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A Singular Limit arising in Combustion Theory: Identification of the Limit

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This is an announcement of results to appear.

Let us consider the family of non-negative solutions for the initial-value problem

$$\partial_t u_\epsilon - \Delta u_\epsilon = -\beta_\epsilon(u_\epsilon) \text{ in } (0,\infty)\times\mathbf{R}^n\,,\ u_\epsilon(0,\cdot) = u_\epsilon^0 \text{ in } \mathbf{R}^n\,. \qquad (1)$$

Here $\epsilon \in (0,1)$, $\beta_\epsilon(z) = \frac{1}{\epsilon}\beta(\frac{z}{\epsilon})$, $\beta \in C_0^1([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$. We assume the initial data $(u_\epsilon^0)_{\epsilon\in(0,1)}$ to be bounded in $C^{0,1}(\mathbf{R}^n)$ and to satisfy $u_\epsilon^0 \to u^0$ in $H^{1,2}(\mathbf{R}^n)$ and $\bigcup_{\epsilon\in(0,1)} \operatorname{supp} u_\epsilon^0 \subset B_S(0)$ for some $S < \infty$.

Formally, each limit u with respect to a sequence $\epsilon_m \to 0$ will be a solution of the free boundary problem

$$\partial_t u - \Delta u = 0 \text{ in } \{u>0\}\cap(0,\infty)\times\mathbf{R}^n,\ |\nabla u| = 1 \text{ on } \partial\{u>0\}\cap(0,\infty)\times\mathbf{R}^n\,. \qquad (2)$$

The singular limit problem (1) has been derived as a model for the propagation of equidiffusional premixed flames with high activation energy ([4]); here $u = \lambda(T_c - T)$, T_c is the flame temperature, which is assumed to be constant, T is the temperature outside the flame and λ is a normalization

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factor.

Let us shortly summarize the mathematical results directly relevant in this context, beginning with the limit problem (2): In the excellent paper [1], H.W. Alt and L.A. Caffarelli proved via minimization of the energy $\int(|\nabla u|^2 + \chi_{\{u>0\}})$ – here $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$ – existence of a stationary solution of (2) in the sense of distributions. They also derived regularity of the free boundary $\partial\{u > 0\}$ up to a set of vanishing $n-1$-dimensional Hausdorff measure. The question of the existence of classical solutions in three dimensions stands still exposed. Existence would however follow by [13], once the non-existence of singular minimizing cones has been established. *Non-minimizing* singular cones *do* in fact appear for $n = 3$ (cf. [1, example 2.7]). Moreover it is known, that solutions of the Dirichlet problem in two space dimensions are not unique (cf. [1, example 2.6]).

For the time-dependent (2), a "trivial non-uniqueness" complicates the matter further, as the positive solution of the heat equation is always another solution of (2). Even for flawless initial data, classical solutions of (2) develop singularities after a finite time span; consider e.g. the example of two colliding traveling waves

$$\begin{aligned} u(t,x) &= \chi_{\{x+t>1\}}(\exp(x+t-1)-1) \\ &+ \chi_{\{-x+t>1\}}(\exp(-x+t-1)-1) \text{ for } t \in [0,1) . \end{aligned} \tag{3}$$

Let us now turn to results concerning the singular perturbation (1): For the stationary problem (1) H. Béréstycki, L.A. Caffarelli and L. Nirenberg obtained in [3] uniform estimates and – assuming the existence of a minimal solution – further results.

L.A. Caffarelli and J.L. Vazquez contributed in [8] among other things the corresponding uniform estimates for the time-dependent case and a convergence result: for initial data u^0 that is strictly mean concave in the interior of its support, a sequence of ϵ-solutions converges to a solution of (2) in the sense of distributions.

Let us finally mention several results on the corresponding two-phase problem, which are relevant as solutions of the one-phase problem are automatically solutions of the corresponding two-phase problem. In [6] and [7], L.A. Caffarelli, C. Lederman and N. Wolanski prove convergence to a sort of barrier solution in the case that $\{u = 0\}^\circ = \emptyset$. In [11], C. Lederman and N. Wolanski show convergence to a viscosity solution in the sense of [5] and derive regularity of the true two-phase part of the free boundary. These results deal quite well with the *true two-phase behavior* of limits, but have – as will become more plain in the examples below – to largely ignore the one-phase behavior. One of the reasons for this is that the limit cannot be expected to be close to a monotone function near free boundary points that are not true two-phase points.

Our result: As an intermediate result we obtain that each limit u of (1) is a solution *in the sense of domain variations*, i.e. u is smooth in $\{u > 0\}$ and satisfies

$$\int_0^\infty \int_{\mathbf{R}^n} [-2\partial_t u \nabla u \cdot \xi + |\nabla u|^2 \mathrm{div}\, \xi - 2\nabla u D\xi \nabla u] \;=\; -\int_0^\infty \int_{R(t)} \xi \cdot \nu \, d\mathcal{H}^{n-1}\, dt \tag{4}$$

for every $\xi \in C_0^{0,1}((0,\infty) \times \mathbf{R}^n; \mathbf{R}^n)$. Here

$$R(t) := \{x \in \partial\{u(t) > 0\} \;:\; \text{there is } \nu(t,x) \in \partial B_1(0) \text{ such that } u_r(s,y) = \frac{u(t + r^2 s, x + ry)}{r} \to \max(-y \cdot \nu(t,x), 0) \text{ locally uniformly in } (s,y) \in \mathbf{R}^{n+1} \text{ as } r \to 0\}$$

is for a.e. $t \in (0,\infty)$ a countably $n-1$-rectifiable subset of the free boundary. Let us remark that already this equation contains information (apart from the rectfiability of $R(t)$) that cannot be inferred from the viscosity notion of solution [11, Definition 4.3]: whereas any function of the form $\alpha \max(x_n, 0) + \beta \max(-x_n, 0)$ with $\alpha, \beta \in (0,1]$ is a viscosity solution in the sense of [11, Definition 4.3], positive α and β have to be equal in order to satisfy (4).

