数理解析研究所講究録
1175 巻 2000 年 107-122

Stanley-Reisner 環における Eisenbud-Goto の不等式

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序

極小自由分解の理論が進むにつれ、regularity という不変量の重要性が認識され、近年その研究が活発に行われてきた。特に、その上限を他の不変量で評価することが、その興味の中心になっている様に思われる。単項式イデアルの regularity に関しては、[Ho-Tr] において、様々な興味深い評価が与えられている。彼らは、単項式イデアルの 算術的次数や、極小生成系の 次数による上限を与えている。[Ho-Tr] のおいては、代数的手法が主に用いられているが、[Fr-Te] や [Te2] においては、Hochster の公式 に基づいて、組合せ論的、位相幾何学的手法がとられている。本小論では、それらについて 解説したい。

多項式環の中の一般の次数付きイデアルの regularity に関して言えば、次の Eisenbud-Goto 予想が、その研究のためのモチヴェーションを与え続けてきた。

Eisenbud-Goto 予想 ([Ei-Go, Introduction]). $A = k[x_1, x_2, \ldots, x_n]$ を n 変数の体 k 上の多項式環とし、P を A の次数付き素イデアルで 1 次式を含まないものとする。このとき、

$$\operatorname{reg} P \leq \deg A/P - \operatorname{codim} A/P + 1。$$

この予想は、今も 可換環論や、代数幾何学で活発に研究されている。興味のある人は、例えば、[Kw] や [Mi-Vo] 及び そこに挙げられている参考文献を見られたい。

本小論においては、Eisenbud により予想された単項式版の Eisenbud-Goto 予想について考察する。

定理 0.1 (Eisenbud の予想)(cf. [Fr-Te], [Te2]). k を体とし、Δ を pure で strongly connected な単体的複体とする。このとき、

$$\mathrm{reg}\ I_\Delta \leq \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1。$$

この定理から Gröbner 基底の理論を用いてもともとの Eisenbud-Goto 予想に対しては次のことが言える。

系 0.2. $A = k[x_1, x_2, \ldots, x_n]$ を n 変数の体 k 上の多項式環とし、P を A の次数付き素イデアルで 1 次式を含まないものとする。さらに $A/\mathrm{in}P$ は reduced であると仮定する。ここで、$\mathrm{in}P$ は、ある項順序に関する P の initial ideal とする。このとき、

$$\mathrm{reg}\ P \leq \deg A/P - \mathrm{codim} A/P + 1。$$

次に Eisenbud-Goto の不等式において等号が成立する場合について考察する。このとき、次の定理が成立する。

定理 0.3 ([Te2]). k を体とし、Δ を pure で strongly connected な $(d-1)$ 次元の単体的複体とする。$r = \mathrm{reg} I_\Delta$ とおく。このとき、

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1$$

であるための必要十分条件は、Δ が次の条件を満たすこと。
(1) もし、$r = 2$ ならば、Δ は、$(d-1)$-tree であること。ただし、$(d-1)$ 単体ではないこと。
(2) もし、$r = 3$ ならば、ある $(d-1)$-tree Δ' とある separated な $v, w \in V(\Delta')$ に対して $\Delta = \Delta'(v \to w)$ となること。
(3) もし、$r \geq 4$ ならば、$\Delta \cong \partial\Delta(r) * \Delta(d-r+1) + ((d-1)\text{-branches})$。

定義されていない用語に対しては、§1 及び §4 を見られたい。

1. PRELIMINARIES

We first fix notation. Let $\mathbf{N}$(resp.$\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). Let $\mid S \mid$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings. We refer the reader to, e.g., [Br–He], [Hi], [Hoc] and [St] for the detailed information about combinatorial and algebraic background.

A *(abstract) simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \le i \le n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. The vertex set of Δ is denoted by $V(\Delta)$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $|F| = i+1$ and we call a maximal face a *facet*. Let F be a face but not a facet. We call F *free* if there is a unique facet G such that $F \subset G$. If $\{x_i\}$ is free, we call x_i free.

We define the *dimension* of $F \in \Delta$ to be $\dim F = |F| - 1$ and the *dimension* of Δ to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. We say that Δ is *pure* if all facets have the same dimension. In a pure $(d-1)$-dimensional complex Δ, we call $(d-2)$- face a *subfacet*. We say that a pure complex Δ is *strongly connected* if for any two facets F and G, there exists a sequence of facets

$$F = F_0, F_1, \ldots, F_m = G$$

such that $F_{i-1} \cap F_i$ is a subfacet for $i = 1, 2, \ldots, m$.

