Stanley-Reisner 環における
Eisenbud-Goto の不等式

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序

極小自由分解の理論が進むにつれ、regularity という不変量の重要性が認識され、近年その研究が活発に行われてきた。特に、その上限を他の不変量で評価することが、その興味の中心になっている様に思われる。単項式イデアルの regularity に関しては、[Ho-Tr] において、様々な興味深い評価が与えられている。彼らは、単項式イデアルの 算術的次数や、極小生成系の 次数による上限を与えている。[Ho-Tr] の中では、代数的手法が主に用いられているが、[Fr-Te] や [Te2] においては、Hochster の公式に基づいて、組合せ論的、位相幾何学的手法がとられている。本小論では、それらについて解説したい。

多項式環の中の一般の次数付きイデアルの regularity に関して言えば、次の Eisenbud-Goto 予想が、その研究のためのモチベーションを与え続けてきた。

Eisenbud-Goto 予想 ([Ei-Go, Introduction]). $A = k[x_1, x_2, \ldots, x_n]$ を $n$ 変数の体 $k$ 上の多項式環とし、$P$ を $A$ の次数付き素イデアルで 1 次式を含まないものとする。このとき、

$$\text{reg } P \leq \deg A/P - \text{codim} A/P + 1.$$  

この予想は、今も可換環論や、代数幾何学で活発に研究されている。興味のある人は、例えば、[Kw] や [Mi-Vo] 及びそこに挙げられている参考文献を見られたい。
本小論においては、Eisenbud により予想された単項式版の Eisenbud-Goto 予想について考察する。

定理 0.1 (Eisenbud の予想)(cf. [Fr-Te], [Te2]). \( k \) を体とし、\( \Delta \) を pure で strongly connected な単体的複体とする。このとき、

\[
\operatorname{reg} I_{\Delta} \leq \deg k[\Delta] - \operatorname{codim} k[\Delta] + 1.
\]

この定理から Gröbner 基底の理論を用いてもともとの Eisenbud-Goto 予想に対しては次のことが言える。

系 0.2. \( A = k[x_1, x_2, \ldots, x_n] \) を \( n \) 変数の \( k \) 上の多項式環とし、\( P \) を \( A \) の次数付き素イデアルで 1 次式を含まないものとする。さらに \( A/\text{in}P \) は reduced であると仮定する。ここで、\( \text{in}P \) は、ある順序に関する \( P \) の initial ideal とする。このとき、

\[
\operatorname{reg} P \leq \deg A/P - \operatorname{codim} A/P + 1.
\]

次に Eisenbud-Goto の不等式において等号が成立する場合について考察する。このとき、次の定理が成立する。

定理 0.3 ([Te2]). \( k \) を体とし、\( \Delta \) を pure で strongly connected な \( (d-1) \) 次元の単体的複体とする。\( r = \operatorname{reg} I_{\Delta} \) とおく。このとき、

\[
\operatorname{reg} I_{\Delta} = \deg k[\Delta] - \operatorname{codim} k[\Delta] + 1
\]

であるための必要十分条件は、\( \Delta \) が次の条件を満たすこと。

(1) もし、\( r = 2 \) ならば、\( \Delta \) は、\( (d-1) \)-tree であること。ただし、\( (d-1) \) 単体ではないこと。

(2) もし、\( r = 3 \) ならば、ある \( (d-1) \)-tree \( \Delta' \) とある separated な \( v, w \in V(\Delta') \) に対して \( \Delta = \Delta'(v \to w) \) となること。

(3) もし、\( r \geq 4 \) ならば、\( \Delta \cong \partial \Delta(r) \ast \Delta(d-r+1) + ((d-1)\text{-branches}) \).

定義されていない用語に対しては、§1 及び §4 を見られたい。

1. PRELIMINARIES

We first fix notation. Let \( \mathbb{N}(\text{resp.} \mathbb{Z}) \) denote the set of nonnegative integers (resp. integers). Let \( |S| \) denote the cardinality of a set \( S \).
We recall some notation on simplicial complexes and Stanley-Reisner rings. We refer the reader to, e.g., [Br-He], [Hi], [Hoc] and [St] for the detailed information about combinatorial and algebraic background.

A (abstract) simplicial complex $\Delta$ on the vertex set $V = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of $V$ such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. The vertex set of $\Delta$ is denoted by $V(\Delta)$. Each element $F$ of $\Delta$ is called a face of $\Delta$. We call $F \in \Delta$ an $i$-face if $|F| = i + 1$ and we call a maximal face a facet. Let $F$ be a face but not a facet. We call $F$ free if there is a unique facet $G$ such that $F \subset G$. If $\{x_i\}$ is free, we call $x_i$ free.

