数理解析研究所講究録
1175 巻 2000 年 123-131

Abelian surfaces in toric 4-folds

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Abstract

We show several properties on embedding of abelian surfaces to projective smooth toric 4-folds.

Introduction

Embedding of an abelian surface into the projective space $\mathbb{P}^4$ of dimension 4 involves two interesting topics: One is on very ampleness of line bundles; the other is on vector bundles of rank 2 on $\mathbb{P}^4$. There are a lot of works on those topics (f.e., [6], [7], [12], [18]). Hulek [8] started to study embedding of abelian surfaces into products of projective spaces. Lange [13] constructed embedding of an abelian surface into $\mathbb{P}^1 \times \mathbb{P}^3$, and [14] studies the vector bundle on $\mathbb{P}^1 \times \mathbb{P}^3$ that arises from his embedding by Serre's construction. Recently, Sankaran [20] studies embedding into toric 4-folds with Picard number ≤ 2.

The aim of this notes is to show several properties on embedding of abelian surfaces into projective toric 4-folds.

We recall some known results and state our main results in Section 1. We prove them in each section.

CONVENTION We work over an algebraically closed field k of any characteristic. We mean by a d-fold a smooth variety over k of dimension d.

1 Main results

In this section, we first review some known results on embedding of abelian surfaces into projective toric 4-folds X with Picard number ≤ 2. We give a problem even if it is of very vague form. We hope to construct an embedding of an abelian surface by using a nice family of theta functions related to a fan. Finally we state our main results.

*The author is supported by the Grants-in-Aid for Encouragement of Young Scientists, the Ministry of Education, Science, Sports and Culture, Japan 1274003.

1.1. For embedding of abelian surfaces into projective spaces, Ramanan [18] shows that a simple abelian variety over $\mathbb{C}$ can be embedding into the projective space of dimension 4 by a very ample line bundle of degree 5:

1.1.1. Theorem (Ramanan [18]). *Let A be a simple abelian surface, and L an ample line bundle of degree $d \geq 5$. Then L is very ample, i.e., the induced morphism $A \to \mathbb{P}^4 \cong \mathbb{P}(H^0(A, L))$ is a closed immersion.*

1.1.2. Remark. (1) For any abelian surface A over $\mathbb{C}$, Reider's theorem says that, for an ample line bundle L of type $(1, d)$ with $d \geq 5$ on A, the line bundle L is very ample if and only if there is no elliptic curve C on A with $(C.L) = 2$ (cf. [4, Chapter 10, (4.1)]).

(2) By using Serre's construction (f.e., cf. [17]), the above embedding gives us an indecomposable vector bundle of rank 2 on the projective 4-fold, so-called Horrocks-Mumford bundle ([5], [6]).

1.2. For $X = \mathbb{P}^1 \times \mathbb{P}^3$, Hulek computed a numerical necessary condition for such an embedding to exist. If A is embedded into $\mathbb{P}^1 \times \mathbb{P}^3$, then the cycle class of $[A]$ in the 2nd Chow group of $\mathbb{P}^1 \times \mathbb{P}^3$ should be $8h_1h_2 + 6h_2^2$. Here h_1 and h_2 are the pull-back of the classes of divisors defined by a point $\in \mathbb{P}^1$ and a plane $\subset \mathbb{P}^3$ respectively. After that, Lange [13] obtained 2-dimensional family of embeddings of abelian surfaces into $\mathbb{P}^1 \times \mathbb{P}^3$.

1.3. Sankaran [20] studies embeddings into projective smooth (4-dimensional) toric varieties with Picard number $= 2$. Note that such toric varieties are projective space bundles over projective spaces [11]. He shows non-existence of totally nondegenerate embedding (cf. 1.8.1) of abelian surfaces into projective toric 4-folds, and explicitly constructed an embedding of an abelian surface into a projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ over $\mathbb{P}^2$. I found a mistake in his computation [20, p. 414, l. 9], by which his proof seems not to be correct. So, we don't know if such an embedding exists or not in this case.

We conclude that (nontrivial) embedding of abelian surfaces into projective toric 4-folds is known at present only in the cases $\mathbb{P}^4$ and $\mathbb{P}^1 \times \mathbb{P}^3$. See [23], [3] and [2] for higher dimensional abelian varieties.