Let $f_i = f_i(\Delta)$, $0 \le i \le d-1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

Next we summarize basic facts on the Hilbert series. Let k be a field and R a homogeneous k-algebra. By a homogeneous k-algebra R we mean a noetherian graded ring $R = \bigoplus_{i \ge 0} R_i$ generated by R_1 with $R_0 = k$. Let M be a graded R-module with $\dim_k M_i < \infty$ for all $i \in \mathbf{Z}$, where $\dim_k M_i$ denotes the dimension of M_i as a k-vector space. The *Hilbert series* of M

is defined by

$$F(M,t) = \sum_{i\in\mathbf{Z}}(\dim_k M_i)t^i.$$

It is well known that the Hilbert series $F(R,t)$ of R can be written in the form

$$F(R,t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{\dim R}},$$

where $h_0(=1), h_1, \ldots, h_s$ are integers with $\deg R := h_0 + h_1 + \cdots + h_s \geq 1$, which is called the *degree* of R. The vector $h(R) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R. We consider $k[\Delta]$ as the graded algebra $k[\Delta] = \bigoplus_{i\geq 0} k[\Delta]_i$ with $\deg x_j = 1$ for $1 \leq j \leq n$. The Hilbert series $F(k[\Delta],t)$ of a Stanley-Reisner ring $k[\Delta]$ can be written as follows:

$$\begin{aligned} F(k[\Delta],t) &= 1 + \sum_{i=1}^{d} \frac{f_{i-1}t^i}{(1-t)^i} \\ &= \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}, \end{aligned}$$

where $\dim \Delta = d-1$, $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of Δ, and $(h_0, h_1, \ldots, h_d)$ is the h-vector of Δ. It is easy to see $\deg k[\Delta] = f_{d-1}$. On the other hand, the *arithmetic degree* of $k[\Delta]$ is defined to be the number of facets in Δ, which is denoted by a-deg$k[\Delta]$. See, e.g., [Ho-Tr] for the definition of the arithmetic degree of a general ring R.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ over a field k. Let $M(\neq 0)$ be a finitely generated graded A-module and let

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A. The length h of this resolution is called the *projective dimension* of M and denoted by $h = \mathrm{pd}M$. We call $\beta_i(M) = \sum_{j\in\mathbf{Z}} \beta_{i,j}(M)$ the i-th *Betti number* of M over A. We define the *Castelnuovo-Mumford regularity* reg M of M by

$$\mathrm{reg}\ M = \max\ \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

See, e.g., [Ei] for further information on regularity. We define the *initial degree* indeg M of M by

$$\mathrm{indeg}\ M = \min\ \{i \mid M_i \neq 0\} = \min\ \{j \mid \beta_{0,j}(M) \neq 0\}.$$

Let l be a natural number. We say that M satisfies (N_l) condition if $\beta_{i,i+s}(M) = 0$ for $i < l,\ s \neq \operatorname{indeg} M$.

We denote the number of generators of M by $\mu(M) = \beta_0(M)$.

The following two theorem are a starting point for our study.

THEOREM 1.1 (Hochster's formula on the Betti numbers [Hoc, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F \subset [n],\ |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

It is easy to see:

COROLLARY 1.2.
reg $I_\Delta = \max\ \{i + 2 \mid \tilde{H}_i(\Delta_F; k) \neq 0$ for some $F \subset V\}$.

If F is a face of Δ, then we define a subcomplex $\operatorname{link}_\Delta F$ by

$$\operatorname{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

THEOREM 1.3 (Hochster's formula on the local cohomology modules (cf. [St, Theorem 4.1])).

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-|F|-1}(\operatorname{link}_\Delta F; k) \left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

where $H^i_{\mathfrak{m}}(k[\Delta])$ *denote the* i*-th local cohomology module of* $k[\Delta]$ *with respect to the graded maximal ideal* $\mathfrak{m}$.

COROLLARY 1.4.
reg $I_\Delta = \max\ \{i + 2 \mid \tilde{H}_i(\operatorname{link}_\Delta F; k) \neq 0$ for some $F \in \Delta\}$.

