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Interface Equations with Nonlocal Effects [1]

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1. Introduction

1.1. **Interface Equation.** Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain. An interface in this article is meant to be an $N-1$ dimensional closed hypersuface $\Gamma \subset \Omega$ which separates Ω into two components Ω_Γ^- and Ω_Γ^+:

$$\Omega = \Omega_\Gamma^- \cup \Gamma \cup \Omega_\Gamma^+.$$

There are varieties of ways for an interface to separate the domain Ω into two parts. For the ease of presentation, we always assume that Γ stays away from $\partial\Omega$ and that the following situation is realized (cf. Figure 1):

$$\text{(1.1)} \qquad \text{(i) } \Omega = \Omega_\Gamma^- \cup \Gamma \cup \Omega_\Gamma^+, \quad \text{(ii) } \partial\Omega_\Gamma^- = \Gamma, \quad \text{(iii) } \partial\Omega_\Gamma^+ = \Gamma \cup \partial\Omega.$$

The subject matter of this article is to describe the motion of interfaces evolving in time (denoted by t) according to certain laws. To be more precise, let $\{\Gamma(t)\}_{t\geq 0}$ be a family of interfaces parameterized by time t, with each $\Gamma(t)$ satisfying the conditions in (1.1). We always assume that $\Gamma(t)$ for each $t \geq 0$ is sufficiently smooth, say, of C^k-class with $k \geq 2$. Then our concern is to find a solution of

$$\text{(1.2)} \qquad \mathbf{V}_{\Gamma(t)}(x) = \mathcal{S}(x : \Gamma(t)) \qquad (x \in \Gamma(t), \quad t \geq 0), \qquad \Gamma(0) = \Gamma_0,$$

where $\mathbf{V}_{\Gamma(t)}(x)$ is the normal speed of $\Gamma(t)$ at $x \in \Gamma(t)$ and $\mathcal{S}(x; \Gamma(t))$ is a scalar function depending on x and the interface $\Gamma(t)$. When we measure the normal speed, we always do so along the unit normal $\nu(x,t)$ of $\Gamma(t)$ which points into $\Omega_{\Gamma(t)}^+$, as in Figure 1. Equations like (1.2) are called an interface equation.

1.2. **Examples.** Before we describe the interface equation which we are concerned with, let us first list some examples that share some common features with our problem. In these examples, we emphasize on wherther the flow preserves the volume of $\Omega_{\Gamma(t)}^-$ or not, and on whether it decreases the area ($N-1$-dimensional volume) of $\Gamma(t)$ or not.

Example 1 (**Mean Curvature Flow**).

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A well-known and extensively studied example of an interface equation is the *mean curvature flow* described by

$$\text{(MC)} \qquad \mathbf{V}_{\Gamma(t)}(x) = -\mathcal{H}(x;\Gamma(t)),$$

where $\mathcal{H}(x;\Gamma(t))$ is the sum of the principal curvatures of $\Gamma(t)$ at $x \in \Gamma(t)$. Our convention for the sign of $\mathcal{H}(x;\Gamma(t))$ is such that it is positive when $\Gamma(t)$ is a sphere and $\Omega^-_{\Gamma(t)}$ is the interior of $\Gamma(t)$.

This problem is derived as a singular limit of the Allen-Cahn equation [3]:

$$\text{(1.3)} \qquad \frac{\partial u}{\partial t} = \Delta u + \frac{1}{\epsilon^2} f(u) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega$$

as $\epsilon \to 0$, where $f(u) = u - u^3$ for example, and $\mathbf{n}$ stands for the outward unit normal. In general, a solution $\Gamma(t)$ of (MC) evolves in such a way that the surface area $|\Gamma(t)|$ decreases. To show this, let dS^Γ_x be the surface element on $\Gamma \subset \Omega$, induced from the standard metric in Ω. Then we compute

$$\begin{aligned}\frac{d}{dt}|\Gamma(t)| &= \int_{\Gamma(t)} \mathcal{H}(x;\Gamma(t))\mathbf{V}_{\Gamma(t)}(x)dS^{\Gamma(t)}_x \\ &= -\int_{\Gamma(t)} \mathcal{H}(x;\Gamma(t))^2 dS^{\Gamma(t)}_x \leq 0.\end{aligned}$$

It is known [12, 9, 11] that if the initial interface Γ_0 is convex, then the solution $\Gamma(t)$ of (MC) shrinks to a point in finite time. Naturally, as long as $\Gamma(t)$ remains convex (or $\mathcal{H}(x;\Gamma(t)) \geq 0$, in general), the volume $|\Omega^-_{\Gamma(t)}|$ is also decreasing:

$$\frac{d}{dt}\left|\Omega^-_{\Gamma(t)}\right| = \int_{\Gamma(t)} \mathbf{V}_{\Gamma(t)}(x)dS^{\Gamma(t)}_x = -\int_{\Gamma(t)} \mathcal{H}(x;\Gamma(t))dS^{\Gamma(t)}_x \leq 0.$$

EXAMPLE 2 (**Averaged Mean Curvature Flow**).

This is a conservation-version of EXAMPLE 1.

$$\text{(AMC)} \qquad \mathbf{V}_{\Gamma(t)}(x) = -\mathcal{H}(x;\Gamma(t)) + \overline{\mathcal{H}}(t)$$

where $\overline{\mathcal{H}}(t)$ is the average of the mearn curvature

$$\overline{\mathcal{H}}(t) = \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} \mathcal{H}(x;\Gamma(t))dS^{\Gamma(t)}_x.$$

(AMC) is derived in [2] as a singular limit of (1.3) with $f(u)$ being replaced by

$$f(u) - \frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx.$$

The flow generated by (AMC) enjoys two properties.

- Volume Preserving. The volume enclosed by $\Gamma(t)$ is preserved:

$$\frac{d}{dt}\left|\Omega^-_{\Gamma(t)}\right| = \int_{\Gamma(t)} \left(\overline{\mathcal{H}}(t) - \mathcal{H}\right) dS^{\Gamma(t)}_x = 0.$$

- Area Decreasing. The area of the interface decreases in time:

$$\frac{d}{dt}\Big|\Gamma(t)\Big| = -\int_{\Gamma(t)} \Big(\overline{\mathcal{H}}(t) - \mathcal{H}\Big)^2 dS_x^{\Gamma(t)} \leq 0.$$

It is shown [13] that if the initial interface Γ_0 is uniformly convex, then the solution $\Gamma(t)$ of (AMC) exists globally for $t \in [0, \infty)$ and it converges (as $t \to \infty$) to a round sphere.

The equation (AMC) has a slight nonlocal effect due to the average $\overline{\mathcal{H}}(t)$. However, the average is determined solely by $\Gamma(t)$ and, in this sense, (AMC) is still of local type. The equations (MC) and (AMC), in fact, do not *feel* the presence of the boundary $\partial\Omega$. This is why (MC) and (AMC), as they stand, are often posed in the entire space $\mathbb{R}^N$.

EXAMPLE 3 (**Mullins-Sekerka Problem**).
The problem is decribed by

$$\text{(M-S)} \qquad \mathbf{V}_{\Gamma(t)}(x) = -\Big(\Big(\Pi^-_{\Gamma(t)} + \Pi^+_{\Gamma(t)}\Big)\mathcal{H}(\cdot\,;\Gamma(t))\Big)(x) \quad (x \in \Gamma(t),\ \ t > 0),$$

where $\Pi^\pm_\Gamma$ are Dirichlet-to-Neumann oprators associated with the boundary value problems:

$$\text{(1.4)} \qquad \left\{\begin{array}{rcll} -\Delta u^\pm &=& 0 & (x \in \Omega^\pm_\Gamma) \\ u^\pm(x) &=& q(x) & (x \in \Gamma) \\ \partial u^+/\partial \mathbf{n} &=& 0 & (x \in \partial\Omega). \end{array}\right.$$

Namely, they are respectively defined by

$$\Big(\Pi^\pm_\Gamma q\Big)(x) = \mp\frac{\partial u^\pm(x)}{\partial \nu} \qquad (x \in \Gamma),$$

with $u^\pm$ being unique solutions of (1.4) on $\Omega^\pm_\Gamma$, respectively. It is known that $\Pi^\pm_\Gamma$ are first order elliptic operators whose principal part is the square root of $-\Delta^\Gamma$:

$$\Pi^\pm_\Gamma \approx \sqrt{-\Delta^\Gamma}.$$

These operators are apparently of nonlocal type and depend strongly on the geometries of Γ and Ω. The existence and uniqueness of time local classical solutions for (M-S) were established by Chen, Hong and Yi [5]. Pego [20] derived (M-S) as a singular limit of the Cahn-Hilliard equation and Chen [4] established the convergence of the solutions of the Cahn-Hilliard equation to those of (M-S) as the singular perturbation parameter tends to 0.