1.4. We remark morphisms, not necessarily embeddings, from algebraic varieties to toric varieties. Several people study this theme from different points of view. We have a generalization of all the results [10]. Here we state only Oda-Sankaran's result because it seems more convenient to construct morphisms concretely:

1.4.1. Theorem (Oda-Sankaran). *Let Δ be a smooth fan, X the associated nonsingular toric variety, and Y a normal variety. Then every morphism $Y \to X$, whose image intersects the open dense torus orbit, corresponds to a pair $(\varphi, \{D_\rho\})$ of a homomorphism*

φ: $M \to \mathbb{C}(Y)^*$ and a set $\{D_\rho\}_{\rho\in\Delta(1)}$ of effective Cartier divisors D_ρ on Y satisfying the following two conditions:

(1) $\mathrm{div}_Y(\varphi(m)) = \sum_{\rho\in\Delta(1)}\langle m, n_\rho\rangle D_\rho$ for all $m \in M$;

(2) $D_{\rho_1} \cap \cdots \cap D_{\rho_s} = \emptyset$ if $\rho_1 + \cdots + \rho_s \notin \Delta$.

Here, $\Delta(1)$ is the set of one-dimensional cones in Δ, and n_ρ is the primitive generator of a one-dimensional cone ρ. We denote by $\langle\ ,\ \rangle$: $M \times N \to \mathbb{Z}$ the duality pairing. See [20, Section 2] for the detail.

1.4.2. From the viewpoint of the above result, we give the following interesting problem:

Problem. Give an embedding of an abelian variety A into a toric variety by constructing a nice family of theta functions on A.

1.5. We now state our results.

1.6. Proposition. *Let X be a proper simplicial toric variety of dimension $r \geq 3$. Then X contains no abelian subvariety of codimension* 1.

1.7. Proposition. *Let $X = X_1 \times X_2$ be the product of projective smooth toric surfaces X_1 and X_2. If an abelian surface A is embedded into X, then A is isomorphic to the product of elliptic curves E_1 and E_2.*

1.8. Since the 4-dimensional projective space $\mathbb{P}^4$ contains an abelian surface A, any blowup of $\mathbb{P}^4$ along a center $\subset \mathbb{P}^4 \setminus A$ also contains A. So, we have to impose some assumption on embedding.

1.8.1. Definition. An embedding of an abelian surface A into a projective smooth toric variety X is *totally nondegenerate* if, for each one-dimensional cone $\rho \in \Delta(1)$, the intersection $D_\rho \cap A$ is of dimension 1. Here D_ρ is the invariant divisor associated to ρ.

1.9. Proposition. *Let X be a projective toric* 4*-fold with Picard number $\geq$* 3. *Then there is no totally nondegenerate embedding of any* simple *abelian surface to X.*

1.10. Proposition. *Let X be the pseudo del-Pezzo toric* 4*-fold. Then, there is no totally nondegenerate embedding of an abelian surface to X.*

1.10.1. Remark. (0) After my talk, I found a mistake in the proof for the del-Pezzo toric 4-fold. So, we haven't done yet in this case.

(1) I don't know whether an abelian surface can be embedded into a toric 4-fold if I don't assume the totally nondegeneracy condition.

(2) Among toric Fano 4-folds, we have $\mathbb{P}^4$, a pseudo del-Pezzo toric 4-fold, and del-Pezzo toric 4-fold as a set of complete representatives with respect to the weak equivalences defined by Sato [21].

(3) Sato [22] studies embedding of abelian surfaces into nonsingular toric Fano 4-folds.

2 No embedding of abelian varieties into complete simplicial toric varieties as divisors

In this section, we show the following proposition:

2.1. Proposition. *Let X be a proper simplicial toric variety of dimension $r \geq 3$. Then X contains no abelian subvariety of codimension* 1.