Next we recall the definition of Alexander dual complexes. For a simplicial complex Δ on the vertex set V, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset V : V \setminus F \notin \Delta\}.$$

THEOREM 1.5 [Tel, Corollary 0.3]. *Let k be a field. Let Δ be a simplicial complex. Then*

$$\operatorname{reg} I_\Delta = \operatorname{pd} k[\Delta^*].$$

2. REGULARITY OF THE SUM OF IDEALS

In this section we give a upper bound for the sums of sqare-free monomial ideals.

In the rest of the paper we always assume that k is a fixed field.

First we prove the following proposition. It seems to be known, but we cannot find it in literature.

PROPOSITION 2.1. *Let I be a monomial ideal in the polynomial ring $A = k[x_1, x_2, \ldots, x_n]$ and m a monomial in A. Then*

$$\operatorname{pd} A/(I + (m)) \leq \operatorname{pd} A/I + 1.$$

The following proof is simplified by suggestion of Eisenbud.

Proof. If we show that

$$\operatorname{pd} A/I \geq \operatorname{pd} (I + (m))/I,$$

then the mapping cone guarantees that

$$\operatorname{pd} A/(I + (m)) \leq \operatorname{pd} A/I + 1.$$

by [Ei, Exercise A.3.30]. We have

$$\begin{aligned}
(I + (m))/I &\cong (m)/((m) \cap I) \\
&\cong (m)/((m) \cap (m_1, \ldots, m_t)) \\
&\cong (m)/(\operatorname{lcm}(m, m_1), \ldots, \operatorname{lcm}(m, m_t)) \\
&\cong A/(m_1', \ldots, m_t') \otimes_A (m),
\end{aligned}$$

where $I = (m_1, \ldots, m_t)$, $m_i' = \frac{\operatorname{lcm}(m, m_i)}{m}$. Hence, we have only to show

$$\operatorname{pd} A/I \geq \operatorname{pd} A/(m_1', \ldots, m_t').$$

Now we have $(A/I)_m \cong A_m/(m'_1, \ldots, m'_t)A_m$. Hence we have

$$\text{pd } A/I \geq \text{pd } (A/I)_m = \text{pd } A_m/(m'_1, \ldots, m'_t)A_m = \text{pd } A/(m'_1, \ldots, m'_t).$$

We are done. **qed**

For the regularity of the sum of square-free monomial ideals, we have the following conjecture:

CONJECTURE 2.2. *Let $\Delta_i(\neq \emptyset)$ be a simplicial complex for $i = 1, 2$. Then we have*

$$\text{reg}(I_{\Delta_1} + I_{\Delta_2}) \leq \text{reg } I_{\Delta_1} + \text{reg } I_{\Delta_2} - 1.$$

If I_{Δ_1} and I_{Δ_2} are complete intersections, then the above inequality holds. The next theorem gives a weaker upper bound.

THEOREM 2.3. *Let $\Delta_i(\neq \emptyset)$ be a simplicial complex for $i = 1, 2$. Then we have*

$$\text{reg}(I_{\Delta_1} + I_{\Delta_2}) \leq \min\{\text{reg } I_{\Delta_1} + \text{a-deg}k[\Delta_2],\ \text{reg } I_{\Delta_2} + \text{a-deg}k[\Delta_1]\} - 1.$$

Proof. Only in this proof, we define a simplicial complex Δ by only the condition (ii) of the definition of a simplicial complex. We do not require the condition (i). Then we have $(\Delta^*)^* = \Delta$. And Theorem 1.5 also holds under this definition.

By the above proposition we have

$$\text{pd } A/(I_{\Delta_1} + I_{\Delta_2}) \leq \text{pd } A/I_{\Delta_1} + \mu(I_{\Delta_2}).$$

Since $I_{\Delta_1} + I_{\Delta_2} = I_{\Delta_1 \cap \Delta_2}$, we have

$$\text{reg } (I_{(\Delta_1 \cap \Delta_2)^*}) \leq \text{reg } I_{\Delta_1^*} + \text{a-deg}k[\Delta_2^*]$$

by Theorem 1.5 and $\mu(I_{\Delta_2}) = \text{a-deg}k[\Delta_2^*]$. Since we have $I_{(\Delta_1 \cap \Delta_2)^*} = I_{\Delta_1^* \cup \Delta_2^*} = I_{\Delta_1^*} \cap I_{\Delta_2^*}$, then we have

$$\text{reg } (I_{\Delta_1^*} \cap I_{\Delta_2^*}) \leq \text{reg } I_{\Delta_1^*} + \text{a-deg}k[\Delta_2^*].$$