The flow generated by (M-S) also enjoys the two properties, volume preserving and area decreasing. In fact, denoting by $u^\pm$ the solutions of (1.4) with $q(x) = \mathcal{H}(x;\Gamma(t))$, one obtains the following:

$$\begin{aligned}\frac{d}{dt}\left|\Omega^-_{\Gamma(t)}\right| &= \int_{\Gamma(t)} \mathbf{V}_{\Gamma(t)} dS^{\Gamma(t)}_x = -\int_{\Gamma(t)} \left(\Pi^-_{\Gamma(t)} + \Pi^+_{\Gamma(t)}\right)\mathcal{H}(\Gamma(t)) dS^{\Gamma(t)}_x \\ &= \int_{\Gamma(t)} \frac{\partial u^+}{\partial \nu} dS^{\Gamma(t)}_x - \int_{\Gamma(t)} \frac{\partial u^-}{\partial \nu} dS^{\Gamma(t)}_x \\ &= -\int_{\Omega^+_{\Gamma(t)}} \Delta u^+ dx + \int_{\partial\Omega} \frac{\partial u^+}{\partial \mathbf{n}} - \int_{\Omega^-_{\Gamma(t)}} \Delta u^- dx = 0,\end{aligned}$$

(note that ν is pointing into Ω^+_Γ)

and

$$\begin{aligned}\frac{d}{dt}\left|\Gamma(t)\right| &= \int_{\Gamma(t)} \mathcal{H}\mathbf{V}_{\Gamma(t)} dS^{\Gamma(t)}_x = -\int_{\Gamma(t)} \mathcal{H}\left(\Pi^-_{\Gamma(t)} + \Pi^+_{\Gamma(t)}\right)\mathcal{H} dS^{\Gamma(t)}_x \\ &= \int_{\Gamma(t)} u^+ \frac{\partial u^+}{\partial \nu} dS^{\Gamma(t)}_x - \int_{\Gamma(t)} u^- \frac{\partial u^-}{\partial \nu} dS^{\Gamma(t)}_x \\ &= -\int_{\Omega^+_{\Gamma(t)}} |\nabla u^+|^2 dx - \int_{\Omega^-_{\Gamma(t)}} |\nabla u^-|^2 dx \leq 0.\end{aligned}$$

EXAMPLE 4 (**Morphology Equation**).

This is an inhomogeneous version of the previous example (M-S) and was derived by Nishiura and Ohnishi [16] (see also [19]) as a singular limit of a nonlocal reaction-diffusion equation. The motion law is given by

$$\text{(Mor)} \qquad \mathbf{V}_{\Gamma(t)}(x) = -\left(\left(\tilde{\Pi}^-_{\Gamma(t)} + \tilde{\Pi}^+_{\Gamma(t)}\right)\mathcal{H}(\cdot;\Gamma(t))\right)(x) \quad (x \in \Gamma(t),\ \ t > 0),$$

in which $\tilde{\Pi}^\pm_\Gamma$ are Dirichlet-to-Neumann operators associated with the following *inhomogeneous* boundary value problems.

$$(1.5) \qquad \left\{\begin{array}{rcll} -\Delta u^\pm &=& \pm 1 - (|\Omega^+_\Gamma| - |\Omega^-_\Gamma|)/|\Omega| & (x \in \Omega^\pm_\Gamma) \\ u^\pm(x) &=& q(x) & (x \in \Gamma) \\ \partial u^+/\partial \mathbf{n} &=& 0 & (x \in \partial\Omega). \end{array}\right.$$

The existence of time local solutions of (Mor) was established by Ohnishi and Imai [18]. The flow generated by (Mor) also enjoys the volume preserving property.

$$\begin{aligned}\frac{d}{dt}\left|\Omega^-_{\Gamma(t)}\right| &= \int_{\Gamma(t)} \frac{\partial u^+}{\partial \nu} dS^{\Gamma(t)}_x - \int_{\Gamma(t)} \frac{\partial u^-}{\partial \nu} dS^{\Gamma(t)}_x \\ &= -\int_{\Omega^+_{\Gamma(t)}} \Delta u^+ dx + \int_{\partial\Omega} \frac{\partial u^+}{\partial \mathbf{n}} - \int_{\Omega^-_{\Gamma(t)}} \Delta u^- dx \\ &= \left|\Omega^+_{\Gamma(t)}\right| \left(1 - \frac{|\Omega^+_{\Gamma(t)}| - |\Omega^-_{\Gamma(t)}|}{|\Omega|}\right) + \left|\Omega^-_{\Gamma(t)}\right| \left(-1 - \frac{|\Omega^+_{\Gamma(t)}| - |\Omega^-_{\Gamma(t)}|}{|\Omega|}\right) = 0.\end{aligned}$$

However, it is not necessarily area decreasing.

1.3. **Interface driven by mean curvature and shape effects.** Interfaces governed by (1.2) may (and often do) develop singularities in finite time and change their topology. In this article, we do not deal with such singularity formation.

The interface equation we study in this article is the following:

$$
(1.6)\quad
\begin{aligned}
&\text{(a)}\quad \mathbf{V}_{\Gamma(t)}(x) = -\mathcal{H}(x;\Gamma(t)) + c_* v(x;\Gamma(t)), \quad (x \in \Gamma(t),\ t \geq 0)\\
&\qquad\ \Gamma(0) = \Gamma_0,\\
&\text{(b)}\quad \left\{
\begin{array}{rcll}
-\Delta v(x;\Gamma(t)) & = & P(x;\Gamma(t)) & (x \in \Omega\backslash\Gamma(t),\ t \geq 0)\\
\dfrac{\partial v}{\partial \mathbf{n}} & = & 0 & (x \in \partial\Omega,\ t \geq 0)\\
v(x;\Gamma(0)) & = & v_0(x) & (x \in \overline{\Omega})\\
v(\cdot;\Gamma(t)) & \in & C^2(\overline{\Omega}\backslash\Gamma(t)) \cap C^1(\overline{\Omega}) & (t \geq 0),
\end{array}
\right.
\end{aligned}
$$

where $c_* > 0$ is a constant,

$$
P(x;\Gamma) = \left\{
\begin{array}{ll}
G^-/D & x \in \Omega_\Gamma^-\\
G^+/D & x \in \Omega_\Gamma^+
\end{array}
\right.
$$

with $G^- < 0 < G^+, D > 0$ being constants. In problem (1.6), the normal speed $\mathbf{V}_{\Gamma(t)}(x)$ of the interface is given as a sum of the mean curvature and the value of the function $v(x;\Gamma)$. The latter function, in turn, is uniquely determined by the shape and relative positions of the interface and the domain. Non-local effects are thus encoded in the values of $v(x;\Gamma)$.

The system (1.6) is derived as a singular limit of the following reaction-diffusion system (cf. [17, 25, 21, 22]).

$$
\text{(RD)}\quad \left\{
\begin{array}{rcll}
\epsilon^2 u_t & = & \epsilon^2 \Delta u + f(u,v) & (x \in \Omega,\ t > 0),\\
\epsilon^2 v_t & = & D\Delta v + \epsilon g(u,v) & (x \in \Omega,\ t > 0),\\
 & & \dfrac{\partial(u,v)}{\partial \mathbf{n}} = 0 & (x \in \partial\Omega,\ t > 0),
\end{array}
\right.
$$

with (f,g) being, for example, $f(u,v) = u - u^3 - v$, $g(u,v) = u - v$, in which case $c_* = 3/\sqrt{2}$.

1.4. **Reformulation.** We will now transform (1.6) into a form of (1.2).

The Neumann boundary value problem

$$
(1.7)\quad \left\{
\begin{array}{rcll}
-\Delta v & = & f & \text{in } \Omega\\
\dfrac{\partial v}{\partial \mathbf{n}} & = & g & \text{on } \partial\Omega,
\end{array}
\right.
$$

for a given pair of data $(f, g) \in C(\overline{\Omega}) \times C(\partial\Omega)$ has a solution in $C^1(\overline{\Omega})$ if and only if the compatiblity condition

$$\int_\Omega f(x)dx = -\int_{\partial\Omega} g(x)dS_x \tag{1.8}$$

is satisfied. In our case (1.6)-(b), $g \equiv 0$, and hence we have to impose the condition

$$\int_\Omega P(x;\Gamma(t))dx = 0,$$

which is equivalent to $|\Omega^-_{\Gamma(t)}|G^- + |\Omega^+_{\Gamma(t)}|G^+ = 0$, where $|\Omega'|$ is the N dimensional Lebesgue measure of a set Ω'. This condition, together with $|\Omega^-_{\Gamma(t)}| + |\Omega^+_{\Gamma(t)}| = |\Omega|$, implies

$$|\Omega^-_{\Gamma(t)}| = \frac{G^+}{[G]}|\Omega|, \qquad |\Omega^+_{\Gamma(t)}| = -\frac{G^-}{[G]}|\Omega| \qquad (\text{with } [G] := G^+ - G^-). \tag{1.9}$$

Let $\mathcal{N}(x;x')$ be the Neumann function associated with the problem (1.7) (cf. [14]). Namely, the unique solution v of (1.7) under the condition $\int_{\partial\Omega} v dS_x = 0$ is given by

$$v(x) = \int_\Omega \mathcal{N}(x;x')f(x')dx' + \int_{\partial\Omega} \mathcal{N}(x;x')g(x')dS_{x'} \tag{1.10}$$

when (1.8) is satisfied. Therefore our problem (1.6) is recast as

$$\mathbf{V}_{\Gamma(t)}(x) = -\mathcal{H}(x;\Gamma(t)) + c_* \int_\Omega \mathcal{N}(x;x')P(x';\Gamma(t))dx' + k. \tag{1.11}$$

Note that the right hand side of (1.11) is well-defined only when the condition (1.9) is satisfied. The constant k on the right hand side of (1.11) is to be determined in such a way that the interfacr $\Gamma(t)$ keeps the condition (1.9) satisfied. If the initial interface satisfies (1.9) with $t = 0$, then $\Gamma(t)$ continues to satisfy it if and only if

$$\begin{aligned} 0 = \frac{d}{dt}|\Omega^-_{\Gamma(t)}| &= \int_{\Gamma(t)} \mathbf{V}_{\Gamma(t)}(x)dS_x^{\Gamma(t)} \\ &= |\Gamma(t)|k - \int_{\Gamma(t)} \mathcal{H}(x;\Gamma(t))dS_x^{\Gamma(t)} \\ &\quad + c_* \int_{\Gamma(t)}\int_\Omega \mathcal{N}(x;x')P(x';\Gamma(t))dx'dS_x^{\Gamma(t)}, \end{aligned}$$

where $|\Gamma(t)|$ is the $N-1$ dimensional Lebesgue measure of $\Gamma(t)$ and $dS_x^{\Gamma(t)}$ is the surface element on $\Gamma(t)$ at $x \in \Gamma(t)$. Therefore the constant k is uniquely determined, and hence so is $\mathcal{S}$ in (1.2) for our problem (1.6). The problem (1.6) is now recast as

$$\begin{aligned}(1.12)\qquad \mathbf{V}_{\Gamma(t)}(x) &= -\mathcal{H}(x;\Gamma(t)) + \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} \mathcal{H}(x;\Gamma(t))dS_x^{\Gamma(t)} \\ &+ c_\times \int_\Omega \mathcal{N}(x;x')P(x';\Gamma(t))dx' \\ &- \frac{c_\times}{|\Gamma(t)|}\int_{\Gamma(t)}\int_\Omega \mathcal{N}(x;x')P(x';\Gamma(t))dx'dS_x^{\Gamma(t)}.\end{aligned}$$

On the right hand side of (1.12), the first line is the averaged mean curvature, while the second and third lines express nonlocal effects.

