a formula similar to (2.1). Set $\widetilde{G}_1 := \mathrm{R}_{E/F} GL_2$. We need the endoscopic liftings in the following three settings:

Standard base change for $\widetilde{G}_1$ The twisted endoscopic data $(G_1, {}^L G_1, 1, \xi_\eta)$ for $(\widetilde{G}_1, \theta_2, \mathbf{1})$ (see [12, Chapt. II]), where

$$\xi_\eta : {}^L G_1 \ni g \rtimes_{\rho_{G_1}} w \longmapsto \begin{cases} (\eta(w) g, \eta(w) g) \times w & \text{if } w \in W_E, \\ (g, g) \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^L \widetilde{G}_1.$$

Also $\theta_2(g) := \mathrm{Ad}(I_2) {}^t\sigma(g)^{-1}$, for $g \in \widetilde{G}_1$.

Twisted base change for $\widetilde{G}_1$ The twisted endoscopic data $(G_1, {}^L G_1, 1, \xi_\mu)$ for the same triple as above, where

$$\xi_\mu : {}^L G_1 \ni g \rtimes_{\rho_{G_1}} w \longmapsto \begin{cases} (\mu(w) g, \mu(w) g) \times w & \text{if } w \in W_E, \\ (g, -g) \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^L \widetilde{G}_1.$$

Endoscopic lift for G_1 The standard endoscopic data $(U(1)^2_{E/F}, {}^L(U(1)^2_{E/F}), s, \lambda_{\mu^{-1}})$ for G_1. Here

$$\lambda_{\mu^{-1}} : {}^L(U(1)^2_{E/F}) \ni (z_1, z_2) \rtimes w \longmapsto \begin{cases} \begin{pmatrix} z_1\mu(w) & \\ & z_2\mu(w) \end{pmatrix} \times w & \text{if } w \in W_E, \\ \begin{pmatrix} & -z_1 \\ z_2 & \end{pmatrix} \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^L G_1.$$

All of these are established in [16].

Recall that we have two $G(F)$-conjugacy classes of F-parabolic subgroups of G other than $\mathbf{B}$ and G itself. Their representatives are $P_i = M_i U_i$, $(i = 1, 2)$, where

$$M_1 = \left\{ m_1(A) := \left(\begin{array}{c|c} A & \\ \hline & \mathrm{Ad}(I_2)^t\sigma(A)^{-1} \end{array} \right) \,\middle|\, A \in \widetilde{G}_1 \right\},$$

$$U_1 = \left\{ \left(\begin{array}{c|c} \mathbf{1}_2 & B \\ \hline & \mathbf{1}_2 \end{array} \right) \,\middle|\, B = -\mathrm{Ad}(I_2)^t\sigma(B) \in \mathbb{M}_2(E) \right\},$$

$$M_2 = \left\{ m_2(a, g) := \left(\begin{array}{c|c|c} a & & \\ \hline & g & \\ \hline & & \sigma(a)^{-1} \end{array} \right) \,\middle|\, \begin{array}{c} a \in \mathrm{R}_{E/F}\mathbb{G}_m \\ g \in G_1 \end{array} \right\},$$

$$U_2 = \left\{ \left(\begin{array}{c|cc|c} 1 & y'' & y' & z + \langle y, y\rangle/2 \\ \hline & 1 & & -\sigma(y') \\ & & 1 & \sigma(y'') \\ \hline & & & 1 \end{array} \right) \,\middle|\, \begin{array}{c} y = (y', y'') \in E^2 \\ z \in F \end{array} \right\}.$$

Here $\langle x, y\rangle = x'\sigma(y'') - y'\sigma(x'')$ denotes the hyperbolic skew hermitian form on E^2. We describe the irreducible representations of various $M(F)$ in the following manner.

$$\chi_1[s_1] \otimes \chi_2[s_2] : \mathbf{T}(F) \ni \mathrm{diag}(a_1, a_2, \sigma(a_2)^{-1}, \sigma(a_1)^{-1}) \longmapsto \chi_1(a_1)|a_1|_E^{s_1/2}\chi_2(a_2)|a_2|_E^{s_2/2} \in \mathbb{C}^\times,$$
$$\pi[s] : M_1(F) \ni m_1(A) \longmapsto |\det A|_E^{s/2}\pi(A) \in GL(V_\pi),$$
$$\chi[s] \otimes \tau : M_2(F) \ni m_2(a, g) \longmapsto \chi(a)|a|_E^{s/2}\tau(g) \in GL(V_\tau).$$

Here χ_i, $\chi \in \Pi(E^\times)$, $\pi \in \Pi(\widetilde{G}_1(F))$, $\tau \in \Pi(G_1(F))$.

Lemma 3.2. *The Langlands data $(P, \Pi_\phi^M := e^\lambda \otimes \Pi_{\varphi^M})$ in Conj.* 3.1 *(C-1) for the local components ϕ_F of the A-parameters listed in Prop.* 2.1 *at v_0 are given by the following.*

(1) For $\phi_F = \phi_\eta$, $P = \mathbf{B}$ and $\Pi_\phi^{\mathbf{T}} = \{\eta[3] \otimes \eta[1]\}$.

(2) For $\phi_F = \phi_{\mu,\eta}$, $P = P_2$ and $\Pi_\phi^{M_2} = \{\mu[2] \otimes \tau_\pm \,|\, \tau_\pm \in \lambda_{\mu^{-1}}(\mathbf{1}, \eta)\}$. $\lambda_{\mu^{-1}}(\mathbf{1}, \eta)$ consists of two irreducible supercuspidal representations if $\eta \neq \mathbf{1}$ and two limit of discrete series representations otherwise. Write them $\tau_\pm$ so that τ_+ is ψ_{U_1}-generic.

(3) For $\phi_F = \phi_{T,\mu}$ with T an L-packet of $G_1(F)$ consisting of infinite dimensional unitarizable representations, $P = P_1$ and $\Pi_\phi^{M_1} = \{\xi_\mu(T)\}$.