Similarly we have

$$\text{reg } (I_{\Delta_1^*} \cap I_{\Delta_2^*}) \leq \text{reg } I_{\Delta_2^*} + \text{a-deg}k[\Delta_1^*].$$

Consider the exact sequence

$$0 \to A/(I_{\Delta_1^*} \cap I_{\Delta_2^*}) \to A/I_{\Delta_1^*} \oplus A/I_{\Delta_2^*} \to A/(I_{\Delta_1^*} + I_{\Delta_2^*}) \to 0.$$

By [Ei, Corollary 20.19], we have

$$\text{reg } A/(I_{\Delta_1^*} + I_{\Delta_2^*}) \le \max\{\text{reg } A/(I_{\Delta_1^*} \cap I_{\Delta_2^*}) - 1, \text{reg } (A/I_{\Delta_1^*} \oplus A/I_{\Delta_2^*})\}.$$

Hence

$$\text{reg } A/(I_{\Delta_1^*} + I_{\Delta_2^*}) \le \min\{\text{reg } I_{\Delta_1^*} + \text{a-deg}k[\Delta_2^*] - 1, \text{reg } I_{\Delta_2^*} + \text{a-deg}k[\Delta_1^*] - 1\}.$$

We obtained the desired result. **qed**

REMARK. Since the inequality reg $I_\Delta \le$ a-deg$k[\Delta]$ holds (cf. [Ho-Tr] and [Fr-Te]), Theorem 2.3 is weaker than Conjecture 2.2.

3. EISENBUD-GOTO INEQUALITY

In this section we prove Eisenbud-Goto inequality for Stanley-Reisner rings of pure and strongly connected simplicial complexes.

First we prove a lemma which is necessary for inductive argument.

LEMMA 3.1. *Let Δ be a pure and strongly connected simplicial complex. Then there exists a facet $F \in \Delta$ such that*

$$\Delta' := \{H \in \Delta \mid H \subset G \textit{ for some facet } G(\neq F) \in \Delta\}$$

is pure and strongly connected.

Proof. We define a graph G_Δ corresponding to Δ as follows: The vertex set $V(G_\Delta)$ consists of $\{y_F \mid F \text{ is a facet of } \Delta\}$. The edge set $E(G_\Delta)$ is defined by: $\{y_F, y_G\} \in E(G_\Delta)$ if and only if $F \cap G$ is a subfacet. If Δ is pure and strongly connected, G_Δ is connected. It is well known that there exists a vertex $y_F \in V(G_\Delta)$ such that $G_{V(G_\Delta)\setminus\{F\}}$ is connected. Then Δ' is pure and strongly connected. **qed**

Now we prove the main result in this section.

THEOREM 3.2(cf. [Fr-Te, Theorem 4.1]). Let Δ be a pure and strongly connected simplicial complex. Then we have

$$\text{reg} I_\Delta \le \deg k[\Delta] - \text{codim} k[\Delta] + 1.$$

Proof. Let V be the vertex set of Δ. Put $\mid V \mid = n$ and $\dim k[\Delta] = d$. We prove the theorem by induction on the number f_{d-1} of facets in Δ.

First if codim $k[\Delta] \leq 1$, then $k[\Delta]$ is a hypersurface. In this case the theorem is clear.

Suppose codim $k[\Delta] \geq 2$ and $f_{d-1} \geq 2$. By the above lemma, there exists a facet $F \in \Delta$ such that

$$\Delta' := \{H \in \Delta \mid H \subset G \text{ for some facet } G(\neq F) \in \Delta\}$$

is pure and strongly connected. Denote by V' the vertex set of Δ' and by f'_{d-1} the number of facets in Δ'. There are two cases.

Case 1 $V \neq V'$. Put $V \setminus V' = \{v\}$. For $W \subset V$ with $v \notin W$ we have $\Delta_W \cong \Delta'_W$. On the other hand, for $W \subset V$ with $v \in W$, $\tilde{H}_i(\Delta_W; k) \cong \tilde{H}_i(\Delta'_{W\setminus\{v\}}; k)$ for $i \geq 1$. Since

$$\text{reg } I_\Delta = \max\{i+2 \mid \tilde{H}_i(\Delta_W; k) \neq 0 \text{ for some } W \subset V\},$$

we have

$$\begin{aligned} \text{reg} I_\Delta &= \text{reg} I_{\Delta'} \\ &\leq f'_{d-1} - (n-1-d) + 1 \\ &= f_{d-1} - (n-d) + 1. \end{aligned}$$