The main result of this article is:

THEOREM 1. *Let Γ_0 be an initial interface of class $C^{2+\alpha}$ $(0 < \alpha < 1)$. Suppose that the condition* (1.9) *is satisfied with $t = 0$. Then the interface equation* (1.12) *has a unique solution of class $C^{2+\alpha,1+\alpha/2}$ on a time interval $[0,T]$ for some $T > 0$.*

THEOREM 1 will be proven in §2, after trasforming the equation (1.12) into a quasilinear parabolic equation. In §3, dynamics of spherical interfaces governed by (1.12) will be discussed. In particular, we will study stability properties of equilibrium interfaces and their possible bifurcations.

2. PROOF

To be more precise in the regularity statement of THEOREM 1, and to prepare the stage for the proof, let us first introduce some geometric tools.

2.1. **Geometric Preliminaries.** Let $\mathcal{M}$ be a manifold of class C^∞ which is $C^{2+\alpha}$-diffeomorphic to the initial interface Γ_0. We choose and fix a C^∞ embedding $\varphi : \mathcal{M} \longmapsto \Omega$ and denote by y a generic point on $\mathcal{M}$ (cf. Figure 2). Let $\nu(y) \in \mathbb{R}^N$ be the unit normal vector of $\varphi(\mathcal{M})$ at $\varphi(y)$, pointing into $\Omega^+_{\varphi(\mathcal{M})}$, where

$$\Omega = \Omega^-_{\varphi(\mathcal{M})} \cup \varphi(\mathcal{M}) \cup \Omega^+_{\varphi(\mathcal{M})}$$

is a decomposition as in (1.1). We also choose the map φ so that $\Omega^\pm_{\varphi(\mathcal{M})}$ satisfy the volume constraint (1.9). For sufficiently small $r_0 > 0$, the map

$$(2.1)\qquad \Phi : \mathcal{M} \times (-r_0, r_0) \to \Omega, \qquad \Phi(y,r) = \varphi(y) + r\nu(y)$$

is a C^∞ embedding. Let $U_{r_0} = \Phi(\mathcal{M}, (-r_0, r_0))$. We denote by $(Y(x), R(x)) \in \mathcal{M} \times (-r_0, r_0)$ the inverse of the embedding Φ. Namely, it holds that

$$x = \varphi(Y(x)) + R(x)\nu(Y(x)) \qquad \text{for } x \in U_{r_0}.$$

By choosing the embedding φ so that $\varphi(\mathcal{M})$ is sufficiently $C^{2+\alpha}$-close to the initial interface Γ_0, one can express $\Gamma(t)$ as the graph of a function $A(y,t)$ on $\varphi(\mathcal{M})$. That is to say, $\Gamma(t)$ is parametrically expressed as

$$\Gamma(t) = \{x = \varphi(y) + A(y,t)\nu(y) \mid y \in \mathcal{M}\}.$$

We will show that the equation (1.12) is written as

$$\text{(2.2)} \qquad \begin{cases} \dfrac{\partial A}{\partial t} = \mathcal{F}(y, A, \nabla A, D^2 A) & (y \in \mathcal{M},\ t > 0) \\ \\ A(y, 0) = A_0(y) & (y \in \mathcal{M}), \end{cases}$$

where $\mathcal{F}(y, A, \nabla A, D^2 A)$ is a *quasi-linear elliptic* operator. The form of $\mathcal{F}$ is rather involved. The complication comes from the expresseion of the mean curvature $\mathcal{H}(\cdot; \Gamma_A)$ in terms of A, where

$$\Gamma_A = \{x = \varphi(y) + A(y)\nu(y) \mid y \in \mathcal{M}\}.$$

Let $g_{ij}(r)$ be the Riemannian metric tensor on $\mathcal{M}$ induced from the Euclidian one on U_{r_0} by the embedding $\Phi(\cdot, r)$. Namely,

$$g_{ij}(r)(y) = \left\langle \frac{\partial \Phi(y,r)}{\partial y_i}, \frac{\partial \Phi(y,r)}{\partial y_j} \right\rangle.$$

We also use below the symbols in the following list.

(L1) $(g^{ij}(r)) = (g_{ij}(r))^{-1}$: contravariant metric tensor.

(L2) $\kappa_j(y)$: the principal curvatures of $\varphi(\mathcal{M}) \subset \Omega$ at $x = \varphi(y)$ $(j = 1, \ldots, N-1)$.

(L3) dV_y^r : the volume element on $\mathcal{M}$ induced from the surface element on $\Phi(\mathcal{M}, r)$ at $x = \Phi(y, r)$ by the embedding $\Phi(\cdot, r)$. Then we have the following expressions.

$$dV_y^r = \Pi_{j=1}^{N-1}(1 + r\kappa_j(y)) dV_y^0$$

$$\Pi_{j=1}^{N-1}(1 + r\kappa_j(y)) =: \sum_{j=0}^{N-1} r^j H_j(y)$$

with $H_0 = 1$ and $H_1(y) = \mathcal{H}(y; \varphi(\mathcal{M}))$. In a local coordinate system y on $\mathcal{M}$,

$$dv_y^0 = \sqrt{\det(g_{ij}(0)(y))}\, dy.$$

(L4) $dV_y^{\Gamma_A}$: the volume element on $\mathcal{M}$ at y induced from the surface element on Γ_A at $x = \varphi(y) + A(y)\nu(y)$ by the embedding

$$\mathcal{M} \ni y \mapsto \varphi(y) + A(y)\nu(y).$$

Note that $dV_y^{\Gamma_A}$ is different from dV_y^r with $r = A(y)$.

(L5) Define

$$\mathcal{D}(y, A, \nabla A) := \sqrt{1 + \sum_{i,j=1}^{N-1} g^{ij}(A(y)) \frac{\partial A}{\partial y_i} \frac{\partial A}{\partial y_j}},$$

then we have

$$
\begin{aligned}
dV_y^{\Gamma_A} &= \mathcal{D}(y, A, \nabla A) dV_y^{A(y)}, \\
|\Gamma_A| &= \int_{\mathcal{M}} dV_y^{\Gamma_A} = \int_{\mathcal{M}} \mathcal{D}(y, A, \nabla A) dV_y^{A(y)} \\
&= \sum_{j=0}^{N-1} \int_{\mathcal{M}} \mathcal{D}(y, A, \nabla A)(A(y))^j H_j(y) dV_y^0.
\end{aligned}
$$

(L6) $\Delta^{\Phi(\cdot,r)}$: the Laplace-Beltrami operator on $\mathcal{M}$ induced from that on $\Phi(\mathcal{M}, r) \subset \Omega$.

(L7) $T^k(y,r) = \mathrm{Hess}(Y^k(x))|_{x=\Phi(y,r)} \quad (k = 1, \dots, N-1)$.

(L8) $\eta^k(y,r) = \nabla_x Y^k(x)|_{x=\Phi(y,r)} \quad (k = 1, \dots, N-1)$.

(L9) $S(y,r) = \mathrm{Hess}(R(x))|_{x=\Phi(y,r)}$.

(L10) $H(y,r) := \sum_{j=1}^{N-1} \frac{\kappa_j(y)}{1+r\kappa_j(y)}$: the sum of principal curvatures of $\Phi(\mathcal{M}; r)$ at $x = \Phi(y,r)$.

With these geometric tools at our disposal, we can now give the precise form of the operator $\mathcal{F}(y, A, \nabla A, D^2 A)$ as follows.