Our main result is then that each limit of (1) – no additional assumptions are necessary – satisfies for a.e. $t \in (0,\infty)$

$$\begin{aligned} \textstyle\int_{\mathbf{R}^n} (\partial_t u(t)\phi \;+\; \nabla u(t)\cdot\nabla\phi) \;&=\; -\textstyle\int_{R(t)} \phi \, d\mathcal{H}^{n-1} \\ -\;\textstyle\int_{\Sigma_*(t)} 2\theta(t,\cdot)\,\phi\, d\mathcal{H}^{n-1} \;&-\; \textstyle\int_{\Sigma_z(t)} \phi\, d\lambda(t) \end{aligned} \tag{5}$$

for every $\phi \in C_0^1(\mathbf{R}^n)$, that the non-degenerate singular set

$$\Sigma_*(t) := \{x \in \partial\{u(t)>0\} : \text{there is } \theta(t,x) \in (0,1] \text{ and } \xi(t,x) \in \partial B_1(0) \text{ such}$$
$$\text{that } u_r(s,y) = \frac{u(t+r^2s, x+ry)}{r} \;\to\; \theta(t,x)|y\cdot\xi(t,x)| \text{ locally uniformly}$$
$$\text{in } (s,y) \in \mathbf{R}^{n+1} \text{ as } r \to 0\}$$

is for a.e. $t \in (0,\infty)$ a countably $n-1$-rectifiable subset of the free boundary whereas $\lambda(t)$ is for a.e. $t \in (0,\infty)$ a Borel measure such that the $n-1$ dimensional Hausdorff measure is on

$$\Sigma_z(t) := \{x \in \partial\{u(t)>0\} \;:\; r^{-n-2}\int_{Q_r(t,x)} |\nabla u|^2 \to 0 \text{ as } r \to 0\}$$

totally singular with respect to $\lambda(t)$, i.e. $r^{1-n}\lambda(t)(B_r(x)) \to 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_z(t)$. Up to a set of vanishing $\mathcal{H}^{n-1}$ measure, $\partial\{u(t)>0\} = R(t) \cup \Sigma_*(t) \cup \Sigma_z(t)$.

Let us shortly describe relevant parts of the proof:
As a first step, we prove convergence of $2B_{\epsilon_m}(u_{\epsilon_m})$ to a characteristic function. We also need some control over the set of *horizontal points,* i.e. the set of points at which the solution's behaviour in the time direction is dominant.
A crucial tool in the local analysis at the free boundary is the *monotonicity formula*

Theorem 1 **(ϵ-Monotonicity Formula)** *Let* $(t_0,x_0) \in (0,\infty)\times\mathbf{R}^n, T_r^-(t_0) = (t_0-4r^2, t_0-r^2)\times\mathbf{R}^n$, $0<\rho<\sigma<\frac{\sqrt{t_0}}{2}$ *and*

$$G_{(t_0,x_0)}(t,x) = 4\pi(t_0-t)\,|4\pi(t_0-t)|^{-\frac{n}{2}-1}\,\exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right) \quad .$$

Then

$$\Psi^\epsilon_{(t_0,x_0)}(r) = r^{-2}\int_{T_r^-(t_0)} \left(|\nabla u_\epsilon|^2 + 2B_\epsilon(u_\epsilon)\right) G_{(t_0,x_0)} + \\ - \frac{1}{2} r^{-2}\int_{T_r^-(t_0)} \frac{1}{t_0 - t} u_\epsilon^2\, G_{(t_0,x_0)}$$

satisfies the monotonicity formula

$$\Psi^\epsilon_{(t_0,x_0)}(\sigma) - \Psi^\epsilon_{(t_0,x_0)}(\rho) \geq \int_\rho^\sigma r^{-1-2}\int_{T_r^-(t_0)} \frac{1}{t_0-t}\Big(\nabla u_\epsilon \cdot (x - x_0) \\ - 2(t_0 - t)\partial_t u_\epsilon - u_\epsilon\Big)^2 G_{(t_0,x_0)}\, dr \geq 0 .$$

The key to our result is then an *estimate for the parabolic mean frequency.*

Proposition 1 *On the closed set* $\Sigma := \{(t,x) \in (0,\infty)\times\mathbf{R}^n : \Psi_{(t,x)}(0+) = 2H_n\}$ *the parabolic mean frequency*

$$2\left(\int_{T_r^-(t)} \frac{1}{t-s} u^2\, G_{(t,x)}\right)^{-1} \int_{T_r^-(t)} |\nabla u|^2\, G_{(t,x)} \geq 1 .$$

The function $r \mapsto r^{-2}\int_{T_r^-(t)} \frac{1}{t-s} u^2\, G_{(t,x)}$ *is non-decreasing and has a right limit* $\theta^2(t,x)\int_{T_1^-(0)} \frac{1}{-s}|x_1|^2 G_{(0,0)}$ *. The function* θ *is upper semicontinuous on* Σ *. At each* $(t,x) \in \Sigma$

$$\int_0^r s^{-3}\int_{T_s^-(t)} (1-\chi) G_{(t,x)}\, ds \to 0 \text{ as } r \to 0 .$$

It is a surprising fact that the parabolic mean frequency is bounded from below at each point of highest density, which includes the set Σ_*. As a consequence we obtain *unique tangent cones* for a.e. time and at $\mathcal{H}^{n-1}$-a.e. point of the graph of u, whence GMT-tools lead to our result.

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