We define the dimension of $F \in \Delta$ to be $\dim F = |F| - 1$ and the dimension of $\Delta$ to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. We say that $\Delta$ is pure if all facets have the same dimension. In a pure $(d - 1)$-dimensional complex $\Delta$, we call $(d - 2)$-face a subfacet. We say that a pure complex $\Delta$ is strongly connected if for any two facets $F$ and $G$, there exists a sequence of facets

$$F = F_0, F_1, \ldots, F_m = G$$

such that $F_{i-1} \cap F_i$ is a subfacet for $i = 1, 2, \ldots, m$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d - 1$, denote the number of $i$-faces in $\Delta$. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the $f$-vector of $\Delta$. Define the $h$-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of $\Delta$ by

$$\sum_{i=0}^{d} f_{i-1}(t - 1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}. $$

Let $\tilde{H}_i(\Delta; k)$ denote the $i$-th reduced simplicial homology group of $\Delta$ with the coefficient field $k$.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in $n$-variables over a field $k$. Define $I_\Delta$ to be the ideal of $A$ which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \not\subseteq \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the Stanley-Reisner ring of $\Delta$ over $k$.

Next we summarize basic facts on the Hilbert series. Let $k$ be a field and $R$ a homogeneous $k$-algebra. By a homogeneous $k$-algebra $R$ we mean a noetherian graded ring $R = \oplus_{i \geq 0} R_i$ generated by $R_1$ with $R_0 = k$. Let $M$ be a graded $R$-module with $\dim_k M_i < \infty$ for all $i \in \mathbb{Z}$, where $\dim_k M_i$ denotes the dimension of $M_i$ as a $k$-vector space. The Hilbert series of $M$
is defined by
\[ F(M, t) = \sum_{i \in \mathbb{Z}} (\dim_k M_i) t^i. \]

It is well known that the Hilbert series \( F(R, t) \) of \( R \) can be written in the form
\[ F(R, t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1 - t)^{\dim R}}, \]
where \( h_0(=1), h_1, \ldots, h_s \) are integers with \( \deg R := h_0 + h_1 + \cdots + h_s \geq 1 \), which is called the degree of \( R \). The vector \( h(R) = (h_0, h_1, \ldots, h_s) \) is called the h-vector of \( R \).

We consider \( k[\Delta] \) as the graded algebra \( k[\Delta] = \bigoplus_{i \geq 0} k[\Delta]_i \) with \( \deg x_j = l \) for \( 1 \leq j \leq n \).

The Hilbert series \( F(k[\Delta], t) \) of a Stanley-Reisner ring \( k[\Delta] \) can be written as follows:
\[ F(k[\Delta], t) = 1 + \sum_{i=1}^{d} \frac{f_{i-1} t^i}{(1 - t)^i} = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1 - t)^d}, \]
where \( \dim \Delta = d - 1 \), \( (f_0, f_1, \ldots, f_{d-1}) \) is the f-vector of \( \Delta \), and \( (h_0, h_1, \ldots, h_d) \) is the h-vector of \( \Delta \). It is easy to see \( \deg k[\Delta] = f_{d-1} \). On the other hand, the arithmetic degree of \( k[\Delta] \) is defined to be the number of facets in \( \Delta \), which is denoted by \( \text{a-deg} k[\Delta] \). See, e.g., [Ho-Tr] for the definition of the arithmetic degree of a general ring \( R \).

Let \( A \) be the polynomial ring \( k[x_1, x_2, \ldots, x_n] \) over a field \( k \). Let \( M(\neq 0) \) be a finitely generated graded \( A \)-module and let
\[ 0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{a,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{a,j}(M)} \longrightarrow M \longrightarrow 0 \]
be a graded minimal free resolution of \( M \) over \( A \). The length \( h \) of this resolution is called the projective dimension of \( M \) and denoted by \( h = \text{pd} M \). We call \( \beta_i(M) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(M) \) the \( i \)-th Betti number of \( M \) over \( A \). We define the Castelnuovo-Mumford regularity \( \text{reg} M \) of \( M \) by
\[ \text{reg} M = \max \{ j - i \mid \beta_{i,j}(M) \neq 0 \}. \]

See, e.g., [Ei] for further information on regularity. We define the initial degree \( \text{indeg} M \) of \( M \) by
\[ \text{indeg} M = \min \{ i \mid M_i \neq 0 \} = \min \{ j \mid \beta_{0,j}(M) \neq 0 \}. \]
Let $l$ be a natural number. We say that $M$ satisfies $(N_l)$ condition if $\beta_{i,i+s}(M) = 0$ for $i < l$, $s \neq \text{indeg } M$.

We denote the number of generators of $M$ by $\mu(M) = \beta_0(M)$.

The following two theorems are a starting point for our study.