(2.3) $\mathcal{F}(y, A, \nabla A, D^2 A)$

$$
\begin{aligned}
&= \mathcal{A}(y, A, \nabla A, D^2 A) - \frac{\mathcal{D}(y, A, \nabla A)}{|\Gamma_A|} \int_{\mathcal{M}} \frac{\mathcal{A}(y', A(y'), \nabla A(y'), D^2 A(y'))}{\mathcal{D}(y', A(y'), \nabla A(y'))} dV_{y'}^{\Gamma_A} \\
&+ c_\times \mathcal{D}(y, A, \nabla A) \int_{\Omega} \mathcal{N}(\Phi(y, A(y)); x') P(x'; \Gamma_A) dx' \\
&- \frac{c_\times \mathcal{D}(y, A(y), \nabla A(y))}{|\Gamma_A|} \int_{\mathcal{M}} \int_{\Omega} \mathcal{N}(\Phi(y', A(y')); x') P(x'; \Gamma_A) dx' dV_{y'}^{\Gamma_A},
\end{aligned}
$$

with $\mathcal{A}(y, A, \nabla A, D^2 A)$ being given by:

$$
\begin{aligned}
\text{(2.4)} \qquad \mathcal{A}(y, A, \nabla A, D^2 A) &= \Delta^{\Phi(\cdot, A(y))} A(y) - H(y, A(y)) \\
&+ \mathcal{D}(y, A, \nabla A)^{-2} \Big[- \sum_{i,j=1}^{N-1} \Big(\sum_{k,l=1}^{N-1} g^{ik}(A) g^{jl}(A) \frac{\partial A}{\partial y_k} \frac{\partial A}{\partial y_l} \Big) \frac{\partial^2 A}{\partial y_i \partial y_j} \\
&- \sum_{i,j=1}^{N-1} {}^T\eta^i S \eta^j \frac{\partial A}{\partial y_i} \frac{\partial A}{\partial y_j} + \sum_{i,j,l=1}^{N-1} {}^T\eta^i T^l \eta^j \frac{\partial A}{\partial y_i} \frac{\partial A}{\partial y_l} \frac{\partial A}{\partial y_j} \Big],
\end{aligned}
$$

where η^i, η^j, T^l and S are evaluated at $(y, r) = (y, A(y))$. The expression in (2.3) is obtained from (1.12) by expressing $\mathbf{V}_\Gamma(x)$ and $\mathcal{H}(x;\Gamma)$ in terms of A as follows [6]:

$$\mathbf{V}_{\Gamma(t)}(x) = \frac{A_t(y,t)}{\mathcal{D}(y, A(y,t), \nabla A(y,t))},$$

$$\mathcal{H}(x;\Gamma(t)) = -\frac{\mathcal{A}(y, A(y,t), \nabla A(y,t), D^2 A(y,t))}{\mathcal{D}(y, A(y,t), \nabla A(y,t))}$$

with $x = \Phi(y, A(y))$.

Note that the elliptic operator $\mathcal{F}$ is not necessarily defined for an arbitrary function A. There is a constraint to be imposed on A, which comes from the condition (1.9):

$$0 = \int_{\Omega^-_{\Gamma_A}} dx - \int_{\Omega^-_{\Gamma_0}} dx = \int_{\mathcal{M}} \int_0^{A(y)} \Pi_{j=1}^{N-1}(1 + r\kappa_j(y))dr dV_y^0$$
$$= \int_{\mathcal{M}} \sum_{j=0}^{N-1} \frac{(A(y))^{j+1}}{j+1} H_j(y) dV_y^0.$$

We emphasize this fact:
The non-local elliptic operator $\mathcal{F}(y, A, \nabla A, D^2 A)$ is defined only for those A that satisfy

$$\mathcal{C}(A) = 0, \qquad \mathcal{C}(A) := \int_{\mathcal{M}} \sum_{j=0}^{N-1} \frac{(A(y))^{j+1}}{j+1} H_j(y) dV_y^0. \tag{2.5}$$

Then the differential equation (2.2) has to be such that the semi-flow generated by it preserves the constraint (2.5). The next proposition guarantees this property.

PROPOSITION 2. *The identities*

$$\text{(i)} \qquad \partial_A \mathcal{C}(A)\langle\tilde{A}\rangle = \int_{\mathcal{M}} \tilde{A}(y) dV_y^{A(y)},$$
$$\text{(ii)} \qquad \partial_A \mathcal{C}(A)\langle\mathcal{F}(\cdot, A, \nabla A, D^2 A)\rangle = 0$$

hold true, where ∂_A stands for the derivative with respect to A.

Proof. (i) By using the definition in (2.5), we have

$$\partial_A \mathcal{C}(A)\langle\tilde{A}\rangle = \int_{\mathcal{M}} \sum_{j=0}^{N-1} \tilde{A}(y)(A(y))^j H_j(y) dV_y^0$$
$$= \int_{\mathcal{M}} \tilde{A}(y) dV_y^{A(y)}$$

in which the identities in item (L3) above are used to obtain the second equality.

(ii) From (i) we have

$$\partial_A \mathcal{C}(A)\langle \mathcal{F}(\cdot, A, \nabla A, D^2 A)\rangle = \int_{\mathcal{M}} \mathcal{F}(y, A(y), \nabla A(y), D^2 A(y)) dV_y^{A(y)}$$

$$= \int_{\mathcal{M}} \frac{1}{\mathcal{D}(y, A, \nabla A)} \mathcal{F}(y, A(y), \nabla A(y), D^2 A(y)) dV_y^{\Gamma_A} = 0,$$

where the first identity in item (L5) is used to obtain the second equality, and the expression in (2.3) to obtain the third one. □

PROPOSITION 2-(ii) says that the differential equation (2.2) is compatible with the constraint (2.5). One can now rephrase THEOREM 1 as follows.

THEOREM 3. *Let $A_0 \in C^{2+\alpha}(\mathcal{M})$ satisfy* (2.5) *and such that $|A_0|_{C^{2+\alpha}(\mathcal{M})}$ is sufficiently small. Then there exist $T > 0$ and a unique solution $A(y,t)$ of* (2.2) *on* $[0, T]$ *such that*

$$A \in C^{2+\alpha, 1+\alpha/2}(\mathcal{M} \times [0, T]).$$

From the arguments above, we find that it suffices to prove THEOREM 3 in order to establish THEOREM 1.

2.2. **Proof of Theorem 3.** We now proceed to the proof of THEOREM 3. It consists of two steps.

S1 : To reduce the problem (2.2) with the non-linear constraint to a problem with a linear constraint.

S2 : To apply the general results on quasi-linear (or fully nonlinear) parabolic equations presented in [15] to the problem with the linear constraint in Step 1.

Step **S1**: Let $\Delta^{\Gamma_{A_0}}$ be the Laplace-Beltrami operator on Γ_{A_0}, or more precisely, it is the Laplacian on $\mathcal{M}$ induced from the Euclidian one by the embedding $\Phi(\cdot, A_0(\cdot))$. The null space of $\Delta^{\Gamma_{A_0}}$ is the set of constant functions. The space $X := C^{2+\alpha}(\mathcal{M})$ is decomposed as the direct sum

$$X = ker(\Delta^{\Gamma_{A_0}}) \oplus ker(\Delta^{\Gamma_{A_0}})^{\perp}, \tag{2.6}$$

where the superscript $\perp$ stands for the orthogonal complement. We denote by $\mathbf{M}$ the set of functions A that satisfy (2.5):

$$\mathbf{M} = \{A \in X \mid \mathcal{C}(A) = 0\}.$$

Since $ker(\Delta^{\Gamma_{A_0}})$ is not contained in the tangent space $T_{A_0}\mathbf{M}$ of $\mathbf{M}$ at A_0, one can express points A on $\mathbf{M}$ near A_0 uniquely as a sum

$$A = A_0 + B + p(B)$$

where $B \in ker(\Delta^{\Gamma_{A_0}})^{\perp}$, $p(B) \in ker(\Delta^{\Gamma_{A_0}})$ and

$$p : ker(\Delta^{\Gamma_{A_0}})^{\perp} \to ker(\Delta^{\Gamma_{A_0}})$$

is an analytic function satisfying $p(0) = 0$. The analyticity of the function p comes from that of $\mathcal{C}$ in (2.5). The problem (2.2) is now recast as:

$$
(2.7) \qquad \begin{cases} B_t &= \Delta^{\Gamma_{A_0}} B + Q\mathcal{G}(y, B) \quad y \in \mathcal{M}, \quad t > 0, \\ & B(y,0) = 0 \qquad\qquad\quad y \in \mathcal{M}, \\ p(B)_t &= (I - Q)\mathcal{G}(\cdot, B) \qquad\quad t > 0, \end{cases}
$$

with $\mathcal{G}(y, B)$ being defined by

$$
\mathcal{G}(y, B) := \mathcal{F}(y, A_0 + B + p(B), \nabla(A_0 + B), D^2(A_0 + B)) - \Delta^{\Gamma_{A_0}} B,
$$

where $(I - Q)B = 1/|\Gamma_0| \int_{\mathcal{M}} B(y) dV_y^{\Gamma_{A_0}}$ is the projection onto $ker(\Delta^{\Gamma_{A_0}})$ associated with the decomposition (2.6). Note that the equation for $p(B)$ in (2.7) is not a differential equation. Once B is found, $p(B)$ is automatically determined.

Step **S2**: It is now easy to prove the local in time existence of solutions to (2.7) by exploiting the general results by Lunardi [15], Chapter 8, Section 5.3. The idea is to use the contraction mapping principle. For each $B \in Y$ with

$$
Y := C^{2+\alpha, 1+\alpha/2}(\mathcal{M} \times [0, T]) \cap C([0, T], range(\Delta^{\Gamma_{A_0}})),
$$

let $\tilde{B} = \Lambda(B)$ be the solution of

$$
\begin{cases} \tilde{B}_t &= \Delta^{\Gamma_{A_0}} \tilde{B} + Q\mathcal{G}(y, B) \quad y \in \mathcal{M}, \quad t > 0, \\ & \tilde{B}(y, 0) = 0 \qquad\qquad\quad y \in \mathcal{M}. \end{cases}
$$

Lunardi then proves that for sufficiently small $T > 0$, $\Lambda : Y \to Y$ is a contraction mapping in a neighborhood of $0 \in Y$. Although the presentation in [15] is done for equations with local terms, the arguments there work equally well for equations with nonlocal lower order terms. Since nonlocal effects appear only as lower order terms in our problem, the proof by Lunardi applies without any modification. This completes the proof of THEOREM 3.

3. Dynamics of Spherical Interfaces

It is not easy to see how the solutions of (1.12) evolve. In this section, we describe the dynamics of spherical interfaces driven by (1.12), when the domain Ω is the unit ball in $\mathbb{R}^N$.