(4) For $\phi_F = \phi_{\boldsymbol{\eta}}$, $P = P_1$ and $\Pi_\phi^{M_1} = \{I_{\widetilde{\mathbf{B}}_1}^{\widetilde{G}_1}(\eta_1 \otimes \eta_2)[1]\}$. Note that η_1 may be η_2 in the local case (cf. Prop. 2.1 *(3) (b)).*

(5) For $\phi_F = \phi_{T,\eta}$ with T an L-packet of $G_1(F)$ consisting of infinite dimensional unitarizable representations, we have $P = P_2$ and $\Pi_\phi^{M_2} = \{\eta[1] \otimes \tau \mid \tau \in T\}$.

Remark 3.3. *(i) It is a result of Keys [8] that $\tau_+ \in \lambda_{\mu^{-1}}(\mathbf{1}, \eta)$ is the* unique *unramified member of the packet when η is trivial.*
(ii) If we assume the generalized Ramanujan conjecture for automorphic forms on GL_2, then the infinite dimensionality and unitarizability conditions in (3) and (5) can be strengthened to the assertion that T is an tempered *L-packet. Same kind of replacements are found in [4].*
(iii) Consider the comment in (4). Returning to the global setting, let η be as in § 2. Regarding it as a character of $\mathrm{R}_{K/k}\mathbb{G}_m(\mathbb{A})$, we have the Eulerian decomposition $\eta = \bigotimes_v \eta_v$. Then η_v must be trivial at all but finite places where the extension K_v/k_v (may be split) and η_v are both unramified.

Now we recall the results of [9] on the composition series of $I_P^G(\pi)$, $\pi \in \Pi_\phi^M$ for (P, Π_ϕ^M) appeared in the above lemma. These will be used also to verify Hiraga's conjecture 5. We write δ^H for the Steinberg representation of a connected quasisplit reductive group $H(F)$. The equalities are those in the Grothendieck group of admissible representations of finite length of $G(F)$.
(1) For ϕ_η we have

$$I_{\mathbf{B}}^G(\eta[3] \otimes \eta[1]) = \eta^G\delta^G + J_{P_1}^G(\eta\delta^{\tilde{G}_1}[2]) + J_{P_2}^G(\eta[3] \otimes \eta^{G_1}\delta^{G_1}) + \eta^G.$$

$\eta^G\delta^G \in \Pi_{\mathrm{disc}}(G(F))$, $\eta^G \in \Pi_{\mathrm{unit}}(G(F))$ and other two constituents are not unitarizable.
(2) For $\phi_{\mu,\eta}$ we have the following two possibilities.

(i) $\eta \neq \mathbf{1}$ and $\tau_\pm \in \lambda_{\mu^{-1}}(\mathbf{1}, \eta)$ are supercuspidal.

$$I_{P_2}^G(\mu[2] \otimes \tau_\pm) = \delta_2^G(\mu, \tau_\pm) + J_{P_2}^G(\mu[2] \otimes \tau_\pm),$$

where $\delta_2^G(\mu, \tau_\pm) \in \Pi_{\mathrm{disc}}(G(F))$ and $J_{P_2}^G(\mu[2] \otimes \tau_\pm) \in \Pi_{\mathrm{unit}}(G(F))$.

(ii) η is trivial and $\tau_\pm = \tau^1(\mu)_\pm$ are the irreducible components of $I_{\mathbf{B}_1}^{G_1}(\mu)$.

$$I_{P_2}^G(\mu[2] \otimes \tau^1(\mu)_\pm) = \delta_0^G(\mu)_\pm + J_{P_1}^G(\mu\delta^{\tilde{G}_1}[1]) + J_{P_2}^G(\mu[2] \otimes \tau^1(\mu)_\pm),$$

where $\delta_0^G(\mu)_\pm \in \Pi_{\mathrm{disc}}(G(F))$, other two constituents are also unitarizable.

(3) For $\phi_{T,\mu}$, we have the following six possibilities.

(i) T consists of one supercuspidal representation. Then $\pi := \xi_\mu(T)$ is an irreducible supercuspidal representation and we have

$$I_{P_1}^G(\pi[1]) = \delta_1^G(\pi) + J_{P_1}^G(\pi[1]).$$

Here $\delta_1^G(\pi) \in \Pi_{\mathrm{disc}}(G(F))$ and $J_{P_1}^G(\pi[1]) \in \Pi_{\mathrm{unit}}(G(F))$.

(ii) $T=\{\eta^{G_1}\delta^{G_1}\}$. Then $\xi_\mu(T)=\eta\mu\delta^{\widetilde{G}_1}$ and

$$I^G_{P_1}(\eta\mu\delta^{\widetilde{G}_1}[1])=\delta^G_0(\eta\mu)_+ +\delta^G_0(\eta\mu)_- + J^G_{P_1}(\eta\mu\delta^{\widetilde{G}_1}[1]).$$

Here $\delta^G_0(\eta\mu)_\pm\in\Pi_{\mathrm{disc}}(G(F))$ and $J^G_{P_1}(\eta\mu\delta^{\widetilde{G}_1}[1])\in\Pi_{\mathrm{unit}}(G(F))$.

(iii) $T=\{I^{G_1}_{\mathbf{B}_1}(\chi)\}$, where $\chi\in\Pi(E^\times)$ is such that $\sigma(\chi)^{-1}\neq\chi$. $\xi_\mu(T)=I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\chi\otimes\mu\sigma(\chi)^{-1})$ and we have

$$I^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\chi\otimes\mu\sigma(\chi)^{-1})[1])=I^G_{P_1}(\mu\chi\delta^{\widetilde{G}_1})+I^G_{P_1}(\mu\chi(\det)).$$

Here $I^G_{P_1}(\mu\chi\delta^{\widetilde{G}_1})\in\Pi_{\mathrm{temp}}(G(F))$ and $I^G_{P_1}(\mu\chi(\det))\in\Pi_{\mathrm{unit}}(G(F))$.

(iv) $T=\{I^{G_1}_{\mathbf{B}_1}(\eta[s])\}$, $0\le s<1$. $\xi_\mu(T)=\{I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\eta[s]\otimes\mu\eta[-s])\}$ and we have

$$I^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\eta[s]\otimes\mu\eta[-s])[1])=I^G_{P_1}(\mu\eta\delta^{\widetilde{G}_1}[s])+I^G_{P_1}(\mu\eta(\det)[s]).$$

Here $I^G_{P_1}(\mu\eta\delta^{\widetilde{G}_1}[s])\in\Pi_{\mathrm{temp}}(G(F))$ if $s=0$ and both constituents are unitarizable.