2.1.1. Remark. If an abelian subvariety defines an ample divisor, then it follows from the Lefschetz theorem.

Proof. Suppose that X contains an abelian subvariety of codimension 1. We first prove that A is a nef and big divisor on X. Suppose $A = \sum_{m \in H^0(X, \mathcal{O}_X(A))} a_m m$. If the convex hull P ($\subset M_{\mathbb{R}}$) of $\{m \in M; a_m \neq 0\}$ is of dimension $< r$, then A is birational to an $(r-1)$-dimensional torus. This is a contradiction. Hence $\dim P = r$. This shows that the convex hull of $m \in H^0(X, \mathcal{O}_X(A))$ is of dimension r, i.e., A is big. Since an abelian surface A doesn't contain any rational curves, one can show that A is nef by [19, (1.7) Corollary].

Using an equivariant resolution of X and [16, Corollary 3.4], we can easily show that, for a nef and big divisor D on X,

$$H^i(X, \mathcal{O}_X(-D)) = 0 \quad \text{for } i < \dim X.$$

We now consider a short exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

By taking its cohomology and using the above vanishing theorem, we have an isomorphism $H^1(A, \mathcal{O}_A) \cong H^2(X, \mathcal{O}_X(-A)) = 0$. Note that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. This contradicts $\dim H^1(A, \mathcal{O}_A) = \dim A$. $\square$

3 Embeddings of abelian surfaces into products of toric surfaces

In this section, we show the following proposition:

3.1. Proposition. *Let $X = X_1 \times X_2$ be the product of projective smooth toric surfaces X_1 and X_2. If an abelian surface A is embedded into X, then A is isomorphic to the product of elliptic curves E_1 and E_2.*

3.1.1. Remark. (1) Hulek [8, Proposition 2.1] shows the case $X = \mathbb{P}^2 \times \mathbb{P}^2$.

(2) We can prove that an elliptic curve in a toric surface X is "essentially" linearly equivalent to $-K_X$. See Proposition 3.4.

(3) If $X \not\cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then one can show that the embedding is defined by $E_i \subsetneq X_i$.

3.2. We show the proposition in a similar way to the one in [8]. We first prove the following lemma:

3.2.1. Lemma. *Let $f\colon A \to C \times X$ be an embedding of an abelian surface A to the product of a projective curve C and a projective toric surface X. Then C is an ellipitic curve, the surface A is the product of C and an elliptic curve E in X, and f is a natural one $C \times D \to C \times X$.*

Proof. We first prove the case where C is nonsingular. By Proposition 2.1, we have the genus $g(C)$ of $C \geq 1$. Since $\mathrm{pr}_1 \circ f\colon A \to C$ must be surjective, this implies that C has only constant 1-forms. So, we conclude $g(C) = 1$.

Suppose that the composite $A \to C \times X \xrightarrow{\mathrm{pr}_2} X$ is surjective. Then, X should have its Picard number ≤ 2, because A has no curve with negative self-intersection (cf. f.e., [9, Theorem 8.5]). Moreover X contains elliptic curves which are fibers of $\mathrm{pr}_1 \circ f$ at distinct points. This is a contradiction. (For X with Picard number=2, use the adjunction formula.) Hence, the image of the above composite is an irreducible curve, say D. Let $\widetilde{D}$ the normalization of D. As in [8, Proof of Proposition 2.1], we can show that $A \to C \times D$ factors through $C \times \widetilde{D}$. So we have $A \cong C \times \widetilde{D} \cong C \times D$.

Next we suppose that C is singular. As in [8, loc. cit.], we can show $A \to C \times X$ factors through $\widetilde{C} \times X$. Here $\widetilde{C}$ is the normalization of C. This case follows from the case where C is nonsingular. $\square$

3.3. *Proof of Proposition* 3.1. By Lemma 3.2.1, we have only to consider the case where $\mathrm{pr}_i \circ f$ is surjective ($i = 1, 2$) for any embedding f. So we can assume that X_i ($i = 1, 2$) is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. We first prove the following lemma:

3.3.1. Lemma. *(1) Let $\pi\colon A \to \mathbb{P}^2$ be a surjective morphism. Then* $\deg \pi$ *is an even integer* > 2.