**Theorem 1.1** (Hochster's formula on the Betti numbers [Hoc, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F \subset [n], |F| = j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{ G \in \Delta \mid G \subset F \}.$$

It is easy to see:

**Corollary 1.2.**

$$\text{reg } I_{\Delta} = \max \{ i + 2 \mid \tilde{H}_i(\Delta_F; k) \neq 0 \text{ for some } F \subset V \}.$$

If $F$ is a face of $\Delta$, then we define a subcomplex $\text{link}_\Delta F$ by

$$\text{link}_\Delta F = \{ G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta \}.$$  

**Theorem 1.3** (Hochster's formula on the local cohomology modules (cf. [St, Theorem 4.1])).

$$F(H^i_m(k[\Delta]), t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-|F|-1}(\text{link}_\Delta F; k) \left( \frac{t^{-1}}{1 - t^{-1}} \right)^{|F|},$$

where $H^i_m(k[\Delta])$ denote the $i$-th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $m$.

**Corollary 1.4.**

$$\text{reg } I_{\Delta} = \max \{ i + 2 \mid \tilde{H}_i(\text{link}_\Delta F; k) \neq 0 \text{ for some } F \in \Delta \}.$$  

Next we recall the definition of Alexander dual complexes. For a simplicial complex $\Delta$ on the vertex set $V$, we define an *Alexander dual complex* $\Delta^*$ as follows:

$$\Delta^* = \{ F \subset V : V \setminus F \notin \Delta \}.$$
THEOREM 1.5 [Te1, Corollary 0.3]. Let $k$ be a field. Let $\Delta$ be a simplicial complex. Then
\[
\text{reg } I_{\Delta} = \text{pd } k[\Delta^*].
\]

2. REGULARITY OF THE SUM OF IDEALS

In this section we give a upper bound for the sums of square-free monomial ideals.

In the rest of the paper we always assume that $k$ is a fixed field.

First we prove the following proposition. It seems to be known, but we cannot find it in literature.

PROPOSITION 2.1. Let $I$ be a monomial ideal in the polynomial ring $A = k[x_1, x_2, \ldots, x_n]$ and $m$ a monomial in $A$. Then
\[
\text{pd} A/(I + (m)) \leq \text{pd} A/I + 1.
\]

The following proof is simplified by suggestion of Eisenbud.

Proof. If we show that
\[
\text{pd } A/I \geq \text{pd} (I + (m))/I,
\]
then the mapping cone guarantees that
\[
\text{pd } A/(I + (m)) \leq \text{pd} A/I + 1.
\]

by [Ei, Exercise A.3.30]. We have,
\[
(I + (m))/I \cong (m)/((m) \cap I) \\
\cong (m)/((m) \cap (m_1, \ldots, m_t)) \\
\cong (m)/\text{lcm}(m, m_1, \ldots, m_t) \\
\cong A/(m_1', \ldots, m_t') \otimes_A (m),
\]
where $I = (m_1, \ldots, m_t)$, $m_i' = \frac{\text{lcm}(m, m_i)}{m}$. Hence, we have only to show
\[
\text{pd } A/I \geq \text{pd } A/(m_1', \ldots, m_t').
\]
Now we have \((A/I)_m \cong A_m/(m_1', \ldots, m_t')A_m\). Hence we have
\[
pd A/I \geq \pd (A/I)_m = \pd A_m/(m_1', \ldots, m_t')A_m = \pd A/(m_1', \ldots, m_t').
\]
We are done. \(\text{q.e.d}\)

For the regularity of the sum of square-free monomial ideals, we have the following conjecture:

**Conjecture 2.2.** Let \(\Delta_i(\neq \emptyset)\) be a simplicial complex for \(i = 1, 2\). Then we have
\[
\reg(I_{\Delta_1} + I_{\Delta_2}) \leq \reg I_{\Delta_1} + \reg I_{\Delta_2} - 1.
\]

If \(I_{\Delta_1}\) and \(I_{\Delta_2}\) are complete intersections, then the above inequality holds. The next theorem gives a weaker upper bound.

**Theorem 2.3.** Let \(\Delta_i(\neq \emptyset)\) be a simplicial complex for \(i = 1, 2\). Then we have
\[
\reg(I_{\Delta_1} + I_{\Delta_2}) \leq \min\{\reg I_{\Delta_1} + a - \deg k[\Delta_2], \ reg I_{\Delta_2} + a - \deg k[\Delta_1]\} - 1.
\]

**Proof.** Only in this proof, we define a simplicial complex \(\Delta\) by only the condition (ii) of the definition of a simplicial complex. We do not require the condition (i). Then we have \((\Delta^\ast)^\ast = \Delta\). And Theorem 1.5 also holds under this definition.