(v) $T=\lambda_{\mu_1^{-1}}(\mathbf{1},\eta)$ with $\eta\neq\mathbf{1}$. $\xi_\mu(T)=\{I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\mu_1,\mu\mu_1\eta)\}$ and the irreducible constituents are given in (4-i) below.

(vi) $T=\{\tau^1(\mu_1)_\pm\}$. $\xi_\mu(T)=\{I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\mu_1,\mu\mu_1)\}$ and the irreducible constituents are given in (4-ii) below.

(4) For $\phi_{\boldsymbol{\eta}}$ we have the following two possibilities.

(i) $\eta_1\neq\eta_2$.

$$\begin{aligned}&I^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{B}_1}(\eta_1\otimes\eta_2)[1])\\&=\delta^G_0(\eta_1,\eta_2)+J^G_{P_2}(\eta_1[1]\otimes\eta_2^{G_1}\delta^{G_1})+J^G_{P_2}(\eta_2[1]\otimes\eta_1^{G_1}\delta^{G_1})+J^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{B}_1}(\eta_1\otimes\eta_2)[1]),\end{aligned}$$

where $\delta_0(\eta_1,\eta_2)\in\Pi_{\mathrm{disc}}(G(F))$ and the other three constituents are all unitarizable.

(ii) $\eta_1=\eta_2$. Write η for this.

$$I^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{B}_1}(\eta\otimes\eta)[1])=\eta^G\tau(\delta^{G_1})+\eta^G\tau(\mathbf{1}_{G_1})+J^G_{P_2}(\eta[1]\otimes\eta^{G_1}\delta^{G_1})+J^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{B}_1}(\eta\otimes\eta)[1]),$$

where $\eta^G\tau(\delta^{G_1})$, $\eta^G\tau(\mathbf{1}_{G_1})\in\Pi_{\mathrm{temp}}(G(F))$ and the other two constituents are also unitarizable.

(5) For $\phi_{T,\eta}$, the following six cases occur.

(i) T consists of supercuspidal representations.

$$I^G_{P_2}(\eta[1]\otimes\tau)=\delta^G_2(\eta,\tau)+J^G_{P_2}(\eta[1]\otimes\tau),\quad \tau\in T,$$

where $\delta^G_2(\eta,\tau)\in\Pi_{\mathrm{disc}}(G(F))$ and $J^G_{P_2}(\eta[1]\otimes\tau)\in\Pi_{\mathrm{unit}}(G(F))$.

(ii) $T=\{\eta'^{G_1}\delta^{G_1}\}$ with $\eta'\neq\eta$.

$$I^G_{P_2}(\eta[1]\otimes\eta'^{G_1}\delta^{G_1})=\delta^G_0(\eta,\eta')+J^G_{P_2}(\eta[1]\otimes\eta'^{G_1}\delta^{G_1}),$$

where both constituents are as in (4-i).

(iii) $T=\{\eta^{G_1}\delta^{G_1}\}$.

$$I^G_{P_2}(\eta[1]\otimes\eta^{G_1}\delta^{G_1})=\eta^G\tau(\delta^{G_1})+J^G_{P_2}(\eta[1]\otimes\eta^{G_1}\delta^{G_1}),$$

where both constituents are as in (4-ii).

(iv) $T=\{I^{G_1}_{\mathbf{B}_1}(\chi)\}$, where $\chi\in\Pi(E^\times)$ is such that $\chi|_{F^\times}\neq\omega_{E/F}$.

$$I^G_{P_2}(\eta[1]\otimes I^{G_1}_{\mathbf{B}_1}(\chi))=I^G_{P_2}(\chi\otimes\eta^{G_1}\delta^{G_1})+I^G_{P_2}(\chi\otimes\eta^{G_1}),$$

where $I^G_{P_2}(\chi\otimes\eta^{G_1}\delta^{G_1})\in\Pi_{\mathrm{temp}}(G(F))$ and $I^G_{P_2}(\chi\otimes\eta^{G_1})\in\Pi_{\mathrm{unit}}(G(F))$.

(v) $T=\{I^{G_1}_{\mathbf{B}_1}(\eta'[s])\}$, $0<s<1$.

$$I^G_{P_2}(\eta[1]\otimes I^{G_1}_{\mathbf{B}_1}(\eta'[s]))=I^G_{P_2}(\eta'[s]\otimes\eta^{G_1}\delta^{G_1})+I^G_{P_2}(\eta'[s]\otimes\eta^{G_1}),$$

where the two constituents are unitarizable.

(vi) $T=\{\tau^1(\mu)_\pm\}$.

$$I^G_{P_2}(\eta[1]\otimes\tau^1(\mu)_\pm)=\tau_0(\mu,\eta)_\pm+J^G_{P_2}(\eta[1]\otimes\tau^1(\mu)_\pm),$$

where $\tau_0(\mu,\eta)_\pm\in\Pi_{\mathrm{temp}}(G(F))$ and the other constituent is unitarizable.

3.3 S-groups and the base points representations

The next lemma follows immediately from the above list.

Lemma 3.4. *(1) For ϕ_η, $S_{\phi_\eta}=\{\pm\mathbf{1}_4\}$, $\mathbb{S}_{\phi_\eta}$ is trivial. In particular the local packet Π_{ϕ_η} consists of the base point representation η^G.*
(2) For $\phi_{\mu,\eta}$, $S_{\phi_{\mu,\eta}}=\{\mathrm{diag}(\epsilon_1\mathbf{1}_3,\epsilon_2)\,|\,\epsilon_i=\pm1\}$, $\mathbb{S}_{\phi_{\mu,\eta}}\simeq\mathbb{Z}/2\mathbb{Z}$. In particular $\Pi_{\phi_{\mu,\eta}}=\{J^G_{P_2}(\mu[2]\otimes\tau_\pm)\,|\,\tau_\pm\in\lambda_{\mu^{-1}}(\mathbf{1},\eta)\}$ and the base point representation is $J^G_{P_2}(\mu[2]\otimes\tau_+)$.
(3) For $\phi_{T,\mu}$, we have the following three cases.