Let $\Gamma(t)$ be the union of k concentric spheres:

$$
\Gamma(t) = \Gamma_1(t) \cup \ldots \cup \Gamma_k(t)
$$

with $\Gamma_j(t) = \{x \in \mathbb{R}^N \mid |x| < \rho_j(t)\}$ $(0 < \rho_1 < \ldots < \rho_k < 1)$.

THEOREM 4. *We assume, for the sake of presentation, that the interior of* $\Gamma_1(t)$ *belongs to* $\Omega^-_{\Gamma(t)}$. *Then the interface equation* (1.12) *is recast as:*

$$(-1)^{j+1}\dot\rho_j = (-1)^j\frac{N-1}{\rho_j} - (N-1)\frac{\sum_{i=1}^k(-1)^i\rho_i^{N-2}}{\sum_{i=1}^k\rho_i^{N-1}} + c_*\left(v(\rho_j) - \frac{\sum_{i=1}^k\rho_i^{N-1}v(\rho_i)}{\sum_{i=1}^k\rho_i^{N-1}}\right), \tag{3.1}$$

in which $v(\rho_j)$ $(j=1,\dots,k)$ *are given by*

$$(\textit{for }\ j=1)\qquad v(\rho_1) = -\frac{G^-}{2DN}\rho_1^2$$

$$(\textit{for }\ j\ge 2)\qquad v(\rho_j) = \left(\frac{1}{2-N}-\frac{1}{2}\right)\frac{[G]}{DN}\sum_{l=1}^{j-1}(-1)^l\rho_l^2 - \rho_j^2\left[\frac{1}{2-N}\frac{[G]}{DN}\sum_{l=1}^{j-1}(-1)^l\left(\frac{\rho_l}{\rho_j}\right)^N + \frac{G_{j-1}}{2DN}\right]\qquad (\textit{if }\ N\ge 3),$$

or

$$= \frac{[G]}{2D}\sum_{l=1}^{j-1}(-1)^l\rho_l^2\left(\log\rho_l - \frac{1}{2}\right) - \rho_j^2\left[\frac{[G]}{2D}\sum_{l=1}^{j-1}(-1)^l\left(\frac{\rho_l}{\rho_j}\right)^2\log\rho_l + \frac{G_{j-1}}{4D}\right]\qquad (\textit{if }\ N=2),$$

where

$$G_j = \begin{cases} G^- & (j:\ \textit{even}) \\ G^+ & (j:\ \textit{odd}). \end{cases}$$

Proof. Since the normal velocity of $\Gamma_j(t)$ is given by $(-1)^{j+1}\dot\rho_j$ and its curvature by $(-1)^j\frac{N-1}{\rho_j}$, the interface equation (1.12) gives rise to the system of ordinary differential equations (3.1) in which $v(r)$ is the solution of the problem (3.2) below. Note that (3.1) does not change even if we add an arbitrary constant to $v(r)$. So, let us normalize it by $v(0)=0$. Let us use the symbols $\rho_0 = 0, \rho_{k+1} = 1$. Then $v(r)$ is the solution of

$$\begin{cases} -(r^{N-1}v_r)_r = r^{N-1}G_j/D & (\rho_j < r < \rho_{j+1}), \\ v_r(0) = 0 = v_r(1), \\ v(\cdot) \in C^1([0,1]). \end{cases} \tag{3.2}$$

In order for (3.2) to have a solution, the volume constraint (1.9) has to be fulfilled;

$$\frac{G^+}{[G]} = \frac{|\Omega^-_{\Gamma(t)}|}{|\Omega|} = \begin{cases} -\sum_{j=1}^{k}(-1)^j\rho_j^N & (k:\text{odd}) \\ -\sum_{j=1}^{k}(-1)^j\rho_j^N + 1 & (k:\text{even}). \end{cases} \tag{3.3}$$

Therefore, in the system (3.1) there are only $k-1$ independent equations. For each $j = 0, 1, \ldots, k$, the equation in (3.2) is easily integrated as

$$r^{N-1}v_r(r) - \rho_j^{N-1}v_r(\rho_j) = -\frac{G_j}{DN}\left(r^n - \rho_j^N\right) \qquad (\rho_j \le r \le \rho_{j+1}) \tag{3.4}$$

which gives rise to

$$\rho_{j+1}^{N-1}\left[v_r(\rho_{j+1}) + \frac{G_{j+1}}{DN}\rho_{j+1}\right] - \rho_j^{N-1}\left[v_r(\rho_j) + \frac{G_j}{DN}\rho_j\right] = \frac{G_{j+1}-G_j}{DN}\rho_{j+1}^N. \tag{3.5}$$

Adding the equations (3.5) on both sides, using $v_r(0) = 0 = v_r(1)$, one obtains

$$G_{k+1} = [G]\sum_{j=0}^{k}(-1)^j\rho_{j+1}^N = -[G]\Big(\sum_{j=1}^{k}(-1)^j\rho_j^N + (-1)^{k+1}\Big),$$

which is nothing but (3.3). Moreover, if the equations in (3.5) are added up to $j-1$, then one also obtains

$$\rho_j^{N-1}\left[v_r(\rho_j) + \frac{G_j}{DN}\rho_j\right] = -\frac{[G]}{DN}\sum_{l=1}^{j}(-1)^l\rho_l^N =: d_j, \tag{3.6}$$

which also gives an explicit formula for $v_r(\rho_j)$:

$$v_r(\rho_j) = -\frac{G_j}{DN}\rho_j + \frac{d_j}{\rho_j^{N-1}}. \tag{3.7}$$

Integrating (3.4) once more, we obtain the values $v(\rho_j)$.

$$v(\rho_1) - v(0) = -\frac{G_0}{2DN}\rho_1^2 \tag{3.8}$$

$$\begin{aligned} v(\rho_j) - v(\rho_{j-1}) =& d_{j-1}\frac{\rho_j^{2-N} - \rho_{j-1}^{2-N}}{2-N} - \frac{G_{j-1}}{2DN}(\rho_j^2 - \rho_{j-1}^2) \qquad (N \ge 3), \\ =& d_{j-1}\log\Big(\frac{\rho_j}{\rho_{j-1}}\Big) - \frac{G_{j-1}}{2DN}(\rho_j^2 - \rho_{j-1}^2) \qquad (N = 2), \\ & (j = 2, \ldots, k+1). \end{aligned} \tag{3.9}$$

Since $v(0) = 0$ is known, one can easily determine the values $v(\rho_j)$ successively for $j = 1, \ldots, k$, which gives the expressions in THEOREM 4. This completes the proof. □

3.1. **Motion of Spherical Interface.** We will now investigate more closely the dynamics of spherical interfaces governed by (3.1) for $k = 1, 2$.

$\boxed{k=1}$: In this case there is no dynamics. In fact (3.1) reduces to $\dot\rho_1 \equiv 0$. So, the single spherical interface is in equilibrium state and it is uniquely given by

$$\rho_1 = \left(\frac{G^+}{[G]}\right)^{1/N} =: E,$$

which comes from the volume constraint (3.3). We also record here

$$v(E) = \frac{[G]}{2ND}(1 - E^N)E^2, \qquad v_r(E) = \frac{[G]}{ND}(1 - E^N)E,$$

which will be used later in stability analysis. The stability property of the equilibrium $\rho_1 = E$ relative to non-radial perturbations will be discussed in §3.2 below. Note that there is no room for radial perturbations in the present situation, again, due to the volume constraint (3.3).

$\boxed{k=2}$: Let us first deal with the case where $N \geq 3$.

The equation for ρ_1 is

$$\dot\rho_1 = -(N-1)\left(\frac{1}{\rho_1} + \frac{\rho_2^{N-2} - \rho_1^{N-2}}{\rho_1^{N-1} + \rho_2^{N-1}}\right) + c_* \frac{(v(\rho_1) - v(\rho_2))\rho_2^{N-1}}{\rho_1^{N-1} + \rho_2^{N-1}}, \tag{3.10}$$

in which ρ_2 is dictated by ρ_1 as follows:

$$\rho_2^N = \rho_1^N + 1 - E^N. \tag{3.11}$$

Moreover, from (3.9) we have

$$\begin{aligned} v(\rho_2) - v(\rho_1) =& \frac{1}{DN}\left(\frac{[G]}{N-2} + \frac{G^+}{2}\right)\rho_1^2 - \frac{1}{DN}\frac{[G]}{N-2}\rho_1^N\left(\rho_1^N + 1 - E^N\right)^{\frac{2-N}{N}} \\ & - \frac{1}{DN}\frac{G^+}{2}\left(\rho_1^N + 1 - E^N\right)^{\frac{2}{N}}. \end{aligned} \tag{3.12}$$

Substituting (3.11)-(3.12) into (3.10), we obtain the differential equation for ρ_1, which we write as

$$\dot\rho_1 = h_N(\rho_1, D), \qquad 0 < \rho_1 < E. \tag{3.13}$$

One can verify the following facts by elementary calculation:

- The principal term of $h_N(\rho_1, D)$ as $\rho_1 \to 0$ is $-(N-1)/\rho_1$.
- $v(\rho_1) - v(\rho_2) > 0$ for ρ_1 near 0 and this is the only positive contribution to $h_N(x, D)$.