By the above proposition we have
\[
pd A/(I_{\Delta_1} + I_{\Delta_2}) \leq \pd A/I_{\Delta_1} + \mu(I_{\Delta_2}).
\]
Since \(I_{\Delta_1} + I_{\Delta_2} = I_{\Delta_1 \cap \Delta_2}\), we have
\[
\reg (I_{(\Delta_1 \cap \Delta_2)^\ast}) \leq \reg I_{\Delta_1^\ast} + a - \deg k[\Delta_2^\ast]
\]
by Theorem 1.5 and \(\mu(I_{\Delta_2}) = a - \deg k[\Delta_2^\ast]\). Since we have \(I_{(\Delta_1 \cap \Delta_2)^\ast} = I_{\Delta_1^\ast \cup \Delta_2^\ast} = I_{\Delta_1^\ast} \cap I_{\Delta_2^\ast}\), then we have
\[
\reg (I_{\Delta_1^\ast} \cap I_{\Delta_2^\ast}) \leq \reg I_{\Delta_1^\ast} + a - \deg k[\Delta_2^\ast].
\]
Similarly we have
\[
\reg (I_{\Delta_2^\ast} \cap I_{\Delta_1^\ast}) \leq \reg I_{\Delta_2^\ast} + a - \deg k[\Delta_1^\ast].
\]
Consider the exact sequence
\[ 0 \to A/(I_{\Delta_1} \cap I_{\Delta_2}) \to A/I_{\Delta_1} \oplus A/I_{\Delta_2} \to A/(I_{\Delta_1} + I_{\Delta_2}) \to 0. \]

By [Ei, Corollary 20.19], we have
\[
\text{reg } A/(I_{\Delta_1} + I_{\Delta_2}) \leq \max\{\text{reg } A/(I_{\Delta_1} \cap I_{\Delta_2}) - 1, \text{reg } (A/I_{\Delta_1} \oplus A/I_{\Delta_2})\}.
\]

Hence
\[
\text{reg } A/(I_{\Delta_1} + I_{\Delta_2}) \leq \min\{\text{reg } I_{\Delta_1} + \text{a-deg } k[\Delta_2] - 1, \text{reg } I_{\Delta_2} + \text{a-deg } k[\Delta_1] - 1\}.
\]
We obtained the desired result.

**REMARK.** Since the inequality \( \text{reg } I_{\Delta} \leq \text{a-deg } k[\Delta] \) holds (cf. [Ho-Tr] and [Fr-Te]), Theorem 2.3 is weaker than Conjecture 2.2.

3. **EISENbud-GOTo INEQUALITY**

In this section we prove Eisenbud-Goto inequality for Stanley-Reisner rings of pure and strongly connected simplicial complexes.

First we prove a lemma which is necessary for inductive argument.

**LEMMA 3.1.** Let \( \Delta \) be a pure and strongly connected simplicial complex. Then there exists a facet \( F \in \Delta \) such that
\[
\Delta' := \{ H \in \Delta \mid H \subset G \text{ for some facet } G(\neq F) \in \Delta \}
\]
is pure and strongly connected.

**Proof.** We define a graph \( G_{\Delta} \) corresponding to \( \Delta \) as follows: The vertex set \( V(G_{\Delta}) \) consists of \( \{y_F \mid F \text{ is a facet of } \Delta\} \). The edge set \( E(G_{\Delta}) \) is defined by: \( \{y_F, y_G\} \in E(G_{\Delta}) \) if and only if \( F \cap G \) is a subfacet. If \( \Delta \) is pure and strongly connected, \( G_{\Delta} \) is connected. It is well known that there exists a vertex \( y_F \in V(G_{\Delta}) \) such that \( G_{V(G_{\Delta}) \setminus \{F\}} \) is connected. Then \( \Delta' \) is pure and strongly connected.

Now we prove the main result in this section.

**THEOREM 3.2** (cf. [Fr-Te, Theorem 4.1]). Let \( \Delta \) be a pure and strongly connected simplicial complex. Then we have
\[
\text{reg } I_{\Delta} \leq \text{deg } k[\Delta] - \text{codim } k[\Delta] + 1.
\]
Proof. Let $V$ be the vertex set of $\Delta$. Put $|V| = n$ and $\dim k[\Delta] = d$. We prove the theorem by induction on the number $f_{d-1}$ of facets in $\Delta$.

First if codim $k[\Delta] \leq 1$, then $k[\Delta]$ is a hypersurface. In this case the theorem is clear.

Suppose codim $k[\Delta] \geq 2$ and $f_{d-1} \geq 2$. By the above lemma, there exists a facet $F \in \Delta$ such that

$$\Delta' := \{ H \in \Delta \mid H \subset G \text{ for some facet } G(\neq F) \in \Delta \}$$

is pure and strongly connected. Denote by $V'$ the vertex set of $\Delta'$ and by $f'_{d-1}$ the number of facets in $\Delta'$. There are two cases.