(3-i,ii) *T consists of only one square integrable representation. $S_{\phi_{T,\mu}}=\{\pm\mathbf{1}_4\}$ and $\mathbb{S}_{\phi_{T,\mu}}$ is trivial. $\Pi_{\phi_{T,\mu}}$ consists of the base point $J^G_{P_1}(\xi_\mu(T)[1])$.*

(3-iii, iv) *T consists of one parabolically induced representation $I^{G_1}_{\mathbf{B}_1}(\chi[s])$.*

$$S_{\phi_{T,\mu}}=\begin{cases}\{\mathrm{diag}(t,t^{-1},t,t^{-1}) & \text{if } \chi[s]\neq\eta,\\ \{\mathrm{diag}(g,g)\,|\,g\in SL_2(\mathbb{C})\} & \text{otherwise,}\end{cases}\qquad \mathbb{S}_{\phi_{T,\mu}}=\{1\}.$$

Again $\Pi_{\phi_{T,\mu}}$ contains only the base point representation $I^G_{P_1}(\mu\chi(\det)[s])$.

(3-v,vi) $T=\lambda_{\mu^{-1}}(\mathbf{1},\eta)$.

$$S_{\phi_{T,\mu}}=\begin{cases}\{\mathrm{diag}(t,t^{-1},t,t^{-1}) & \textit{if } \eta \textit{ is not trivial,}\\ \{\mathrm{diag}(g,g)\,|\,g\in SL_2(\mathbb{C})\} & \textit{otherwise,}\end{cases}\qquad \mathbb{S}_{\phi_{T,\mu}}=\{1\}.$$

$\Pi_{\phi_{T,\mu}}$ *consists of the base point representation* $J^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\mu\mu_1,\mu\mu_1\eta)[1])$.

(4) For $\phi_{\boldsymbol{\eta}}$,

$$S_{\phi_{\boldsymbol{\eta}}}=\begin{cases}\{\mathrm{diag}(\epsilon_1,\epsilon_2,\epsilon_2,\epsilon_1)\,|\,\epsilon_i=\pm1\} & \textit{if } \eta_1\neq\eta_2,\\ \{\mathrm{diag}(g,\theta_2(g))\,|\,g\in O_2(\mathbb{C})\} & \textit{otherwise,}\end{cases}\qquad \mathbb{S}_{\phi_{\boldsymbol{\eta}}}\simeq\mathbb{Z}/2\mathbb{Z}.$$

The base point representation is $J^G_{P_1}(I^{\widetilde{G}_1}_{\widetilde{\mathbf{B}}_1}(\eta_1\otimes\eta_2)[1])$.
(5) For $\phi_{T,\eta}$, *we have the following three cases.*

(5-i,ii,iii) T *consists of square integrable representations.*

$$S_{\phi_{T,\eta}}=\begin{cases}\{\mathrm{diag}(\epsilon_1,\epsilon_2,\epsilon_2,\epsilon_1\,|\,\epsilon_i=\pm1\} & \textit{if } T \textit{ is stable,}\\ \{\mathrm{diag}(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_1\,|\,\epsilon_i=\pm1\} & \textit{if } T=\lambda_{\mu^{-1}}(\mathbf{1},\eta'),\end{cases}\qquad \mathbb{S}_{\phi_{T,\mu}}\simeq\mathbb{S}_{\varphi_T}\times\mathbb{Z}/2\mathbb{Z}.$$

The base point representation is $J^G_{P_2}(\eta[1]\otimes\tau_+)$, *where* $\tau_+\in T$ *is the unique* $\psi_{\mathbf{U}_1}$-*generic element.*

(5-iv,v) T *consists of a principal or complementary series representation* $I^{G_1}_{\mathbf{B}_1}(\chi[s])$. *Then*

$$S_{\phi_{T,\eta}}=\begin{cases}\{\mathrm{diag}(\epsilon,t,t^{-1},\epsilon)\,|\,\epsilon=\pm1,\ t\in\mathbb{C}^\times\} & \textit{if } \chi[s]\neq\eta,\\ \{\mathrm{diag}(\epsilon,g,\epsilon)\,|\,\epsilon=\pm1,\ g\in SL_2(\mathbb{C})\} & \textit{otherwise,}\end{cases}$$

and $\mathbb{S}_{\phi_{T,\eta}}$ *is trivial.* $\Pi_{\phi_{T,\eta}}$ *consists of the base point representation* $I^G_{P_2}(\chi[s]\otimes\eta^{G_1})$.

(5-vi) $T=\{\tau^1(\mu)_\pm\}$.

$$S_{\phi_{T,\eta}}=\{\mathrm{diag}(\epsilon,g,\epsilon)\,|\,\epsilon=\pm1,\ g\in O_2(\mathbb{C})\},\quad \mathbb{S}_{\phi_{T,\eta}}\simeq\mathbb{S}_{\varphi_T}\times\mathbb{Z}/2\mathbb{Z}.$$

The base point representation is $J^G_{P_2}(\eta[1]\otimes\tau^1(\mu)_+)$.

Remark 3.5. *Note that these representations are exactly the local components of the residual discrete spectrum of G. The correspondence is illustrated as follows:*

A-parameter	*Residual representation in [11]*
ϕ_η	*(1) with* $\chi=\eta_u$
$\phi_{\mu,\eta}$, $\eta\neq\mathbf{1}$	$J^{G(\mathbb{A})}_{P_2(\mathbb{A})}(\mu\|\ \|_{A_K}\otimes\theta_{\mu^{-1}}(\eta_u,W))$ *in (5)*
$\phi_{\mu,\mathbf{1}}$	*(2) with* $\chi=\mu$
$\phi_{T,\mu}$	*(4) with* $\pi=\xi_\mu(T)$
$\phi_{\boldsymbol{\eta}}$	*(3)*
$\phi_{T,\eta}$	$J^{G(\mathbb{A})}_{P_2(\mathbb{A})}(\eta\|\ \|^{1/2}_{\mathbb{A}_K}\otimes\tau)$ *in (5),* $\tau\in T$.

The theta lift $\theta_{\mu^{-1}}(\eta_u,W)$ *is defined below §4.1. The fact that these representations appeared in the discrete spectrum with multiplicity one justifies the choice of our "base point" representations.*

4 Theta correspondences

In Lem. 3.4, the A-packets are completely determined except for the cases (4) and (5-i), (5-ii), (5-iii), (5-vi). Among these excluded cases, (4-ii) and (5-vi) can be treated by the construction of [1, § 7] since the A-parameters are not elliptic. But in the other cases, the rest members of the packets must be *supercuspidal*. In this section, we construct the candidates for these representations by the local theta correspondences. We begin with a brief review of the Weil representations and local theta correspondences for unitary dual pairs of our concern.

