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Asymptotic Behavior of Solutions to One-phase Stefan Problems for Sublinear Heat Equations

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1 Introduction

Let us consider the following one-dimensional one-phase Stefan problem for sublinear heat equations. The problem is to find a curve $x=\ell(t)>0$ on $[0,T]$, $0<T<\infty$ and a function $u=u(t,x)$ on $(0,T)\times(0,\infty)$ satisfying

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in } Q(T;\ell) := \{(t,x) : 0<t<T, 0<x<\ell(t)\}, \tag{1.1}$$

$$u(t,0) = 0 \quad \text{for } 0<t<T, \tag{1.2}$$

$$u(t,x) = 0 \quad \text{for } 0<t<T \text{ and } x \geq \ell(t), \tag{1.3}$$

$$\ell'(t) \left(= \frac{d}{dt}\ell(t)\right) = -u_x(t,\ell(t)) \quad \text{for } 0<t<T, \tag{1.4}$$

$$u(0,x) = u_0(x) \quad \text{for } x \geq 0, \tag{1.5}$$

$$\ell(0) = \ell_0, \tag{1.6}$$

where $-1<\alpha<0$, $\ell_0>0$ and u_0 is a given non-negative initial function.

Throughout this paper we put $f(r)=r^{1+\alpha}$ if $r\geq 0$, $=0$ otherwise, and denote by SP $:=$ SP(f,u_0,ℓ_0) the above system (1.1) $\sim$ (1.6).

In our problem α is supposed to be negative. Hence, we need a careful treatment for sublinear heat equations because of the lack of the Lipschitz continuity of the nonlinear term. Here, we list some results concerned with uniqueness for the following initial boundary value problem (P):

$$v_t = \Delta v + v^{1+\alpha} \quad \text{in } (0,T)\times\Omega,$$

$$v = 0 \quad \text{on } (0,T)\times\partial\Omega,$$

$$v(0,x) = v_0(x) \quad \text{for } x\in\Omega,$$

where Ω is a domain in $\mathbb{R}^N$ and v_0 is a given initial function on Ω. It is well-known that the uniqueness theorem dose not hold generally. Fujita and Watanabe [13] have shown that (P) admits a positive solution for $v_0 \equiv 0$. If the initial function v_0 is not identically zero, then the nonnegative solution of (P) with $\Omega = \mathbb{R}^N$ is unique (see Aguirre and Escobedo [1]). Recently, Cazenave, Dickstein and Escobedo have also established the uniqueness of solutions of (P) for any bounded domain Ω in [11]. Also, we refer to, for instance, [9, 8, 17], for sublinear elliptic problems.

In case $\alpha > 0$ the authors have considered the above one-phase Stefan problems and have shown behaviors of the free boundary of a blow-up solution, and global existence and decay of a solution with a small initial data [4, 6, 2, 5]. Moreover, the following result A) concerned with the large-time behavior of a solution to SP for $\alpha \in (-1, 0)$ was already obtained in [14].

A) If u grows up, then $\ell(t) \to \infty$ as $t \to \infty$.

In this paper we consider only non-negative solutions because the uniqueness theorem holds for only them. Our main results are stated as follows:

1) (Global existence and uniqueness of a solution) The problem SP admits one and only one non-negative solution on the time interval $[0, \infty)$.

2) (Comparison principle) Let $\{u_i, \ell_i\}$ be a non-negative solution of $\mathrm{SP}(f, u_{0i}, \ell_{0i})$ on $[0, T]$ for $i = 1, 2$. If $u_{01} \leq u_{02}$ and $\ell_{01} \leq \ell_{02}$, then $u_1 \leq u_2$ and $\ell_1 \leq \ell_2$.

3) (Growing up of a solution) $\ell(t) \to \infty$ and $u(t, x) \to \infty$ for any $x > 0$ as $t \to \infty$, if $u_0 \neq 0$.

In section 2 we give precise assumptions for data, a definition of a solution of SP and a theorem concerned with the global existence and the uniqueness without their proof. We consider the classical solutions, which means that u_{xx} and u_t are continuous, because the strong maximum principle is applied in the proof of uniqueness. When we study large-time behavior we do not need uniqueness. Then it is not necessary to deal with classical solutions. In order to simplify our argument we give another definition of a solution of SP in section 3. Also, we shall provide some lemmas to investigate large time behavior. Finally, we shall prove that the solution with non-zero initial data always grows up.