Case 1 $V \neq V'$. Put $V \setminus V' = \{v\}$. For $W \subset V$ with $v \notin W$ we have $\Delta_W \cong \Delta'_W$. On the other hand, for $W \subset V$ with $v \in W$, $H_i(\Delta_W; k) \cong H_i(\Delta'_W \setminus \{v\}; k)$ for $i \geq 1$. Since

$$\reg I_{\Delta} = \max\{i + 2 \mid H_i(\Delta_W; k) \neq 0 \text{ for some } W \subset V\},$$

we have

$$\reg I_{\Delta} = \reg I_{\Delta'} + 1 \leq f'_{d-1} - (n - d) + 2 = f_{d-1} - (n - d) + 1.$$ 

Case 2 $V = V'$. We have $\reg I_{\Delta} = \pd k[\Delta^*]$. Now we see that $k[\Delta^*] = k[(\Delta')^*]/(m)$, where $m = \prod_{x_i \in V \setminus F} x_i$. By Proposition 2.1, we have

$$\reg I_{\Delta} \leq \reg I_{\Delta'} + 1 \leq f'_{d-1} - (n - d) + 2 = f_{d-1} - (n - d) + 1.$$ 

qed

Corollary 3.3. Let $\Delta$ be a simplicial complex such that codim $k[\Delta] \geq 2$. Assume $I_{\Delta}$ satisfies $(N_2)$ condition. Then we have

$$\pd k[\Delta] \leq \mu(I_{\Delta}) - \deg I_{\Delta} + 1.$$ 

Proof. If $I_{\Delta}$ satisfies $(N_2)$ condition, then $k[\Delta^*]$ satisfies $(S_2)$ condition by [Ya, Corollary 3.7] and then $\Delta^*$ is pure and strongly connected. If $\Delta^*$ is pure, then deg $k[\Delta^*] = \mu(I_{\Delta})$. If codim $k[\Delta] \geq 2$, then $\deg I_{\Delta} = \text{codim} k[\Delta^*]$. We are done by Theorems 1.5 and 2.4. qed
Proof of Corollary 0.2. Put $I_\Delta = \text{in} P$. Then by [Ka-St, Theorem 1], $\Delta$ is pure and strongly connected. By Theorem 3.2, we have
\[
\begin{align*}
\text{reg} P & \leq \text{reg} I_\Delta \\
& \leq \deg k[\Delta] - \text{codim} k[\Delta] + 1 \\
& = \deg A/P - \text{codim} A/P + 1.
\end{align*}
\]
\[\text{qed}\]

4. EQUALITY CASE

In this section, we classify pure and strongly connected simplicial complexes $\Delta$ which satisfy $\text{reg} I_\Delta = \deg k[\Delta] - \text{codim} k[\Delta] + 1$, and give some characterization for such complexes.

First we introduce some notation. Put $[m] = \{1, 2, \ldots, m\}$. We denote the elementary $(m - 1)$-simplex by $\Delta(m) = 2^m$ and put $\Delta(0) = \emptyset$. We put $\partial \Delta(m) = 2^m \setminus \{[m]\}$, which is the boundary complex of $\Delta(m)$.

Let $\Delta_i$ be a $(d - 1)$-dimensional pure simplicial complex for $i = 1, 2$. If $\Delta_1 \cap \Delta_2 = 2^F$ for some $F$ with $\dim F = d - 2$, we write $\Delta_1 \cup_F \Delta_2$ for $\Delta_1 \cup \Delta_2$. We sometimes write $\Delta_1 \cup_F \Delta_2$ for $\Delta_1 \cup_F \Delta_2$ if we do not need to express $F$ explicitly.

We define a $(d - 1)$-tree inductively as follows.

(1) $\Delta(d)$ is a $(d - 1)$-tree.
(2) If $T$ is a $(d - 1)$-tree, then so is $T \cup \Delta(d)$.

If $T_1, T_2, \ldots, T_m$ are $(d-1)$-trees, we abbreviate $\Delta \cup T_1 \cup T_2 \cup \cdots \cup T_m$ as $\Delta + ((d - 1)$-branches).

Let $\Delta$ be a $(d - 1)$-dimensional pure and strongly connected complex. Take $v, w \in V(\Delta)$. We say $v$ and $w$ are separated in $\Delta$ if $\{v, w\} \not\in \Delta$ and that there exists no subfacet $F$ in $\Delta$ with $\{v\} \cup F$, $\{w\} \cup F \in \Delta$. If $v$ and $w$ are separated in $\Delta$, We denote $\Delta(v \to w)$ for the abstract simplicial complex which is obtained by substitution of $w$ for every $v$ in $\Delta$. The vertex set of $\Delta(v \to w)$ is $V(\Delta) \setminus \{v\}$.