Case 2 $V = V'$. We have reg $I_\Delta = $ pd $k[\Delta^*]$. Now we see that $k[\Delta^*] = k[(\Delta')^*]/(m)$, where $m = \prod_{x_i \in V\setminus F} x_i$. By Proposition 2.1, we have

$$\begin{aligned} \text{reg } I_\Delta &\leq \text{reg } I_{\Delta'} + 1 \\ &\leq f'_{d-1} - (n-d) + 2 \\ &= f_{d-1} - (n-d) + 1. \end{aligned}$$

qed

COROLLARY 3.3. *Let Δ be a simplicial complex such that* codim$k[\Delta] \geq 2$. *Assume I_Δ satisfies* (N_2) *condition. Then we have*

$$\text{pd}k[\Delta] \leq \mu(I_\Delta) - \text{indeg} I_\Delta + 1.$$

Proof. If I_Δ satisfies (N_2) condition, then $k[\Delta^*]$ satisfies (S_2) condition by [Ya, Corollary 3.7] and then Δ^* is pure and strongly connected. If Δ^* is pure, then $\deg k[\Delta^*] = \mu(I_\Delta)$. If codim$k[\Delta] \geq 2$, then indeg$I_\Delta = $ codim$k[\Delta^*]$. We are done by Theorems 1.5 and 2.4. **qed**

Proof of Corollary 0.2. Put $I_\Delta = \text{in}P$. Then by [Ka-St, Theorem 1], Δ is pure and strongly connected. By Theorem 3.2, we have

$$\begin{aligned} \text{reg}P &\leq \text{reg}I_\Delta \\ &\leq \deg k[\Delta] - \text{codim}k[\Delta] + 1 \\ &= \deg A/P - \text{codim}A/P + 1. \end{aligned}$$

qed

4. EQUALITY CASE

In this section, we classify pure and strongly conneceted simplicial complexes Δ which satisfy $\text{reg}I_\Delta = \deg k[\Delta] - \text{codim}k[\Delta] + 1$, and give some characterization for such complexes.

First we introduce some notation. Put $[m] = \{1, 2, \ldots, m\}$. We denote the elementary $(m-1)$-simplex by $\Delta(m) = \mathbf{2}^{[m]}$ and put $\Delta(0) = \{\emptyset\}$. We put $\partial\Delta(m) = \mathbf{2}^{[m]} \setminus \{[m]\}$, which is the boundary complex of $\Delta(m)$.

Let Δ_i be a $(d-1)$-dimensional pure simplicial complex for $i = 1, 2$. If $\Delta_1 \cap \Delta_2 = \mathbf{2}^F$ for some F with $\dim F = d-2$, we write $\Delta_1 \cup_F \Delta_2$ for $\Delta_1 \cup \Delta_2$. We sometimes write $\Delta_1 \cup_* \Delta_2$ for $\Delta_1 \cup_F \Delta_2$ if we do not need to express F explicitly.

We define a $(d-1)$-tree inductively as follows.
(1)$\Delta(d)$ is a $(d-1)$-tree.
(2)if Υ is a $(d-1)$-tree, then so is $\Upsilon \cup_* \Delta(d)$.

If $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$ are $(d-1)$-trees, we abbreviate $\Delta \cup_* \Upsilon_1 \cup_* \Upsilon_2 \cup_* \cdots \cup_* \Upsilon_m$ as $\Delta + ((d-1)\text{-branches})$.

Let Δ be a $(d-1)$-dimensional pure and strongly connected complex. Take $v,\ w \in V(\Delta)$. We say v and w are *separated* in Δ if $\{v,\ w\} \notin \Delta$ and that there exists no subfacet F in Δ with $\{v\} \cup F,\ \{w\} \cup F \in \Delta$. If v and w are separated in Δ, We denote $\Delta(v \to w)$ for the abstract simplicial complex which is obtained by substitution of w for every v in Δ. The vertex set of $\Delta(v \to w)$ is $V(\Delta) \setminus \{v\}$.