4.1 Weil representations to be used

We consider the local theta correspondences of unitary groups defined with respect to a quadratic extension E/F of p-adic fields [13], [6].

Fix a generator δ of E over F such that $\Delta := \delta^2 \in F^\times$. Let $(W_n, \langle\,,\,\rangle_n)$ be the skew-hermitian space

$$W_n = E^{2n}, \quad \langle (x, x'), (y, y') \rangle_n = x\,{}^t\sigma(y') - x'\,{}^t\sigma(y),$$

and $(V_\pm, (\,,\,)_\pm)$ be the hermitian planes E^2 with the forms

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)_+ := \delta(\sigma(x_1)y_2 - \sigma(x_2)y_1), \quad \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)_- := -\sigma(x_1)y_1 + \gamma\sigma(x_2)y_2.$$

Here we have fixed $\gamma \in F^\times \setminus \mathrm{N}_{E/F}(E^\times)$. We write $G = G_2 := U(W_2)$, $G_1 := U(W_1) = U(V_+)$, $G_1' := U(V_-)$. Note that G and G_1 are quasisplit while G_1' is anisotropic.

For $(W_n, \langle\,,\,\rangle_n)$ and $(V_\pm, (\,,\,)_\pm)$ as above, define

$$\mathbb{W} := V_\pm \otimes_E W_n, \quad \langle\!\langle v \otimes w, v' \otimes w' \rangle\!\rangle := \frac{1}{2}\mathrm{Tr}_{E/F}[(v, v')\sigma(\langle w, w' \rangle)],$$

an $8n$-dimensional symplectic space. We have a homomorphism

$$\iota : G_1^\bullet(F) \times G_n(F) \ni (h, g) \longmapsto h \otimes g \in \mathrm{Sp}(\mathbb{W}).$$

Write $Y := \{(0, \dots, 0, y_1, \dots, y_n) \in W_n\}$, $Y' := \{(y_1', \dots, y_n', 0, \dots, 0) \in W_n\}$, two maximal isotropic subspaces dual to each other. These give the Lagrangians $\mathbb{Y} := V_\pm \otimes_E Y$, $\mathbb{Y}' := V_\pm \otimes_E Y'$ of $\mathbb{W}$. Let $P = MU$ be the Siegel parabolic subgroup:

$$P := \mathrm{Stab}(Y, G), \quad M := \mathrm{Stab}(Y \oplus Y', G), \quad U := \{g \in P \,|\, g|_Y = \mathrm{id}_Y\}.$$

More explicitly, we have

$$M = \left\{ \begin{pmatrix} a & \\ & {}^t\sigma(a)^{-1} \end{pmatrix} \,\middle|\, a \in \mathrm{R}_{E/F}GL_n \right\}, \quad U = \left\{ \begin{pmatrix} \mathbf{1}_n & b \\ & \mathbf{1}_n \end{pmatrix} \,\middle|\, b = {}^t\sigma(b) \right\}$$

Recall the metaplectic group $\mathrm{Mp}(\mathbb{W})$ of $\mathrm{Sp}(\mathbb{W})$:

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(\mathbb{W}) \longrightarrow \mathrm{Sp}(\mathbb{W}) \longrightarrow 1.$$

The Lagrangian $\mathbb{Y}$ specifies a continuous embedding $\mathrm{Sp}(\mathbb{W}) \hookrightarrow \mathrm{Mp}(\mathbb{W})$ so that the multiplication of $\mathrm{Mp}(\mathbb{W}) = \mathrm{Sp}(\mathbb{W}) \times \mathbb{C}^1$ is given by

$$(g_1, \varepsilon_1)(g_2, \varepsilon_2) = (g_1 g_2, \varepsilon_1 \varepsilon_2 c_{\mathbb{Y}}(g_1, g_2)), \quad c_{\mathbb{Y}}(g_1, g_2) = \gamma_{\psi_F}(L(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)).$$

Here $L(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)$ is the Leray invariant [15, Defn. 2.10] and $\gamma_{\psi_F}(\bullet)$ denotes the Weil constant.

Using the Bruhat decomposition $G_n = \coprod_{r=0}^n P w_r P$,

$$w_r = \left(\begin{array}{cc|cc} \mathbf{0}_r & & -\mathbf{1}_r & \\ & \mathbf{1}_{n-r} & & \\ \hline \mathbf{1}_r & & \mathbf{0}_r & \\ & & & \mathbf{1}_{n-r} \end{array}\right)$$

write $g \in G_n(F)$ as

$$g = \begin{pmatrix} a_1 & * \\ & {}^t\sigma(a_1)^{-1} \end{pmatrix} w_r \begin{pmatrix} a_2 & * \\ & {}^t\sigma(a_2)^{-1} \end{pmatrix}.$$

Define $r(g) := r$ and $d(g) := \det(a_1 a_2) \in E^\times / \mathrm{N}_{E/F}(E^\times)$. Fix $\eta \in \Pi(E^\times)$ such that $\eta|_{F^\times} = \mathbf{1}$ and recall Langlands' λ-factor $\lambda(E/F, \psi_F) = \gamma_{\psi_F}(1)/\gamma_{\psi_F}(\Delta)$. If we set

$$\begin{aligned} \beta_{V_\pm}(g) :&= (\lambda(E/F, \psi_F)^2 \omega_{E/F}(\det V_\pm))^{-r(g)} \eta(d(g)) \\ &= \begin{cases} \eta(d(g)) & \text{in the case of } V_+ \\ (-1)^{r(g)} \eta(d(g)) & \text{in the case of } V_-, \end{cases} \end{aligned}$$

then

$$\widetilde{\iota}_\eta : G_1^\bullet(F) \times G_n(F) \ni (h, g) \longmapsto (\iota(h, g), \beta_{V_\pm}(g)) \in \mathrm{Mp}(\mathbb{W})$$

is a continuous homomorphism lifting ι [13, Th. 3.1].