(2) Let $\pi\colon A \to \mathbb{P}^1 \times \mathbb{P}^1$ be a surjective morphism, H_1 (resp. H_2) the divisor $\pi^*(\{0\} \times \mathbb{P}^1)$ (resp. $\pi^*(\mathbb{P}^1 \times \{0\})$) on A. Then $H_i^2 = 0$ for $i = 1, 2$. Moreover $H_1.H_2$ is greater than 3, and is not prime.

Proof. (1) Since $\pi^*(L)^2$ is even and equals $\deg \pi$ for a line on $\mathbb{P}^2$, we have only to show $\deg \pi > 2$. Suppose $\deg \pi = 2$. By the Riemann-Roch theorem, we have $h^0(A, \pi^*(L)) =$

$(\pi^*(L)^2)/2 = 1$. Hence $\pi^*(L_1) = \pi^*(L_2)$ as divisors for distict two lines L_1, L_2 on $\mathbb{P}^2$. This is absurd.

(2) Set $L_1 = \{0\} \times \mathbb{P}^1$ and $L_2 = \mathbb{P}^1 \times \{0\}$. Since $H_i^2 = \pi^*(L_i)^2 = L_i^2 = 0$ $(i = 1, 2)$, the divisor H_i is numerically equivalent to $\nu_i E_i$ for an elliptic curve E_i on A and $\nu_i \in \mathbb{Z}_{>0}$ $(i = 1, 2)$. If $\nu_1 = 1$, then $h^0(H_1) = 1$ because the homomorphism $H^1(A, \mathcal{O}_A) \to H^1(E, \mathcal{O}_E)$ of tangent spaces of their dual abelian varieties is surjective. This contradicts $\pi^*(L_1) \neq \pi^*(\{\infty\} \times \mathbb{P}^1)$. So, $\nu_1 > 1$. Likewise, we have $\nu_2 > 1$.

Suppose $H_1.H_2$ is prime. Then either ν_1 or ν_2 must be 1 because $H_1.H_2 = (\nu_1\nu_2)E_1.E_2$ is prime. This is a contradiction as we have seen above. □

3.3.2. We prove the case $X = (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^2$. Suppose $\mathrm{pr}_i \circ f$ is surjective for $i = 1, 2$. Let x_1 (resp. x_2, x_3) be the cycle class defined by $\mathrm{pr}_1{}^*(\{(0:1)\})$ (resp. $\mathrm{pr}_2{}^*(\{(0:1)\})$, $\mathrm{pr}_3{}^*(\{(0:*:*)\})$). Suppose that $[A] = ax_1x_2 + bx_1x_3 + cx_2x_3 + dx_3^2$. Then we have the following equations: $[A]x_1x_2 = d$; $[A]x_3^2 = a$; $[A]x_1x_3 = c$; $[A]x_2x_3 = b$. The self-intersection formula, cf. [20, p. 411], implies

$$(a-2)(d-2) + (a/2) + (b-3)(c-3) = 13.$$

Lemma 3.3.1 shows that a is an even integer greater than 2, and that d is greater than 4 and not prime. One can verify that $b, c > 1$, and either of b or c is not prime. If c is prime, then we have a bijective morphism from an elliptic curve to $X_1 \cong \mathbb{P}^1$ or a line in $X_3 \cong \mathbb{P}^2$. This is a contradiction. So $b, c > 1$ are not prime. It is easy to see that there is no solution of the above equation satisfying all the conditions. This completes the proof. □

3.3.3. *Proof of the case* $X = (\mathbb{P}^1)^4$. Let x_i be the cycle classes defined by the divisor $\mathrm{pr}_i^*(\{(0:1)\})$ on X $(i = 1, 2, 3, 4)$. Let's set $[A]x_ix_j = a_{ij}$ $(1 \leq i < j \leq 4)$. Then the self-intersection formula implies

$$(a_{12}-2)(a_{34}-2) + (a_{13}-2)(a_{24}-2) + (a_{14}-2)(a_{23}-2) = 12.$$

Since we have $a_{i,j} \geq 4$ by the above lemma (2), we conclude that $a_{ij} = 4$ for any i, j. Taking the Stein factorization of $A \to X_1 = \mathbb{P}^1$, We have a diagram with exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & & & & & \big\downarrow{\scriptstyle q_1} & & \\ & & & & & & \mathbb{P}^1. & & \end{array}$$