By Lemma 3.1 we know that every $(d-1)$-dimensional pure and strongly connected simplicial complex can be constructed from the $(d-1)$-dimensional elementary simplex $\Delta(d)$ by a succession

$$\Delta(d) = \Delta_1 \to \Delta_2 \to \cdots \to \Delta_{f_{d-1}}$$

of either of the following operations :
(1)$\Delta_{i+1} = \Delta_i \cup_{F'} \mathbf{2}^F$, where $x \notin V(\Delta_i)$, F' is a subfacet of Δ_i and $F =$

$F' \cup \{x\}$.
(2)$\Delta_{i+1} = (\Delta_i \cup_{F'} \mathbf{2}^F)(x \to y)$, where $x \notin V(\Delta_i)$, F' is a subfacet of Δ_i and $y \in V(\Delta_i)$ such that x and y are separated and $F = F' \cup \{x\}$.

Let Δ_i be a simplicial complex for $i = 1, 2$ such that $V(\Delta_1) \cap V(\Delta_2) = \emptyset$. We define the simplicial join $\Delta_1 * \Delta_2$ of Δ_1 and Δ_2 by

$$\Delta_1 * \Delta_2 = \{F \cup G \mid F \in \Delta_1, G \in \Delta_2\}.$$

LEMMA 4.1. *Let Δ be a $(d-1)$-dimensional pure and strongly connected complex. We assume*

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1 = 3.$$

Then Δ can be expressed as follows:

$$\Delta \cong \Delta'(x \to y) * \Delta(d-s) + ((d-1)\text{-branches})$$

for some $(s-1)$-tree Δ' and for some separated x, $y \in V(\Delta')$ with $\tilde{H}_1(\Delta'(x \to y); k) \neq 0$.

Proof. We may assume Δ has no branches. Then Δ can be expressed as $\Delta = \Delta'(x \to y)$, where Δ' is a $(d-1)$-tree and $x, y \in V(\Delta')$ are the only free vertices in Δ'. Let F be the facet with $x \in F$ and G the facet with $y \in G$ in Δ'.

Let $G_{\Delta'}$ be the graph introduce in the proof of Lemma 3.1. Since Δ' is a $(d-1)$-tree with only two free vertices x, y, then $G_{\Delta'}$ is a line with the end points y_F and y_G. Hence there exists a sequence of facets

$$F = F_0, F_1, \ldots, F_m = G$$

such that $F_{i-1} \cap F_i$ is a subfacet for $i = 1, 2, \ldots, m$. Then $F_0, F_1, \ldots, F_m$ are all facets in Δ'. We put $W = F \cap G$. If we have $z \in F_i$ and $z \notin F_{i+1}$, then $z \notin F_{i+2}$, since Δ' is a $(d-1)$-tree. Then we have $W \subset F_i$ for $i = 0, 1, 2, \ldots, m$. Then we have $\Delta' = \Delta_1 * \mathbf{2}^W$, and $\Delta'(x \to y) = \Delta_1(x \to y) * \mathbf{2}^W$, where Δ_1 is an $(s-1)$-tree and $s = d - \mid W \mid$. It is easy to check that $\Delta_1(x \to y)$ is contractible to the circle $\mathbf{S}^1$. **qed**

THEOREM 4.2. *Let Δ be a $(d-1)$-dimensional pure and strongly connected complex. We put $r = \mathrm{reg} I_\Delta$. Then*

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

if and only if Δ satisfies the following condition:
(1) Δ is a $(d-1)$-tree which is not the $(d-1)$-simplex if r=2.
(2) $\Delta = \Delta'(v \to w)$ for some $(d-1)$-tree Δ' and for some separated $v, w \in V(\Delta')$ if r=3.
(3)$\Delta \cong \partial\Delta(r) * \Delta(d-r+1) + ((d-1)\text{-branches})$ *if* $r \geq 4$.

Proof. First we assume that Δ satisfies $r = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1$.

We use induction on r. If $r = 2$, then Δ is a $(d-1)$-tree by [Fr].

If $r = 3$, then by the procedure to construct pure and strongly connected complexes, (3) is easy to check.

We assume $r = 4$. We prove the statement by induction on $\dim \Delta$. We may assume Δ has no branches. Then Δ is of the form