The Heisenberg group $\mathcal{H}(\mathbb{W})$ associated to $\mathbb{W}$ is $\mathbb{W} \oplus F$ with the multiplication

$$(w; z)(w'; z') = (w + w'; z + z' + \frac{\langle w, w' \rangle}{2}).$$

By Stone-von Neumann theorem, there exists, up to isomorphisms, unique irreducible unitary representation ρ_{ψ_F} of $\mathcal{H}(\mathbb{W})$ on which the center F acts by ψ_F. Its underlying admissible representation is realized on $\mathcal{S}(\mathbb{Y}') = \mathcal{S}(V_\pm^n)$:

$$\rho_{\psi_F}(y', y; z)\phi(x') = \psi_F(z + \frac{\langle 2x' + y', y \rangle}{2})\phi(x' + y'), \quad \phi \in \mathcal{S}(\mathbb{Y}').$$

This extends uniquely to an irreducible admissible representation ρ_{ψ_F} of $\mathrm{Mp}(\mathbb{W}) \ltimes \mathcal{H}(\mathbb{W})$, the metaplectic Jacobi group. Here the action of $\mathrm{Mp}(\mathbb{W})$ on $\mathcal{H}(\mathbb{W})$ is through the $\mathrm{Sp}(\mathbb{W})$-action on $\mathbb{W}$. The composite

$$\omega_{V_\pm, \eta}^n : G^\bullet(F) \times G_n(F) \xrightarrow{\widetilde{\iota}_\eta} \mathrm{Mp}(\mathbb{W}) \xrightarrow{\rho_{\psi_F}} U(\mathcal{S}(V_\pm^n))$$

is the Weil representation of $G^\bullet(F) \times G_n(F)$ associated to η. It is characterized by the formulae [13, § 5]:

$$\omega^n_{V_\pm,\eta}(\begin{pmatrix} a & \\ & {}^t\sigma(a)^{-1}\end{pmatrix})\phi(v) = \eta(\det a)|\det a|_E\phi(v.a), \quad a \in \mathrm{GL}_n(E) \tag{4.1}$$

$$\omega^n_{V_\pm,\eta}(\begin{pmatrix} \mathbf{1}_n & b \\ & \mathbf{1}_n\end{pmatrix})\phi(v) = \psi_F(\frac{\mathrm{tr}(v,v)b}{2})\phi(v), \quad b = {}^t\sigma(b) \in \mathbb{M}_n(E) \tag{4.2}$$

$$\omega^n_{V_\pm,\eta}(w_n)\phi(v) = (\pm 1)^n \mathcal{F}_{V_\pm}\phi(-v) \tag{4.3}$$

$$\omega^n_{V_\pm,\eta}(h)\phi(v) = \phi(h^{-1}v), \quad h \in G^\bullet(F) \tag{4.4}$$

where

$$\mathcal{F}_{V_\pm}\phi(v) := \int_{V^n_\pm} \phi(v')\psi_E(\frac{\mathrm{tr}(v,v')_\pm}{2})dv', \quad \psi_E = \psi_F \circ \mathrm{Tr}_{E/F}.$$

For $\pi \in \Pi(G_n(F))$, let $\mathcal{S}(V^n_\pm)_\pi$ be the maximal quotient (possibly zero) of $\mathcal{S}(V^n_\pm)$ on which $G_n(F)$ acts by some copy of π. There exists an algebraic representation $\Theta_\eta(\pi, V_\pm)$ of $G^\bullet(F)$ such that $\mathcal{S}(V^n_\pm)_\pi \simeq \Theta_\eta(\pi, V_\pm) \otimes \pi$.

Conjecture 4.1 (Local Howe duality). *(i) $\Theta_\eta(\pi, V_\pm)$ is a finitely generated admissible representation.*
(ii) It admits a unique irreducible quotient $\theta_\eta(\pi, V_\pm)$.
(iii) $\Pi(G_n(F)) \ni \pi \mapsto \theta_\eta(\pi, V_\pm) \in \Pi(G^\bullet(F))$ is an bijection between the subsets of elements of $\Pi(G_n(F))$ and $\Pi(G^\bullet(F))$ which appear as quotients of $\omega^n_{V_\pm,\eta}$.

Of course, this is now a theorem of Waldspurger if the residual characteristic of F is odd [17]. We make use of the result of [6] which is still valid in the even residual characteristic case (see the remark in the beginning of section 3 of that paper). This justifies our use of notation $\theta_\eta(\pi, V_\pm)$ in any case. Similarly we consider the lifting $\theta_\eta(\tau, W_n)$ from $G^\bullet(F)$ to $G_n(F)$ under the same Weil representation.

4.2 Local theta correspondences

Let $\phi : \mathcal{L}_k \times SL_2(\mathbb{C}) \to {}^LG$ be a global A-parameter. Assume that the local A-packets Π_{ϕ_v} associated to its local components ϕ_v are defined. At all but a finite number of places, the base point representation $\pi^1_v \in \Pi_{\phi_v}$ is unramified. Then we can form the global A-packet Π_ϕ as the restricted tensor product $\bigotimes_v \Pi_{\phi_v}$ with respect to the base point representations. The following hypothesis is one of the naive goals of the Arthur conjecture.

Assumption 4.2. *The strong multiplicity one property holds for A-packets. That is, two irreducible discrete automorphic representations sharing all but a finite number of local components belong to a same A-packet.*

We combine this with the theta correspondence to construct candidates of A-packets. The key is the following result of M. Harris.

Proposition 4.3 ([5] Th. 4.1). *Write $\varepsilon(s, \tau \times \chi, \psi_F)$ for the standard ε-factor for $\tau \times \chi$. Then $\theta_\eta(\tau, V_\varepsilon) \neq 0$ if and only if*

$$\varepsilon(1/2, \tau \times \eta^{-1}, \psi_F)\omega_\tau(-1) = \varepsilon \tag{4.5}$$

For such ε, *we have*

$$\theta_\eta(\tau, V_\varepsilon) = \begin{cases} \eta^{G_1}\tau^\vee & \text{if } \varepsilon = 1 \\ \eta^{G_1'}\mathrm{JL}(\tau)^\vee & \text{ohterwise.} \end{cases}$$

Here $\tau^\vee$ *is the contragredient of* τ *and* $\mathrm{JL}(\tau)$ *denotes the Shimizu-Jacquet-Langlands correspondent of* τ.