By Lemma 3.1 we know that every $(d-1)$-dimensional pure and strongly connected simplicial complex can be constructed from the $(d-1)$-dimensional elementary simplex $\Delta(d)$ by a succession
\[
\Delta(d) = \Delta_1 \to \Delta_2 \to \cdots \to \Delta_{f_{d-1}}
\]
of either of the following operations:
(1) $\Delta_{i+1} = \Delta_i \cup_F 2^F$, where $x \not\in V(\Delta_i)$, $F'$ is a subfacet of $\Delta_i$ and $F = \{x\}$. 

$F' \cup \{x\}$.

(2) $\Delta_{i+1} = (\Delta_i \cup_{F'} 2^F)(x \rightarrow y)$, where $x \notin V(\Delta_i)$, $F'$ is a subfacet of $\Delta_i$ and $y \in V(\Delta_i)$ such that $x$ and $y$ are separated and $F = F' \cup \{x\}$.

Let $\Delta_i$ be a simplicial join for $i = 1, 2$ such that $V(\Delta_1) \cap V(\Delta_2) = \emptyset$. We define the simplicial join $\Delta_1 \ast \Delta_2$ of $\Delta_1$ and $\Delta_2$ by

$$\Delta_1 \ast \Delta_2 = \{F \cup G \mid F \in \Delta_1, G \in \Delta_2\}.$$  

**Lemma 4.1.** Let $\Delta$ be a $(d-1)$-dimensional pure and strongly connected complex. We assume

$$\text{reg} I_{\Delta} = \deg k[\Delta] - \text{codim} k[\Delta] + 1 = 3.$$  

Then $\Delta$ can be expressed as follows:

$$\Delta \cong \Delta'(x \rightarrow y) \ast \Delta(d-s) + ((d-1)\text{-branches})$$  

for some $(s-1)$-tree $\Delta'$ and for some separated $x, y \in V(\Delta')$ with $H_1(\Delta'(x \rightarrow y); k) \neq 0$.

**Proof.** We may assume $\Delta$ has no branches. Then $\Delta$ can be expressed as $\Delta = \Delta'(x \rightarrow y)$, where $\Delta'$ is a $(d-1)$-tree and $x, y \in V(\Delta')$ are the only free vertices in $\Delta'$. Let $F$ be the facet with $x \in F$ and $G$ the facet with $y \in G$ in $\Delta'$.

Let $G_{\Delta'}$ be the graph introduced in the proof of Lemma 3.1. Since $\Delta'$ is a $(d-1)$-tree with only two free vertices $x$, $y$, then $G_{\Delta'}$ is a line with the end points $y_F$ and $y_G$. Hence there exists a sequence of facets

$$F = F_0, F_1, \ldots, F_m = G$$  

such that $F_{i-1} \cap F_i$ is a subfacet for $i = 1, 2, \ldots, m$. Then $F_0, F_1, \ldots, F_m$ are all facets in $\Delta'$. We put $W = F \cap G$. If we have $z \in F_i$ and $z \notin F_{i+1}$, then $z \notin F_{i+2}$, since $\Delta'$ is a $(d-1)$-tree. Then we have $W \subset F_i$ for $i = 0, 1, 2, \ldots, m$. Then we have $\Delta' = \Delta_1 \ast 2^W$, and $\Delta'(x \rightarrow y) = \Delta_1(x \rightarrow y) \ast 2^W$, where $\Delta_1$ is an $(s-1)$-tree and $s = d - \left| W \right|$. It is easy to check that $\Delta_1(x \rightarrow y)$ is contractible to the circle $S^1$.  

**Theorem 4.2.** Let $\Delta$ be a $(d-1)$-dimensional pure and strongly connected complex. We put $r = \text{reg} I_{\Delta}$. Then

$$\text{reg} I_{\Delta} = \deg k[\Delta] - \text{codim} k[\Delta] + 1.$$
if and only if $\Delta$ satisfies the following condition:

1. $\Delta$ is a $(d-1)$-tree which is not the $(d-1)$-simplex if $r = 2$.
2. $\Delta = \Delta'(v \to w)$ for some $(d-1)$-tree $\Delta'$ and for some separated $v, w \in V(\Delta')$ if $r = 3$.
3. $\Delta \cong \partial \Delta(r) \ast \Delta(d - r + 1) + ((d - 1)\text{-branches})$ if $r \geq 4$.

**Proof.** First we assume that $\Delta$ satisfies $r = \deg k[\Delta] - \text{codim} k[\Delta] + 1$.

We use induction on $r$. If $r = 2$, then $\Delta$ is a $(d-1)$-tree by [Fr].