$$\Delta = (\Delta' \cup_{F'} \mathbf{2}^F)(x \to y)$$

where Δ' is pure and strongly connected and $F = F' \cup \{x\}$ is a facet of Δ and $y \in V(\Delta')$ such that x and y are separated in $\Delta' \cup_{F'} \mathbf{2}^F$. We have $\deg k[\Delta'] - \mathrm{codim} k[\Delta'] + 1 = 3$ and hence from the proof of Theorem 3.2 we get $\mathrm{reg} I_{\Delta'} = 3$. By the assumption of induction and the previous lemma, Δ' is of the form $\Delta' = \Delta''(v \to w) * \Delta(d-s) + ((d-1)\text{-branches})$ for some $(s-1)$-tree Δ'' and for some separated $v, w \in V(\Delta'')$ with $\tilde{H}_1(\Delta''(v \to w); k) \neq 0$. If $x \notin V(\Delta''(v \to w) * \Delta(d-s))$ or if $F' \notin \Delta''(v \to w) * \Delta(d-s)$, then the branch part can be contractible to a 1-dimensional subcomplex, then we have $\tilde{H}_2(\Delta_X; k) = 0$ for each $X \subset V(\Delta)$. Contradiction. Since Δ has no branches, we have $\Delta' = \Delta''(v \to w) * \Delta(d-s)$ and $x \in V(\Delta''(v \to w) * \Delta(d-s))$ and $F' \in \Delta''(v \to w) * \Delta(d-s)$.

Case 1. We assume $F' \cap V(\Delta(d-s)) \neq \emptyset$. In this case Δ is a cone . Hence we are done by induction.

Case 2. We assume $F' \cap V(\Delta(d-s)) = \emptyset$. In this case $d - s \leq 1$. Then we have $d = s$ or $d = s+1$. Then Δ and its subcomplexes of the form Δ_X for $X \subset V(\Delta)$ are contractible or contractible to a 1-dimensional complex, unless $d = s+1$ and $\Delta''(v \to w) = \partial\Delta(3)$. Here we omit a detail. Only case we must consider is $s = 1$, $d = 2$ and $\Delta = \partial\Delta(4)$. In this case $r = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1 = 4$.

If $r \geq 5$, we prove the statement by induction on $\dim \Delta$. We may assume Δ has no branches. Then Δ is of the form

$$\Delta = (\Delta' \cup_{F'} \mathbf{2}^F)(x \to y)$$

where Δ' is pure and strongly connected and $F = F' \cup \{x\}$ is a facet of Δ and $y \in V(\Delta')$ such that x and y are separated. We have $\deg k[\Delta'] -$

codim$k[\Delta'] + 1 = r - 1$ and hence from the proof of Theorem 3.2 we get reg$I_{\Delta'} = r - 1$. By the assumption of induction, Δ' is of the form $\Delta' = \partial\Delta(r-1) * \Delta(d-r+2) + ((d-1)\text{-branches})$. If $x \notin V(\partial\Delta(r-1) * \Delta(d-r+2))$ or if $F' \notin \partial\Delta(r-1) * \Delta(d-r+2)$, then the branch part can be contractible to a 1-dimensional subcomplex, then we have $\tilde{H}_{r-2}(\Delta_X; k) = 0$ for each $X \subset V(\Delta)$. Contradiction. Since Δ has no branches, we have $\Delta' = \partial\Delta(r-1) * \Delta(d-r+2)$ and $x \in V(\partial\Delta(r-1) * \Delta(d-r+2))$ and $F' \in \partial\Delta(r-1) * \Delta(d-r+2)$.

Case 1. We assume $F' \cap V(\Delta(d-r+2)) \neq \emptyset$. In this case Δ is a cone. Hence we are done by induction.

Case 2. We assume $F' \cap V(\Delta(d-r+2)) = \emptyset$. In this case $d-r+2 \leq 1$. Then we have $d = r-1$ or $d = r-2$. If $d = r-2$, then reg $I_\Delta \leq d+1 = r-1$. Contradiction. Hence we have $d = r-1$. In this case, for $F (\neq \emptyset)$, $\dim \text{link}_\Delta F \leq d-2 = r-3$. Then $\tilde{H}_{r-2}(\Delta; k) \neq 0$. Hence we have $\Delta = \partial(\Delta(r-1) * \Delta(d-r+2)) \cup_{F'} \mathbf{2}^F)(x \to y) \cong \partial\Delta(r)$. In this case $r = \deg k[\Delta] - \text{codim}k[\Delta] + 1$.

On the other hand, if Δ satisfies (1), (2), or (3), then it is easy to check $r = \deg k[\Delta] - \text{codim}k[\Delta] + 1$. **qed**

COROLLARY 4.3. *Let Δ be a $(d-1)$-dimensional pure and strongly connected complex on the vertex set $[n]$. Assume $r :=$ reg$I_\Delta \geq 4$. Then the following conditions are equivalent:*

(1)

reg$I_\Delta = \deg k[\Delta] - \text{codim}k[\Delta] + 1$.