2 Global existence and uniqueness

We begin with assumptions for data, the definition of a solution and the statement of our result. Throughout this paper, we use the following notations of function spaces and norms, $C^{N+\nu}([0,T])$, $|z|_{C^{N+\nu}([0,T])}$, $C^{N+\nu}(\Omega)$ and $|z|_{C^{N+\nu}(\Omega)}$, where $N = 0, 1, 2$, $\nu \in (0, 1)$, $0 < T < \infty$ and Ω is a non-cylindrical domain, in general. The precise definitions of these notations are given in [12; Section 2].

Now we give the definition of a solution of $\mathrm{SP}(f, u_0, \ell_0)$ in the following way.

Definition 2.1. We call that a pair $\{u, \ell\}$ of functions u on $(0, T) \times (0, \infty)$ and ℓ on $[0, T]$ is a solution of $\mathrm{SP}(f, u_0, \ell_0)$ on $[0, T]$, if the following conditions (S1) $\sim$ (S3) hold:
(S1) $u \geq 0$ on $(0, T) \times (0, \infty)$, $u \in W^{1,2}(0, T; L^2(0, \infty))$, $u_x \in C^\nu(\overline{Q(T; \ell)})$, $u_{xx} \in C(\overline{Q(t_0, T; \ell)})$ for any $t_0 \in (0, T]$,

$$\ell > 0 \text{ on } [0, T] \text{ and } \ell \in C^{1+\nu}([0, T]),$$

where $\nu \in (0,1)$ and $Q(t_0,T;\ell) = Q(T;\ell) \cap \{t > t_0\}$;
(S2) $u_t = u_{xx} + f(u)$ in $Q(T;\ell)$;
(S3) (1.2) $\sim$ (1.6) hold.

Moreover, a couple $\{u,\ell\}$ is said to be a solution of SP(f,u_0,ℓ_0) on an interval $[0,T')$, $0 < T' \leq \infty$, if it is the solution of SP(f,u_0,ℓ_0) on $[0,T]$ for any $0 < T < T'$.

The next theorem is our main result which shows the existence and uniqueness of our problem SP(f,u_0,ℓ_0).

Theorem 2.1. *Assume that $\ell_0 > 0$ and a nonnegative function u_0 on $[0,\infty)$ satisfy $u_0 \in C^{1+\beta}([0,\ell_0])$ for some $\beta \in (0,1)$,. $u_0 > 0$ on $(0,\ell_0)$, $u_0(0) = 0$, $u_0 = 0$ on $[\ell_0,\infty)$, $u_{0x}(0) > 0$ and $u_{0x}(\ell_0) < 0$. Then, there exists one and only one solution $\{u,\ell\}$ of SP(f,u_0,ℓ_0) on $[0,\infty)$.*

By the proof of the existence and uniqueness, it is easy to get the comparison principle for SP.

Corollary 2.1. *Let $T > 0$. For $i = 1,2$ assume that $\ell_{0i} > 0$ and a nonnegative function u_{0i} on $[0,\infty)$ satisfy $u_{0i} \in C^{1+\beta}([0,\ell_{0i}])$ for some $\beta \in (0,1)$, $u_{0i} > 0$ on $(0,\ell_{0i})$, $u_{0i}(0) = 0$, $u_{0i} = 0$ on $[\ell_{0i},\infty)$, $u_{0ix}(0) > 0$ and $u_{0ix}(\ell_{0i}) < 0$. Let $\{u_i,\ell_i\}$ be a solution of SP(f,u_{0i},ℓ_{0i}) on $[0,T]$ for $i = 1,2$. If $\ell_{01} \leq \ell_{02}$ and $u_{01} \leq u_{02}$ on $[0,\infty)$, then $\ell_1 \leq \ell_2$ on $[0,T]$ and $u_1 \leq u_2$ on $(0,T) \times (0,\infty)$.*

Here, we omit the proofs of the above theorem and corollary because the complete proofs are given in [7].