We are now ready to give the case-by-case construction.
(5-i) We need to find the partner for $J^G_{P_2}(\eta[1]\otimes\tau)$, $\tau \in \Pi_0(G_1(F))$. Take $\varepsilon \in \{\pm 1\}$ satisfying (4.5) and write $\tau' := \theta_\eta(\tau, V_\varepsilon)$. The tower property of theta correspondence yields

$$\theta_\eta(\tau', W_2) \simeq J^G_{P_2}(\eta[1]\otimes\tau).$$

It follows from Prop. 4.3 that $\theta_\eta(\mathrm{JL}(\tau'), W_1) = 0$, and hence (the early lift)

$$\pi(\tau, \eta) := \theta_\eta(\mathrm{JL}(\tau'), W_2) \in \Pi_0(G(F)).$$

We set $\Pi_{\phi_{T,\eta}} = \{J^G_{P_2}(\eta[1]\otimes\tau), \pi(\tau,\eta)\}$.
(5-ii) Construct the partner for $J^G_{P_2}(\eta[1]\otimes\eta'^{G_1}\delta^{G_1})$, $\eta \neq \eta'$. We know that $\varepsilon(\frac{1}{2}, \eta'^{G_1}\delta^{G_1}\times\eta^{-1}, \psi_F) = 1$, and $\theta_\eta(\eta'^{G_1}\delta^{G_1}, V_+) = (\eta\eta'^{-1})^{G_1}\delta^{G_1}$. Thus

$$\theta_\eta((\eta\eta'^{-1})^{G_1}\delta^{G_1}, W_2) = J^G_{P_2}(\eta[1]\otimes\eta'^{G_1}\delta^{G_1}).$$

$\mathrm{JL}((\eta\eta'^{-1})^{G_1}\delta^{G_1}) = (\eta\eta'^{-1})^{G_1'}$ and

$$\pi(\eta'^{G_1}\delta^{G_1}, \eta) := \theta_\eta((\eta\eta'^{-1})^{G_1'}, W_2) \in \Pi_0(G(F)).$$

We set $\Pi_{\phi_{T,\eta}} = \{J^G_{P_2}(\eta[1]\otimes\eta'^{G_1}\delta^{G_1}), \pi(\eta'^{G_1}\delta^{G_1}, \eta)\}$.
(5-iii) Construct the partner of $J^G_{P_2}(\eta[1]\otimes\eta^{G_1}\delta^{G_1})$. In this case $\varepsilon(\frac{1}{2}, \eta^{G_1}\delta^{G_1}\times\eta^{-1}, \psi_F) = -1$ and $\theta_\eta(\eta^{G_1}\delta^{G_1}, V_-) = \mathbf{1}_{G_1'}$. It follows that

$$\theta_\eta(\mathbf{1}_{G_1'}, W_2) = J^G_{P_2}(\eta[1]\otimes\eta^{G_1}\delta^{G_1}).$$

(This can also be deduced from the result of [14].) We have $\theta_\eta(\eta^{G_1}\delta^{G_1}, W_2) = \eta^G\tau(\mathbf{1}_{G_1})$ and set $\Pi_{\phi_{T,\eta}} = \{J^G_{P_2}(\eta[1]\otimes\eta^{G_1}\delta^{G_1}), \eta^G\tau(\mathbf{1}_{G_1})\}$.

These three cases form the local theory of the theta correspondence of *infinite dimensional* automorphic representations of G_1' to G.
(5-vi) In this case the A-parameter becomes

$$\phi_{T,\eta}|_{\mathcal{L}_E} = \mathrm{diag}(\mu, \eta, \eta, \mu)\times p_{W_E}, \quad \phi_{T,\eta}(w_\sigma) = \mathrm{diag}(-1, \mathbf{1}_2, 1)\rtimes w_\sigma$$

$$\phi_{T,\eta}(g) = \begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix} \times 1, \quad g \in SL_2(\mathbb{C}).$$

This certainly passes through ${}^LM_2$ and the corresponding A-packet of $M_2(F)$ is $\Pi_{\phi^{M_2}_{T,\eta}} = \{\mu\otimes\eta^{G_1}\}$. Thus by [1, § 7], the induced packet $\Pi_{\phi_{T,\eta}}$ becomes

$$\Pi_{\phi_{T,\eta}} = \{J^G_{P_2}(\eta[1]\otimes\tau^1(\mu)_\pm)\},$$

the set of irreducible constituents of $I_{P_2}^G(\mu \otimes \eta^{G_1})$.
(4-i) We need to construct the partner of $J_{P_1}^G(I_{\tilde{\mathbf{B}}_1}^{\tilde{G}_1}(\eta_1 \otimes \eta_2)[1])$, $\eta_1 \neq \eta_2$. We know that

$$\theta_{\eta_1}((\eta_1\eta_2^{-1})^{G_1}, W_2) = J_{P_1}^G(I_{\tilde{\mathbf{B}}_1}^{\tilde{G}_1}(\eta_1 \otimes \eta_2)[1]).$$

We set $\Pi_{\phi_{\boldsymbol{\eta}}} = \{J_{P_1}^G(I_{\tilde{\mathbf{B}}_1}^{\tilde{G}_1}(\eta_1 \otimes \eta_2)[1]), \pi(\eta_2^{G_1}\delta^{G_1}, \eta_1)\}$ (see (5-ii) above).
(4-ii) In this case the parameter is given by

$$\phi_{\boldsymbol{\eta}}|_{\mathcal{L}_E} = \eta \mathbf{1}_4 \times p_{W_E}, \quad \phi_{\boldsymbol{\eta}}(w_\sigma) = \mathbf{1}_4 \rtimes w_\sigma,$$
$$\phi_{\boldsymbol{\eta}}(g) = \left(\begin{array}{c|c} g & \\ \hline & g \end{array}\right) \times 1.$$

This passes through ${}^LM_1$ and the corresponding A-packet for M_1 is $\Pi_{\phi_{\boldsymbol{\eta}}^{M_1}} = \{\eta(\det)\}$. The induced packet becomes

$$\Pi_{\phi_{\boldsymbol{\eta}}} = \{J_{P_1}^G(I_{\tilde{\mathbf{B}}_1}^{\tilde{G}_1}(\eta \otimes \eta)[1]), J_{P_2}^G(\eta[1] \otimes \eta^{G_1}\delta^{G_1})\}.$$

These two cases form the local theory of the theta correspondence of *one-dimensional* automorphic representations of G_1' to G.