Based upon these observations, one can in fact show the existence of $D_N^* > 0$ such that (cf. Figure 3):

$$\begin{aligned}
&\text{for } D > D_N^*: && h_N(\rho_1, D) < 0 \qquad (0 < \rho_1 < E);\\
&\text{for } D = D_N^*: && h_N(\rho_1, D_N^*) = 0 \text{ has a unique solution} \rho_1^* \in (0, E);\\
&\text{for } D < D_N^*: && h_N(\rho_1, D) = 0 \text{ has two solutions } \rho_1^-(D) < \rho_1^+(D) < E \text{ such that}\\
& && h_N'(\rho_1^-(D), D) > 0 \text{ and } h_N'(\rho_1^+(D), D) < 0.
\end{aligned}$$

That is to say, the solution of (3.13) converges to 0 in finite time for $D > D_*$, giving rise to a single interface, and a saddle-node bifurcation takes place at $D = D_*$. The bifurcated equilibria $\rho_1^+(D)$ and $\rho_1^-(D)$ are, respectively, stable and unstable relative to radially symmetric perturbations. The stability property of $\rho_1^+(D)$, relative to non-radial perturbations, will be discussed in §3.2.

When $N = 2$, the same statements as above for $N \geq 3$ are still valid. The equation for ρ_1 in this case is given by

$$\dot{\rho}_1 = h_2(\rho_1, D) \tag{3.14}$$

with

$$h_2(\rho_1, D) = -\frac{1}{\rho_1} - \frac{c_*[G]}{4D} \frac{\sqrt{\rho_1^2 + 1 - E^2}\Big(\rho_1^2 \log(1 + \frac{1-E^2}{\rho_1^2}) - (1 - E^2)E^2\Big)}{\rho_1 + \sqrt{\rho_1^2 + 1 - E^2}}.$$

It is of interest to compare the dynamics of our equation with that of (Mor) in Example 4. As noted earlier, the semi-flow generated by (Mor) preserves the volume of $\Omega^-_{\Gamma(t)}$. According to the ratio $|\Omega^-_{\Gamma(0)}|/|\Omega^+_{\Gamma(0)}|$ of initial volumes, the first equation in (1.5) has different inhomogeneous terms. For the sake of comparison, let us choose the ratio to be equal to 1. Moreover, we replace ± 1 in the first equation of (1.5) by $\pm 1/D$ with $D > 0$ being a parameter. The equation of ρ_1 for (Mor) is then given by ($N \geq 3$)

$$\begin{aligned}
\dot{\rho}_1 = &-\frac{2\rho_1}{DN} - \frac{(N-1)(N-2)}{\rho_1^2} \frac{\rho_2^{N-3}(\rho_2 + \rho_1)}{\rho_2^{N-2} - \rho_1^{N-2}}\\
&+ \frac{(N-2)}{DN\rho_1} \frac{\rho_2^2 - \rho_1^2}{\rho_2^{N-2} - \rho_1^{N-2}} =: k_N(\rho_1, D)
\end{aligned} \tag{3.15}$$

where $\rho_2^N = \rho_1^N + 1 - E^N$.

The profile of $k_3(\rho_1, D)$ is depicted in Figure 4 for several values of D, which shows that the qualitative dynamics of spherical interface for (Mor) is similar to that of (1.12).

3.2. **Stability of Equilibria.** We consider the equilibrium problem associated with (1.6):

$$
(3.16)\qquad \begin{cases} c_* v(x;\Gamma) = \mathcal{H}(x;\Gamma), & (x \in \Gamma) \\ -\Delta v(x;\Gamma) = P(x;\Gamma) & (x \in \Omega\backslash\Gamma) \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 & (x \in \partial\Omega) \\ v(\cdot;\Gamma) \in C^2(\overline{\Omega}\backslash\Gamma) \cap C^1(\overline{\Omega}). \end{cases}
$$

The resolution of the problem (3.16) for a general domain Ω is much harder than that of (1.6). When Ω is the unit ball, one can easily obtain radially symmetric solutions of (3.16). Let us, however, start with a general setting.

Let us assume that (3.16) has a smooth solution (Γ, v). We choose the embedding $\varphi : \mathcal{M} \to \Omega$ so that $\varphi(\mathcal{M}) = \Gamma$. We now linearize (1.12) around $A = 0$ and consider the associated eigenvalue problem. It is given by

$$
\begin{aligned}
(3.17)\qquad \lambda A =& \Delta^\Gamma A + (\sum_{j=1}^{N-1} \kappa_j(y)^2)A - \frac{1}{|\Gamma|}\int_{\mathcal{M}} (\sum_{j=1}^{N-1} \kappa_j(y')^2)A(y')dV^0_{y'} \\
&+ c_*\left(\int_\Omega \frac{\partial \mathcal{N}}{\partial \nu_x}(\varphi(y); x')P(x';\Gamma)dx'\right)A \\
&\qquad - \frac{c_*}{|\Gamma|}\int_{\mathcal{M}}\left(\int_\Omega \frac{\partial \mathcal{N}}{\partial \nu_x}(\varphi(y'); x')P(x';\Gamma)dx'\right)A(y')dV^0_{y'} \\
&- c_*\frac{[G]}{D}\int_{\mathcal{M}} \mathcal{N}(\varphi(y);\varphi(y'))A(y')dV^0_{y'} \\
&\qquad + \frac{c_*}{|\Gamma|}\frac{[G]}{D}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{N}(\varphi(y);\varphi(y'))A(y')dV^0_{y'}dV^0_{y}.
\end{aligned}
$$

Note that the first line on the right hand side of (3.17) is the averaged Jacobi operator on Γ, which is the linearization of the first line on the right hand side of (1.12). When one tries to linearize $\int_\Omega \mathcal{N}(\varphi(y'); x')P(x';\Gamma)dx'$ in Γ, the jump discontinuity of $P(x';\Gamma)$ at Γ introduces a *delta function* supported on Γ, which is expressed in the fourth line of (3.17).

To determine the stability of the equilibrium solution (Γ, v), we will study the eigenvalues of (3.17). Before we treat the problem, let us first present a useful result related to one of the nonlocal terms in (3.17).

By using the Dirichlet-to-Neumann operators $\Pi^\pm_\Gamma$ associated with (1.4), let us define $\hat{\Pi}^\pm$ by

$$
\left(\hat{\Pi}^\pm A\right)(y) := \left(\Pi^\pm_\Gamma A \circ \varphi^{-1}\right)(\varphi(y)) \qquad A \in C^{2+\alpha}(\mathcal{M}).
$$

Proposition 5. *A right inverse of $\hat{\Pi}^- + \hat{\Pi}^+$ has an explicit expression:*

$$(\hat{\Pi}^- + \hat{\Pi}^+)^{-1}A(y) = \int_{\mathcal{M}} \mathcal{N}(\varphi(y);\varphi(y'))A(y')dV^0_{y'} + constant.$$

If we impose the normailzation

$$\int_{\mathcal{M}} (\hat{\Pi}^- + \hat{\Pi}^+)^{-1}A(y)dV^0_y = 0$$

on the right inverse, it is uniquely detemined as

$$\begin{aligned}\hat{\Pi}^{-1}A(y) :=&(\hat{\Pi}^- + \hat{\Pi}^+)^{-1}A(y) \\ =&\int_{\mathcal{M}} \mathcal{N}(\varphi(y);\varphi(y'))A(y')dV^0_{y'} \\ &-\frac{1}{|\Gamma|}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{N}(\varphi(y);\varphi(y'))A(y')dV^0_{y'}dV^0_y.\end{aligned} \tag{3.18}$$

Proof. The proof is essentially given in [8] (Chapter 3).

Let us define $u^{\pm}(x)$ by

$$\begin{aligned}u^{\pm}(x) &= \int_{\mathcal{M}} \mathcal{N}(x;\varphi(y'))A(y')dV^0_{y'}, \\ &= \int_{\Gamma} \mathcal{N}(x;x')A(\varphi^{-1}(x'))dS^{\Gamma}_{x'}, \qquad (x \in \Omega^{\pm}_{\Gamma}).\end{aligned}$$

Note that $u^-(x) \equiv u^+(x)$ for $x \in \Gamma$, where $u^{\pm}(x)$ for $x \in \Gamma$ is well-defined as the limit

$$u^{\pm}(x) = \lim_{x' \to x} u^{\pm}(x') \qquad (x' \in \Omega^{\pm}_{\Gamma}).$$

Arguing as in the proof of Theorem 3.28 in [8], we obtain for $x \in \Gamma$

$$\begin{aligned}\frac{\partial u^+(x)}{\partial \nu} &= -\frac{1}{2}A(\varphi^{-1}(x)) + \int_{\Gamma} \mathcal{K}(x';x)A(\varphi^{-1}(x))dS^{\Gamma}_{x'}, \\ \frac{\partial u^-(x)}{\partial \nu} &= \frac{1}{2}A(\varphi^{-1}(x)) + \int_{\Gamma} \mathcal{K}(x';x)A(\varphi^{-1}(x))dS^{\Gamma}_{x'},\end{aligned} \tag{3.19}$$

where

$$\mathcal{K}(x';x) := \frac{\partial}{\partial \nu_x}\mathcal{N}(x';x).$$

Note that our Neumann function plays the role of $-N(x,y)$ in Chapter 3 of [8], and this is why the sign of the first terms on the right hand side of (3.19) is opposite to that in Theorem 3.28 of [8]. Now, (3.19) immediately gives

$$A(\varphi^{-1}(x)) = \frac{\partial u^-(x)}{\partial \nu} - \frac{\partial u^+(x)}{\partial \nu} \qquad \text{on } \Gamma,$$

or equivalently

$$\Big((\Pi_\Gamma^- + \Pi_\Gamma^+)^{-1} A \circ \varphi^{-1}\Big)(x) = \int_{\mathcal{M}} \mathcal{N}(x;\varphi(y'))A(y')dV_{y'}^0 + \text{constant},$$

since identities in (3.19) are valid even if we replace $u^{\pm}(x)$ by $u^{\pm}(x)+$ constant. This completes the proof. □

In the sequel, we always use the normalized right inverse in (3.18).