For the projection $\mathrm{pr}_2 \colon (\mathbb{P}^1)^4 \to \mathbb{P}^1$, we define an elliptic curve E_2 in a similar way. Since $(\deg q_1)(\deg(\mathrm{pr}_2 \circ f)|_{E_1}) = a_{12} = 4$, we have an isomorphism $E_1 \times E_2 \xrightarrow{\cong} A$. □

3.4. Proposition. *Let E be an elliptic curve contained in a smooth projective toric surface X. Then, there is an equivariant morphism π from X to a projective toric surface Y satisfying that $C \cong \pi(E)$ via π and that $\pi(E)$ is linearly equivalent to $-K_Y$.*

Proof. Since E is nef and big, we have a birational morphism $\pi\colon X \to Y$ such that $\pi^*(E') = E$ for some ample $\mathbb{Q}$-Cartier divisor on Y (f.e., cf. [16, Proposition 3.3]). Here Y is a projective toric surface. We now show that π induces an isomorphism $E \to \pi(E)$. If $E.D_\rho > 0$ for $\rho \in \Delta(1)$, the projection formula implies $E'.\pi_*(D_\rho) > 0$. So we have $D_\rho \cong \pi(D_\rho)$ for D_ρ with $E.D_\rho > 0$. Hence $C.D_\rho = 0$ for any D_ρ such that $\pi(D_\rho)$ is a point. This means $E \cong \pi(E)$.

Using a short exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + E) \to \mathcal{O}_E(K_X + E) \to 0,$$

we have a unique effective divisor D linearly equivalent to $K_X + E$. Since $E'.\pi_*(D) = E'.\pi_*(K_X+E) = E.(K_X+E) = 0$, we have $\pi_*(D) = 0$. So, $K_Y+\pi(E) = \pi_*(K_X)+\pi_*(E) = \pi_*(D) = 0$, i.e., $\pi(E) = -K_Y$. □

4 Embedding of simple abelian surfaces

In this section, we show that a smooth proper 4-dimensional toric variety with Picard number ≥ 3 contains no simple abelian surface.

4.1. Proposition. *Let X be a projective toric 4-fold with Picard number ≥ 3. Then there is no totally nondegenerate embedding of any* simple *abelian surface to X.*

4.1.1. Remark. In the case (2), I don't know if an abelian surface, not necessarily simple, can be embedded totally nondegenerately into the toric variety.

Proof. By the smoothness of X and combinatorial argument on fans, one can easily verify that there exists a primitive collection consisting of just 2 vertices. This implies that X has no totally nondegenerate embedding of a simple abelian surface.

□

5 Totally nondegenerate Embeddings into del Pezzo and pseudo del Pezzo 4-folds

In this section, we show the following proposition:

5.1. Proposition. *Let X be the pseudo del-Pezzo toric 4-fold. Then, there is no totally nondegenerate embedding of an abelian surface to X.*

5.2. Definition. Let $\{e_i\}_i$ be a standard basis of $N \cong \mathbb{Z}^4$. A pseudo del-Pezzo (resp. del-Pezzo) toric 4-fold is the toric variety associated to the fan defined by the convex hull of $\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, e_1+e_2+e_3+e_4\}$ (resp. $\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm(e_1+e_2+e_3+e_4)\}$).

Proof. Let D_i (resp. E_i, D_0) ($i = 1, 2, 3, 4$) be the effective divisor on A defined as the intersection of A with the invariant divisor associated to $\mathbb{R}_{\geq 0} e_i$ (resp. $\mathbb{R}_{\geq 0}(-e_i)$, $\mathbb{R}_{\geq 0}(e_1 + \cdots + e_4)$). Computing linearly equivalence and primitive collections, we have $D_i + D_0 \sim E_i$ and $D_i.E_i = 0$. Since an effective divisor D on an abelian surface A with $D^2 > 0$ is ample, we can easily show that $D_i^2 = E_i^2 = D_0^2 = D_0.D_i = D_0.E_i = 0$ for $i = 1, \ldots, 4$. This contradicts that $-K_X|_A = \sum_{i=0}^4 D_i + \sum_{i=1}^4 E_i$ is ample. $\square$

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