(2)

$\Delta \cong \partial\Delta(r) * \Delta(d-r+1) + ((d-1)\text{-branches})$.

(3) $k[\Delta]$ *is Cohen-Macaulay with h-vector* $(1, n-d, 1, \ldots, 1(= h_{r-1}))$.

(4)

$$\beta_{i,i+j}(k[\Delta]) = \begin{cases} 1, & \textit{for } i = j = 0 \\ (n-d-1)\binom{n-d}{i} - \binom{n-d-1}{i+1}, & \textit{for } j = 1, i = 1, 2, \ldots, n-d \\ \binom{n-d-1}{i-1}, & \textit{for } j = r-1, i = 1, 2, \ldots, n-d \\ 0, & \textit{otherwise.} \end{cases}$$

(5)

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \begin{cases} 0, & \textit{for } i \neq d \\ \frac{t^{-d+r-1} + t^{-d+r-2} + \cdots + t^{-d+2} + (n-d)t^{-d+1} + t^{-d}}{(1-t^{-1})^d}, & \textit{for } i = d. \end{cases}$$

Proof. (1)$\Rightarrow$ (2) follows by Theorem 4.2. (2)$\Rightarrow$ (3) is easy to show, since Δ is shellable. (2)$\Rightarrow$(4). It is easy to see that $\beta_{i,i+j}(k[\Delta]) = 0$ unless $j = 0, 1, or, r-1$ by Hochster's formula. We see that

$$\beta_{i,i+r-1}(k[\Delta]) = \sum_{\substack{V(\partial\Delta(r)) \subset W \subset V(\Delta) \setminus V(\Delta(d-r+1)) \\ |W| = i+r-1}} \dim \tilde{H}_i(\Delta_W; k) = \binom{n-d-1}{i-1},$$

for $i = 1, 2, \ldots, n-d$. We can compute $\beta_{i,i+1}(k[\Delta])$ by the Hilbert series of $k[\Delta]$. (3)$\Rightarrow$ (5) follows from [St, Theorem 6.4]. (4)$\Rightarrow$ (3), (5)$\Rightarrow$ (3), and, (3)$\Rightarrow$ (1) are trivial. **qed**

COROLLARY 4.4. *Let Δ be a $(d-1)$-dimensional pure and strongly connected complex on the vertex set $[n]$. Assume* $\mathrm{reg} I_\Delta = 3$ *and $k[\Delta]$ satisfies* (S_2) *condition. Then the following conditions are equivalent:*
(1)

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

(2)

$$\Delta = \Delta(l\text{-gon}) * \Delta(d-2) + ((d-1)\text{-branches})$$

for some $l \geq 3$, where $\Delta(l$-gon$)$ is the boundary complex of the l-gon.
(3) $k[\Delta]$ is Cohen-Macaulay with h-vector $(1, n-d, 1)$.
(4)

$$\beta_{i,i+j}(k[\Delta]) = \begin{cases} 1, & \text{for } i = j = 0 \\ \frac{i(n-d-i)}{n-d+i}\binom{n-d+2}{i+1} + \binom{n-d-l+2}{i-l+1}, & \text{for } j = 1,\ i = 1, 2, \ldots, n-d \\ \binom{n-d-l+2}{i-l+2}, & \text{for } j = 2,\ i = 1, 2, \ldots, n-d \\ 0, & \text{otherwise} \end{cases}$$

for some $l \geq 3$.
(5)

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \begin{cases} 0, & \text{for } i \neq d \\ \frac{t^{-d+2} + (n-d)t^{-d+1} + t^{-d}}{(1-t^{-1})^d}, & \text{for } i = d. \end{cases}$$

Proof. Note that $k[\Delta]$ satisfies (S_2) if and only if (a)Δ is pure and (b)$\mathrm{link}_\Delta F$ is connected for every $F \in \Delta$ with $\dim \mathrm{link}_\Delta F \geq 1$. Then (1)$\Rightarrow$ (2) follows by Lemma 4.1. The rest is similar to the proof of the above corollary. **qed**

REMARK. A Cohen-Macaulay homogeneous ring R with h-vector $h(R) = (1, h_1, 1, 1, \ldots, 1)$ is called a *stretched* Cohen-Macalay ring (cf.[Oo]). These corollaries also give the classification of stretched Cohen-Macaulay Stanley-Reisner rings.

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