The eigenvalue problem (3.17) is now rewritten as

$$\begin{aligned}(3.20)\qquad \lambda A =& \Delta^\Gamma A + \Big(\sum_{j=1}^{N-1} \kappa_j(y)^2\Big)A - \frac{1}{|\Gamma|}\int_{\mathcal{M}} \Big(\sum_{j=1}^{N-1} \kappa_j(y')^2\Big)A(y')dV_{y'}^0 \\ &+ \left(c_\times \int_\Omega \frac{\partial \mathcal{N}}{\partial \nu_x}(\varphi(y);x')P(x';\Gamma)dx'\right)A \\ &\qquad - \frac{c_\times}{|\Gamma|}\int_{\mathcal{M}}\left(\int_\Omega \frac{\partial \mathcal{N}}{\partial \nu_x}(\varphi(y');x')P(x';\Gamma)dx'\right)A(y')dV_{y'}^0 \\ &- c_\times \frac{[G]}{D}\hat{\Pi}^{-1}A(y).\end{aligned}$$

On the right side of (3.20), the first line is nothing but the *averaged Jacobi operator* on Γ. We consider the eigenvalue problem for (3.20), with Γ being the equilibrium sphere given in §3.1.

THEOREM 6 (**Eigenvalues for Single Interface**). *Let* $\Gamma = \{x \in \Omega \mid |x| = E\}$ *be the equilibrium interface given in* §3.1. *Then eigenfuctions of the associtaed eigenvalue problem* (3.20) *are all spherical harmonic functions of degree* $j \geq 1$. *Moreover, the corresponding eigenvalue is given by*

$$\begin{aligned}\lambda_j =& -\frac{(j-1)(j-1+N)}{E^2} \\ &+ \frac{c_\times [G]}{D}E\left[\frac{1-E^N}{N} - \frac{j+(j+N-2)E^{2j+N-2}}{j(2j+N-2)}\right].\end{aligned}$$

Proof. Since perturbation A has to satisfy (cf. PROPOSITION 2-(i) with $A \equiv 0, \tilde{A} = A$)

$$\int_{\mathcal{M}} A(y)dV_y^0 = 0,$$

and in the present situation the following identities hold true

$$\sum_{j=1}^{N-1} \kappa_j^2 = (N-1)/E^2, \qquad \int_\Omega \frac{\partial \mathcal{N}}{\partial \nu}(\varphi(y);x')P(x';\Gamma)dx' = v_r(E) = \frac{[G]}{ND}(1-E^N)E,$$

the eigenvalue problem (3.20) is reduced to

$$\lambda A = \Delta^\Gamma A + \frac{N-1}{E^2} A + \frac{c_*[G]}{D}\Big(\frac{(1-E^N)E}{N} - \hat{\Pi}^{-1}\Big) A.$$

The right hand side of the latter equation becomes a multiplication operator when A is spherically harmonic. By using the completeness of the sysmem of spherical harmonics of degree $j \geq 0$ ([8]), we conclude that the eigenfunctions are all spherical harmonic functions of some degree.

Moreover, one can easily verify that the eigenvalue of the Jacobi operator for spherical harmonics of degree $j \geq 1$ is $-(j-1)(j-1+N)/E^2$. By recalling the definition of $\hat{\Pi}^{-1}$, it is also verified by elementary computations that the eigenvalue of the inverse $\hat{\Pi}^{-1}$ associated with spherical harmonics of degree $j \geq 1$ is givn by

$$\frac{j + (j+N-2)E^{2j+N-2}}{j(2j+N-2)} E.$$

Combining these together, we complete the proof of the theorem. □

Remark. The result in THEOREM 6 is exactly the same as Theorem 2.2 in [24] for the eigenvalue of transition layer solutions for the reaction-diffusion system (RD). This indicates that the interface equation (1.12) does approximate the motion of internal layers for the reaction-diffusion system (RD).

THEOREM 6 also shows that as $D > 0$ gets smaller, λ_j crosses zero at $D = D_j > 0$ for all $j \geq j_0 > 1$. This indicates that a symmetry-breaking static bifurcation takes place at such values of D. Indeed, one can prove by using the equivariant bifurcation theory [10] that there exist *non-radial* solutions of (3.16). The latter statement has been proven valid in [24] for radially symmetric internal layer solutions of (RD). Applying the equivariant bifurcation theory of [10] as in [24], we have recently estabished in [23] that symmetry breaking bifurcations take place at $D = D_j$ for each $j \geq 1$.

REFERENCES

[1] N. Alikakos, P. W. Bates and X. Chen. Convergence of the Cahn-Hilliard Equation to the Hele-Shaw model. *Arch. Rational Mech. Anal.* **128**(1994), 164-205.

[2] L. Bronsard and B. Stoth, Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation, *SIAM J. Math. Anal.* **28**(1997), 769-807.

[3] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations. *J. Differential Equations* **96** (1992), 116-141.

[4] X. Chen, Global asymptotic limit of solutions of the Cahn-Hillard Equation. *J. Differential Geom.* **44** (1996), 262-311.

[5] X. Chen, J. Hong and F. Yi, Existence, uniqueness, and regularity of classical solutions of the Mullins-Sekerka problem, *Comm. Partial Diff. Equations* **21**(1996), 1705-1727.

[6] X. Chen and F. Reitich, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling, *J. Math. Anal. Appl.* **164**(1992), 350-362.

[7] J. Escher and G. Simonett, Classical solutions for Hele-Shaw models with surface tension, *Adv. Differential Equations* **2**(1997), 619-642.

[8] G. B. Folland, *Introduction to Partial Differential Equations* 2nd Ed., Princeton University Press, Princeton, New Jersey (1995).

[9] M. Gage and R. Hamilton, The heat equation shrinking convex plane curves, *J. Differential Geom.* **23**(1986), 69-96.

[10] M. Golubitsky, I. Stewart and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theorey*: Volume II, Springer-Verlag New York-Berlin-Heidelberg, 1988.

[11] M. Grayson, The heat equation shrinks embedded plane curves to round points, *J. Differential Geom.* **26**(1987), no. 2, 285–314.

[12] G. Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Equations* **20** (1984), 237-266.

[13] G. Huisken, The volume preserving mean curvature flow, *J. Reine Angew. Math.* **382** (1987), 35-48.

[14] S. Ito, *Diffusion Equations* (Kinokuniya-AMS, 1979).

[15] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser Basel-Boston-Berlin, 1995.

[16] Y. Nishiura and I. Ohnishi, Some mathematical aspects of the micro-phase separation in diblock copolymers, *Physica D* **84**(1995), no. 1-2, 31-39.

[17] Y. Nishiura and H. Suzuki, Nonexistence of higher-dimensional stable Turing patterns in the singular limit, *SIAM J. Math. Anal.* **29**(1998), no. 5, 1087-1105.

[18] I. Ohnishi and H. Imai, Modified Hele-Shaw moving boundary problem related to some phase transition phenomena, *Bull. Univ. Electro-Comm.* **11**(1998), no. 1, 17-28.

[19] I. Ohnishi, Y. Nishiura, M. Imai and Y. Matsushita, Analytical solutions describing the phase separation driven by a free energy functional containing a long-range interaction term, *Chaos* **9**(1999), no. 2, 329-341.

[20] R. L. Pego, Front migration in the nonlinear Cahn-Hilliard equation, *Proc. Roy. Soc. London Ser. A* **422**(1989), 261-278.

[21] K. Sakamoto; Asymptotic expansion of interface equation for a reaction-diffusion system, *Tohoku Math. Publications* **8**(1998), 149 -158.

[22] K. Sakamoto, Spatial homogenization and internal layers in a reaction-diffusion system, *Preprint*(1999), (to appear in *Hiroshima Math. J.* **30**).

[23] K. Sakamoto and T. Sogabe, Domain Perturbation and Bifurcation in Free Interface Problems, in preparation.

[24] K. Sakamoto and H. Suzuki, Symmetry Breaking from Radially Symmetric Internal Layers, *Preprint, 1999.*

[25] H. Suzuki, Asymptotic characterization of stationary interfacial patterns for reaction diffusion systems, *Hokkaido Math. J.* **26**(1997), no. 3, 631-667.

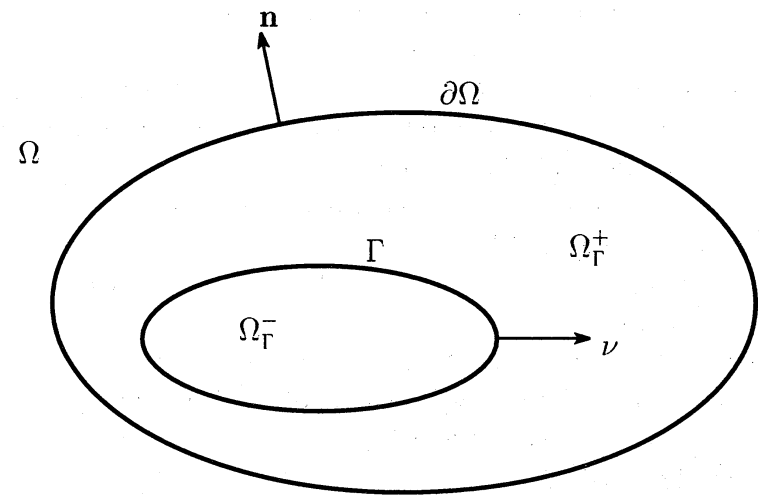

FIGURE 1. Γ divides Ω into two parts, Ω_Γ^- and Ω_Γ^+.