If $r = 3$, then by the procedure to construct pure and strongly connected complexes, (3) is easy to check.

We assume $r = 4$. We prove the statement by induction on $\dim \Delta$. We may assume $\Delta$ has no branches. Then $\Delta$ is of the form

$$\Delta = (\Delta' \cup F, 2^F)(x \to y)$$

where $\Delta'$ is pure and strongly connected and $F = F' \cup \{x\}$ is a facet of $\Delta$ and $y \in V(\Delta')$ such that $x$ and $y$ are separated in $\Delta' \cup_{Fr} 2^F$. We have $\deg k[\Delta'] - \text{codim} k[\Delta'] + 1 = 3$ and hence from the proof of Theorem 3.2 we get $\text{reg}_{\Delta} = 3$. By the assumption of induction and the previous lemma, $\Delta'$ is of the form $\Delta' = \Delta''(v \to w) \ast \Delta(d - s) + ((d - 1)\text{-branches})$ for some $(s-1)$-tree $\Delta''$ and for some separated $v, w \in V(\Delta'')$ with $\tilde{H}_1(\Delta''(v \to w); k) \neq 0$.

If $x \not\in V(\Delta''(v \to w) \ast \Delta(d - s))$ or if $F' \not\in \Delta''(v \to w) \ast \Delta(d - s)$, then the branch part can be contractible to a 1-dimensional subcomplex, then we have $\tilde{H}_2(\Delta X; k) = 0$ for each $X \subset V(\Delta)$. Contradiction. Since $\Delta$ has no branches, we have $\Delta' = \Delta''(v \to w) \ast \Delta(d - s)$ and $x \in V(\Delta''(v \to w) \ast \Delta(d - s))$.

Case 1. We assume $F' \cap V(\Delta(d - s)) \neq \emptyset$. In this case $\Delta$ is a cone. Hence we are done by induction.

Case 2. We assume $F' \cap V(\Delta(d - s)) = \emptyset$. In this case $d - s \leq 1$. Then we have $d = s$ or $d = s + 1$. Then $\Delta$ and its subcomplexes of the form $\Delta X$ for $X \subset V(\Delta)$ are contractible or contractible to a 1-dimensional complex, unless $d = s + 1$ and $\Delta''(v \to w) = \partial \Delta(3)$. Here we omit a detail. Only case we must consider is $s = 1, d = 2$ and $\Delta = \partial \Delta(4)$. In this case

$$r = \deg k[\Delta] - \text{codim} k[\Delta] + 1 = 4.$$  

If $r \geq 5$, we prove the statement by induction on $\dim \Delta$. We may assume $\Delta$ has no branches. Then $\Delta$ is of the form

$$\Delta = (\Delta' \cup F, 2^F)(x \to y)$$

where $\Delta'$ is pure and strongly connected and $F = F' \cup \{x\}$ is a facet of $\Delta$ and $y \in V(\Delta')$ such that $x$ and $y$ are separated. We have $\deg k[\Delta'] -
codim$k[\Delta'] + 1 = r - 1$ and hence from the proof of Theorem 3.2 we get
\[ \text{reg} I_{\Delta'} = r - 1. \]
By the assumption of induction, \( \Delta' = \partial \Delta(r - 1) * \Delta(d - r + 2) + ((d - 1)\text{-branches}) \).
If \( x \notin V(\partial \Delta(r - 1) * \Delta(d - r + 2)) \) or \( F' \notin \partial \Delta(r - 1) * \Delta(d - r + 2) \),
then the branch part can be contractible to a 1-dimensional subcomplex, then we have \( \tilde{H}_{r-2}(\Delta_X; k) = 0 \).
Hence we are done by induction.

\textbf{Case 1.} We assume \( F' \cap V(\Delta(d - r + 2)) \neq \emptyset \). In this case \( \Delta \) is a cone.

\textbf{Case 2.} We assume \( F' \cap V(\Delta(d - r + 2)) = \emptyset \). In this case \( d - r + 2 \leq 1 \).
Then we have \( d = r - 1 \) or \( d = r - 2 \). If \( d = r - 2 \), then \( \text{reg} I_{\Delta} \leq d + 1 = r - 1 \).
Contradiction. Hence we have \( d = r - 1 \). In this case, for \( F(\neq \emptyset) \), \( \dim \text{link}_A F \leq d - 2 = r - 3 \).
Then \( \tilde{H}_{r-2}(\Delta; k) \neq 0 \). Hence we have \( \Delta = \partial(\Delta(r - 1) * \Delta(d - r + 2)) \cup_{F'} 2^F(x \rightarrow y) \cong \partial \Delta(r) \).
In this case \( r = \deg k[\Delta] - \text{codim} k[\Delta] + 1 \).