5 Zelevinskii duality and Hiraga's conjecture

Let G be a connected reductive group over a p-adic field F. We write $\mathrm{Adm}(G(F))$ for the category of admissible representations of finite length of $G(F)$ and $K\Pi(G(F))$ for its Grothendieck group. If $P = MU$ be a parabolic subgroup of G, then we have the parabolic induction functor

$$I_P^G : \mathrm{Adm}(M(F)) \longrightarrow \mathrm{Adm}(G(F)),$$

and the Jacquet functor

$$r_P^G : \mathrm{Adm}(G(F)) \longrightarrow \mathrm{Adm}(M(F)).$$

r_P^G is the left adjoint of I_P^G. The homomorphisms between Grothendieck groups induced by these functors are denoted by the same symbols.

In [18, 9.16], Zelevinskii introduced certain involution $\mathcal{D}_G$ on $K\Pi(GL_n(F))$. For a general reductive group G, its definition is given by [2]

$$\mathcal{D}_G(\pi) := \sum_P (-1)^{\mathrm{rk}_F(Z_M/Z_G)} I_P^G \circ r_P^G(\pi).$$

Extending the result of Zelevinskii for $GL(n)$, Waldspurger proved that this sends irreducible representations to irreducible representations [3]. Recently Hiraga gave the following conjecture on the relation of this involution with A-packets.

Conjecture 5.1. *$\mathcal{D}_G$ sends A-packets to A-packets. Moreover if we write $\mathcal{D}_G(\phi)$ for the A-parameter of the A-packet $\mathcal{D}_G(\Pi_\phi)$ and*

$$\phi : W_F \times SU(2) \times SL_2(\mathbb{C}) \ni (w,h,g) \longmapsto \rho(w)\lambda(h)\tau(g) \in {}^LG,$$

then $\mathcal{D}_G(\phi)(w,h,g) = \rho(w)\tau(h)\lambda(g)$. Here rational representations of $SL_2(\mathbb{C})$ are identified with those of $SU(2)$ by restriction.

As a corollary of our calculation, we deduce

Corollary 5.2. *The above conjecture is valid for $U_{E/F}(4)$.*

We end this note by giving some examples of this corollary.

(1) In the notation of 3.2 (4-i), $\mathcal{D}_G$ transposes $\delta_0^G(\eta_1,\eta_2)$, $J_{P_2}^G(\eta_1[1]\otimes\eta_2^{G_1}\delta^{G_1})$ and $J_{P_1}^G(I_{\widetilde{\mathbf{B}}_1}^{\widetilde{G}_1}(\eta_1\otimes\eta_2)[1])$, $J_{P_2}^G(\eta_2[1]\otimes\eta_1^{G_1}\delta^{G_1})$, respectively. First consider the case (4-i). The elliptic Langlands parameter

$$\varphi_{\boldsymbol{\eta}}|_{W_E} = \mathrm{diag}(\eta_1,\eta_2,\eta_2,\eta_1)\times p_{W_E}, \quad \varphi_{\boldsymbol{\eta}}(w_\sigma) = \mathbf{1}_4 \rtimes w_\sigma,$$

$$\varphi_{\boldsymbol{\eta}}\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \begin{pmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d\end{pmatrix} \times 1, \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SU(2)$$

corresponds to the square integrable L-packet $\Pi_{\varphi_{\boldsymbol{\eta}}} = \{\delta_0^G(\eta_1,\eta_2), \pi(\eta_2^{G_1}\delta^{G_1},\eta_1)\}$. One finds that $\mathcal{D}_G(\varphi_{\boldsymbol{\eta}}) = \phi_{\boldsymbol{\eta}}$ while $\mathcal{D}_G(\Pi_{\varphi_{\boldsymbol{\eta}}}) = \Pi(\phi_{\boldsymbol{\eta}})$, since $\mathcal{D}_G$ fixes the supercuspidal representations. This suggests that we might construct some A-packets by applying $\mathcal{D}_G$ to elliptic L-packets. This is the original motivation of Hiraga's conjecture. On the other hand in the case (5-ii), we have $\mathcal{D}_G(\phi_{\eta_2^{G_1}\delta^{G_1},\eta_1}) = \phi_{\eta_1^{G_1}\delta^{G_1},\eta_2}$. Again the conjecture is valid because the associated A-packets share the supercuspidal $\pi(\eta_2^{G_1}\delta^{G_1},\eta_1)$. In such a case, Conj. 5.1 works little for constructing A-packets.

(2) Next in the notation of 3.2 (4-ii), $\mathcal{D}_G$ transposes $\eta^G\tau(\delta^{G_1})$, $\eta^G\tau(\mathbf{1}_{G_1})$ and $J_{P_1}^G(I_{\widetilde{\mathbf{B}}_1}^{\widetilde{G}_1}(\eta\otimes\eta)[1])$, $J_{P_2}^G(\eta[1]\otimes\eta^{G_1}\delta^{G_1})$, respectively. The tempered Langlands parameter $\varphi_{\boldsymbol{\eta}}$ in this case corresponds to the tempered L-packet $\Pi_{\varphi_{\boldsymbol{\eta}}} = \{\eta^G\tau(\delta^{G_1}), \eta^G\tau(\mathbf{1}_{G_1})\}$. As is conjectured, the A-packet corresponding to $\mathcal{D}_G(\varphi_{\boldsymbol{\eta}}) = \phi_{\boldsymbol{\eta}}$ is $\{J_{P_1}^G(I_{\widetilde{\mathbf{B}}_1}^{\widetilde{G}_1}(\eta\otimes\eta)[1]), J_{P_2}^G(\eta[1]\otimes\eta^{G_1}\delta^{G_1})\}$. On the other hand, $\phi_{\eta^{G_1}\delta^{G_1},\eta}$ is unchanged under $\mathcal{D}_G$ while the two members of the corresponding A-packet are transposed with each other. Also this is the first example that one representation which is not square integrable is shared by two distinct A-packets.

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