3 Auxiliary lemmas

In the final section we shall discuss the asymptotic behavior of solutions of SP. We do not need a uniqueness theorem when we study the large-time behavior. Hence, we give another definition of a solution to SP and an existence theorem.

Definition 3.1. A pair $\{u,\ell\}$ is a solution of SP(f,u_0,ℓ_0) on $[0,T]$ if the following conditions hold:

(S1') $u \geq 0$ on $(0,T) \times (0,\infty)$, $u \in W^{1,2}(0,T;L^2(0,\infty)) \cap L^\infty(0,T;H^1(0,\infty))$, $\ell > 0$ on $[0,T]$ and $\ell \in W^{1,3}(0,T)$.

(S2') (1.1) holds for a.e. $(t,x) \in Q(T;\ell)$, (1.2) and (1.3) hold, and $\ell'(t) = -u_x(t,\ell(t)-)$ for a.e. $t \in [0,T]$.

(S3') $u(0,x) = u_0(x)$ for $x \geq 0$ and $\ell(0) = \ell_0$.

Also, we call that a couple $\{u,\ell\}$ is a solution of SP(f,u_0,ℓ_0) on an interval $[0,T')$, $0 < T' \leq \infty$, if it is the solution of SP(f,u_0,ℓ_0) on $[0,T]$ in the above sense for any $0 < T < T'$.

From now on we always consider a solution of SP in the sense of Definition 3.1. The next theorem guarantees the existence of a solution of SP.

Theorem 3.1. *Assume that $\ell_0 > 0$, $u_0 \in H^1(0,\infty) \subset C([0,\infty))$ with $u_0 \geq 0$ on $[0,\infty)$, $u(0) = 0$ and $u_0 = 0$ on $[\ell_0,\infty)$. Then, there exists a solution $\{u,\ell\}$ of SP(f,u_0,ℓ_0) on $[0,\infty)$.*

In order to prove the theorem we introduce the following approximate problem $\mathrm{SP}_\varepsilon := \mathrm{SP}(f_\varepsilon, u_0, \ell_0)$ where $\varepsilon > 0$ and

$$f_\varepsilon(r) = \begin{cases} (r+\varepsilon)^{1+\alpha} - \varepsilon^{1+\alpha} & \text{if } r \geq 0, \\ (1+\alpha)\varepsilon^\alpha r & \text{otherwise.} \end{cases}$$

Clearly, for each $\varepsilon > 0$ $f_\varepsilon(0) = 0$ and f_ε is Lipschitz continuous on $\mathbb{R}$. Hence, we obtain the local existence in time and uniqueness for the approximate problems.

Lemma 3.1. *(cf. [15, 3]) Let $\varepsilon > 0$. Under the same assumptions as in Theorem 3.1 there exists a positive constant $T_0 > 0$ such that the problem SP_ε has a unique solution on $[0, T_0]$.*

In order to prove Theorem 3.1 we use the following energy inequality.

Lemma 3.2. *(cf. [16; theorem 2.3 and lemma 5.1]) Suppose that the same assumptions as in Theorem 3.1 hold. Let $\{u_\varepsilon, \ell_\varepsilon\}$ be a solution of SP_ε on $[0,T]$, $T > 0$. Then, the following inequality holds:*

$$|u_{\varepsilon t}(t)|^2_{L^2(0,\infty)} + \frac{1}{2}\frac{d}{dt}|u_{\varepsilon x}(t)|^2_{L^2(0,\infty)} + \frac{1}{2}|\ell'_\varepsilon(t)|^3 \leq \frac{d}{dt}\int_0^\infty \hat{f}_\varepsilon(u_\varepsilon(t))dx \text{ for a.e. } t \in [0,T], \quad (3.1)$$

where $\hat{f}_\varepsilon(r) = \int_0^r f_\varepsilon(\xi)d\xi$ for $r \in \mathbb{R}$.