On the other hand, if \( \Delta \) satisfies (1), (2), or (3), then it is easy to check \( r = \deg k[\Delta] - \text{codim} k[\Delta] + 1 \).

\textbf{Corollary 4.3.} \textit{Let \( \Delta \) be a \((d - 1)\text{-dimensional pure and strongly connected complex on the vertex set } [n] \). Assume } r := \text{reg} I_{\Delta} \geq 4 \text{. Then the following conditions are equivalent:}

\begin{enumerate}
\item[(1)] \( \text{reg} I_{\Delta} = \deg k[\Delta] - \text{codim} k[\Delta] + 1 \).
\item[(2)] \( \Delta \cong \partial \Delta(r) * \Delta(d - r + 1) + ((d - 1)\text{-branches}) \).
\item[(3)] \( k[\Delta] \) is Cohen-Macaulay with \( h \)-vector \((1, n - d, 1, \ldots, 1(= h_{r-1}))\).
\item[(4)] \( \beta_{i,i+j}(k[\Delta]) = \begin{cases} 1, & \text{for } i = j = 0 \\
(n - d - 1)(n - i) - (n - d - 1), & \text{for } j = 1, i = 1, 2, \ldots, n - d \\
\frac{(n - d - 1)}{i - 1}, & \text{for } j = r - 1, i = 1, 2, \ldots, n - d \\
0, & \text{otherwise.}\end{cases} \)
\item[(5)] \( F(H^i_m(k[\Delta]), t) = \begin{cases} 0, & \text{for } i \neq d \\
\frac{(d + 1) \cdots (d + i - 1)}{(1 - t)^i} & \text{for } i = d.\end{cases} \)
\end{enumerate}
Proof. (1)⇒(2) follows by Theorem 4.2. (2)⇒(3) is easy to show, since Δ is shellable. (2)⇒(4). It is easy to see that $\beta_{i,i+j}(k[\Delta]) = 0$ unless $j = 0, 1, or, r - 1$ by Hochster's formula. We see that

$$\beta_{i,i+r-1}(k[\Delta]) = \sum_{V(\partial \Delta(r)) \subset W \subset V(\Delta) \setminus V(\Delta(d-r+1))} \dim H_i(\Delta_W; k) = \binom{n-d-1}{i-1},$$

for $i = 1, 2, \ldots, n - d$. We can compute $\beta_{i,i+1}(k[\Delta])$ by the Hilbert series of $k[\Delta]$. (3)⇒(5) follows from [St, Theorem 6.4]. (4)⇒(3), (5)⇒(3), and, (3)⇒(1) are trivial.

**Corollary 4.4.** Let Δ be a $(d - 1)$-dimensional pure and strongly connected complex on the vertex set [n]. Assume $\text{reg} \Delta = 3$ and $k[\Delta]$ satisfies $(S_2)$ condition. Then the following conditions are equivalent:

1. $\text{reg} \Delta = \deg k[\Delta] - \text{codim} k[\Delta] + 1.$
2. $\Delta = \Delta(l\text{-gon}) \ast \Delta(d - 2) + ((d - 1)\text{-branches})$
for some $l \geq 3$, where $\Delta(l\text{-gon})$ is the boundary complex of the $l\text{-gon}$.
3. $k[\Delta]$ is Cohen-Macaulay with $h$-vector $(1, n-d, 1)$.
4. $\beta_{i,i+j}(k[\Delta]) = \begin{cases} 1, & \text{for } i = j = 0 \\ \frac{i(n-d-j)}{n-d+j} + \binom{n-d-1}{i-1}, & \text{for } j = 1, i = 1, 2, \ldots, n - d \\ \frac{n-d-l+2}{i-l+2}, & \text{for } j = 2, \text{ } i = 1, 2, \ldots, n - d \\ 0, & \text{otherwise} \end{cases}$
for some $l \geq 3$.
5. $F(H^i_{\mathfrak{m}}(k[\Delta]), t) = \begin{cases} 0, & \text{for } i \neq d \\ \frac{t^{d+2} - (n-d)t^{d+1} + t - d}{(1-t^{-1})^d}, & \text{for } i = d. \end{cases}$

Proof. Note that $k[\Delta]$ satisfies $(S_2)$ if and only if (a)Δ is pure and (b)link$_{\Delta} F$ is connected for every $F \in \Delta$ with $\text{dim} \text{link}_\Delta F \geq 1$. Then (1)⇒(2) follows by Lemma 4.1. The rest is similar to the proof of the above corollary. qed
Remark. A Cohen-Macaulay homogeneous ring $R$ with $h$-vector $h(R) = (1, h_1, 1, 1, \ldots, 1)$ is called a stretched Cohen-Macaulay ring (cf. [Oo]). These corollaries also give the classification of stretched Cohen-Macaulay Stanley-Reisner rings.

References


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