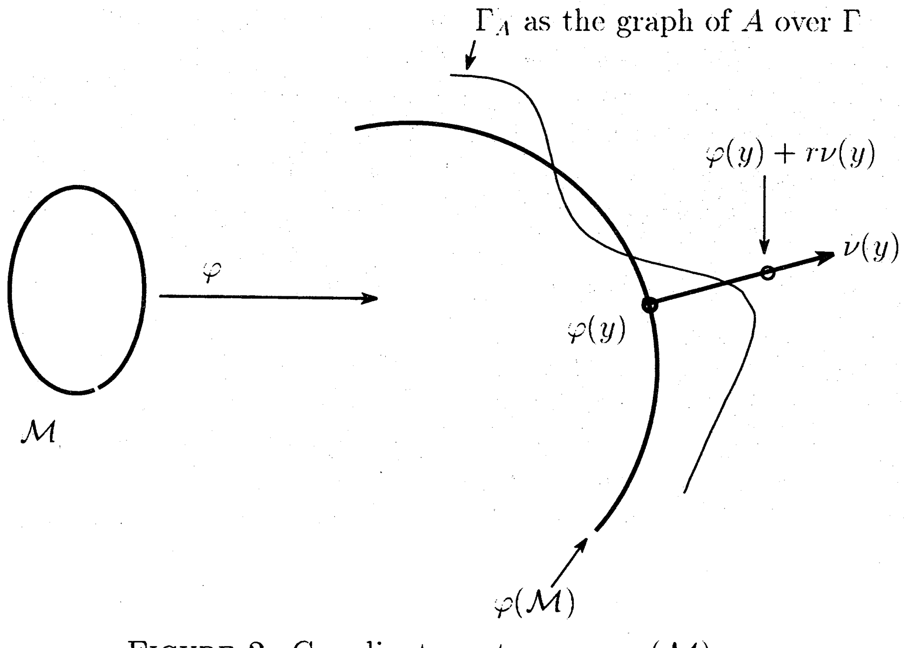

FIGURE 2. Coordinate system near $\varphi(\mathcal{M})$.

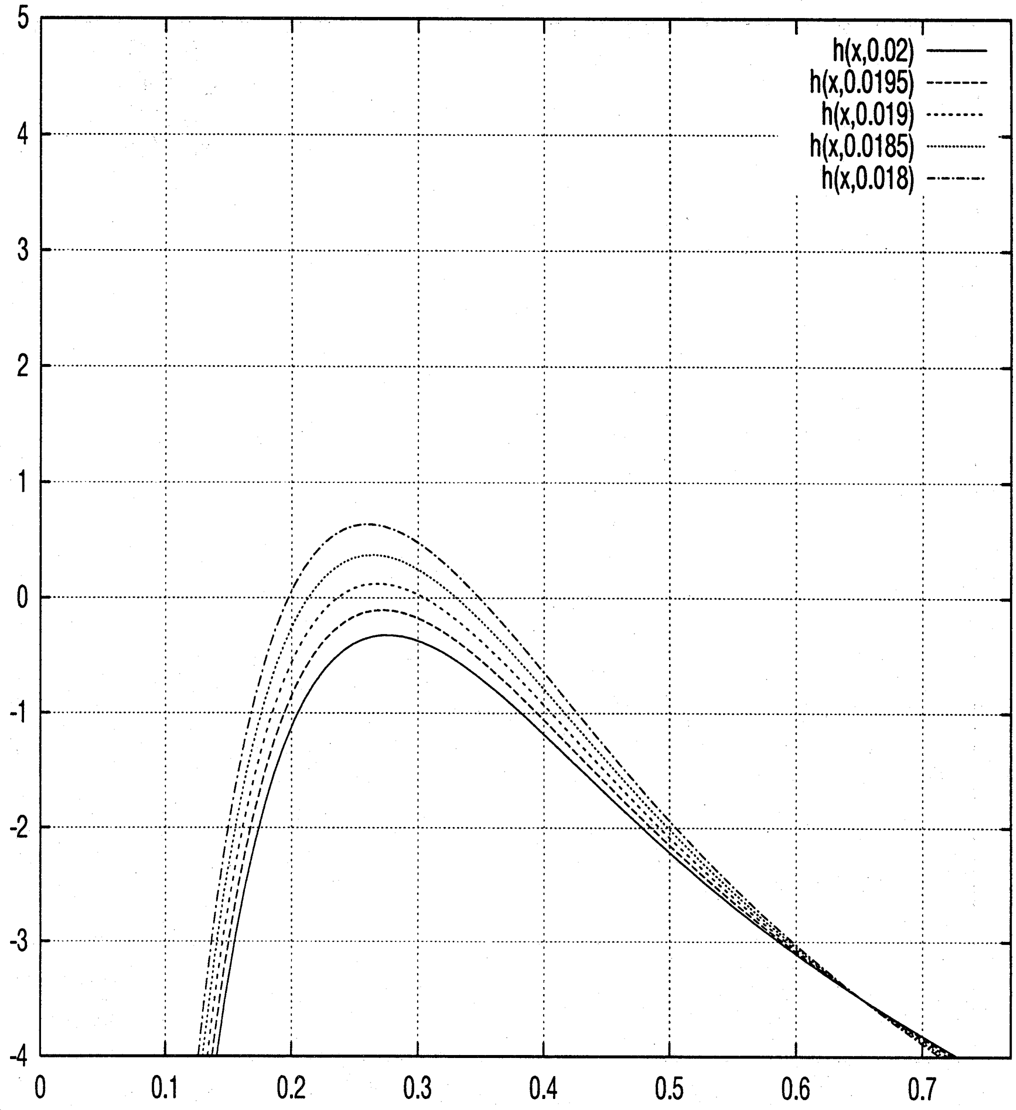

FIGURE 3. Profiles of $h_N(\rho_1, D)$ for $N = 3$, $G^{\pm} = \pm 1$, $c_* = 3/\sqrt{2}$ and for various values of D.

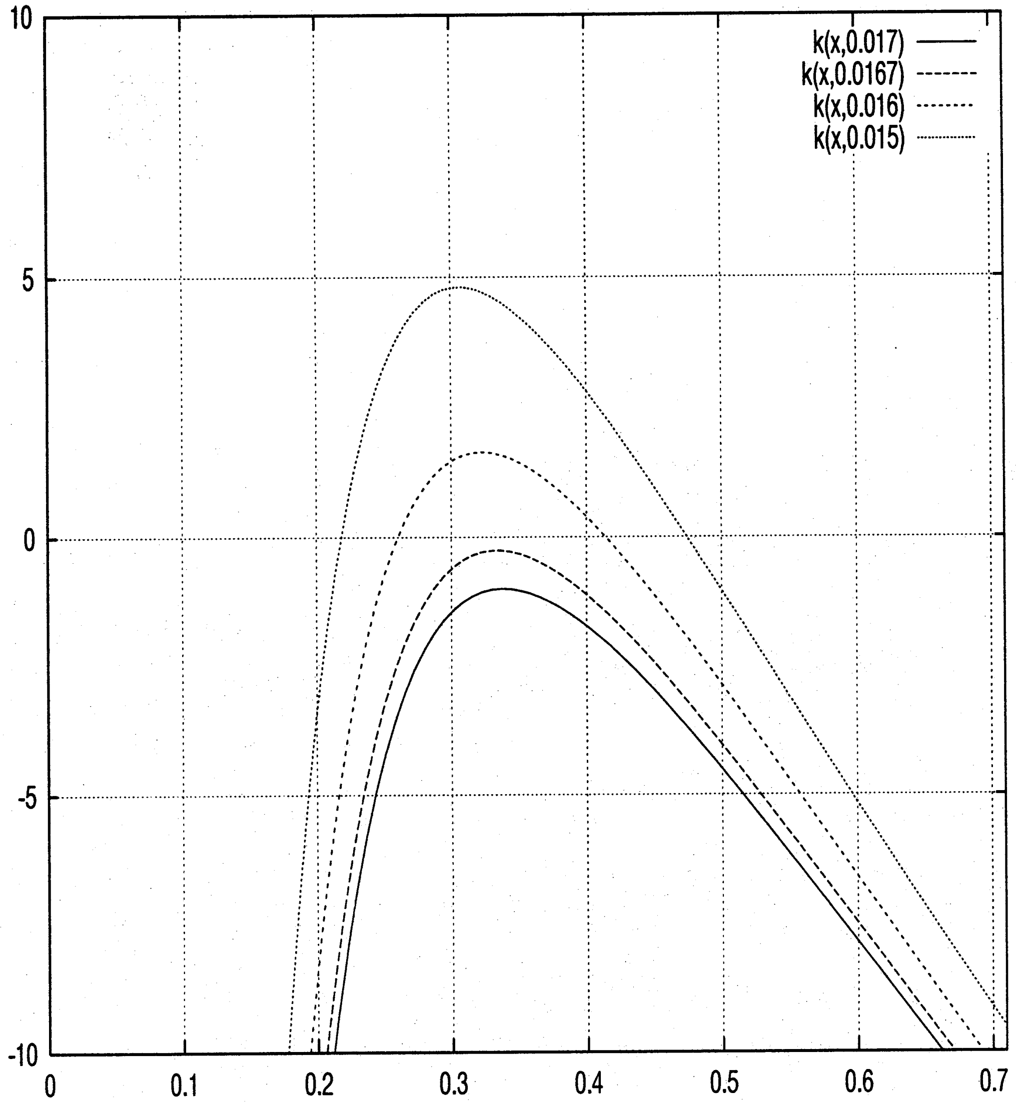

FIGURE 4. Profiles of $k_3(\rho_1, D)$ for $|\Omega_\Gamma^-|/|\Omega_\Gamma^+| = 1$ and for various values of D.