Proof of Theorem 3.1. Let $T > 0$ and $\varepsilon \in (0,1]$. By Lemma 3.1 we have a solution $\{u_\varepsilon, \ell_\varepsilon\}$ on $[0, \bar{T}_\varepsilon]$ where $\bar{T}_\varepsilon > 0$. Let $[0, T_\varepsilon)$ be a maximal interval of the exisitence of a solution of SP_ε. We assume that $T_\varepsilon < T$ and put $\hat{T}_\varepsilon = \min\{T, T_\varepsilon\}$. Immediately, we obtain

$$\int_0^{\ell_\varepsilon(t)} \hat{f}_\varepsilon(u_\varepsilon)(t)dx \leq \delta\int_0^{\ell_\varepsilon(t)} |u_\varepsilon(t)|^2 dx + C_\delta \ell_\varepsilon(t),$$

where $\delta > 0$ and C_δ is a positive constant depending only on δ. Integrating it we get

$$\begin{aligned} &\int_0^t |u_{\varepsilon\tau}(\tau)|^2_{L^2(0,\ell_\varepsilon(\tau))}d\tau + \frac{1}{2}|u_{\varepsilon x}(t)|^2_{L^2(0,\ell_\varepsilon(t))} + \frac{1}{2}\int_0^t |\ell'_\varepsilon(\tau)|^3 d\tau \\ \leq\ &\frac{1}{2}|u_{0\varepsilon x}|^2_{L^2(0,\ell_0)} + \delta\int_0^{\ell_\varepsilon(t)} |u_\varepsilon(t)|^2 dx + C_\delta\ell_\varepsilon(t) \quad \text{for a.e. } t \in [0, \hat{T}_\varepsilon]. \end{aligned}$$

Here, we note that

$$\int_0^{\ell_\varepsilon(t)} |u_\varepsilon(t)|^2 dx \leq 2t\int_0^t\int_0^{\ell_\varepsilon(\tau)} |u_{\varepsilon\tau}(\tau)|^2 dx d\tau + 2\int_0^{\ell_0} |u_0|^2 dx \quad \text{for } t \in [0, \hat{T}_\varepsilon],$$

and

$$\ell_\varepsilon(t) \leq \eta\int_0^t |\ell'_\varepsilon(\tau)|^3 d\tau + C_\eta t + \ell_0 \quad \text{for } 0 \leq t \leq \hat{T}_\varepsilon,$$

where $\eta > 0$ and $C_\eta > 0$.

Consequently,

$$\begin{aligned} &\int_0^t |u_{\varepsilon\tau}(\tau)|^2_{L^2(0,\ell_\varepsilon(\tau))}d\tau + \frac{1}{2}|u_{\varepsilon x}(t)|^2_{L^2(0,\ell_\varepsilon(t))} + \frac{1}{2}\int_0^t |\ell'_\varepsilon(\tau)|^3 d\tau \\ \leq\ &\frac{1}{2}|u_{0x}|^2_{L^2(0,\ell_0)} + 2\delta(T\int_0^t\int_0^{\ell_\varepsilon(\tau)} |u_{\varepsilon\tau}(\tau)|^2 dx d\tau + \int_0^\infty |u_{0\varepsilon}|^2 dx) \\ &+C_\delta(\eta\int_0^t |\ell'_\varepsilon(\tau)|^3 d\tau + C_\eta T + \ell_0) \quad \text{for } 0 \leq t \leq \hat{T}_\varepsilon, \end{aligned}$$

By choosing δ and η as sufficiently small numbers it holds that

$$\begin{aligned}&\frac{1}{2}\int_0^t |u_{\varepsilon\tau}(\tau)|^2_{L^2(0,\infty)}d\tau + \frac{1}{2}|u_{\varepsilon x}(t)|^2_{L^2(0,\infty)} + \frac{1}{4}\int_0^t |\ell'_\varepsilon(\tau)|^3 d\tau \\ \leq\ & \frac{1}{2}\int_0^{\ell_0} |u_{0\varepsilon x}|^2 dx + \frac{1}{2}\int_0^{\ell_0} |u_{0\varepsilon}|^2 dx + C_\delta C_\eta T + C_\delta \ell_0 \quad \text{for } 0 \leq t \leq \hat{T}_\varepsilon.\end{aligned}$$

In particular, there is a positive constant L_1 independent of ε such that

$$\ell_\varepsilon(t) \leq L_1 \quad \text{for } 0 \leq t \leq \hat{T}_\varepsilon.$$

From the above estimates and Lemma 3.1 the solution $\{u_\varepsilon, \ell_\varepsilon\}$ can be extended beyond time $\hat{T}_\varepsilon$ for each $\varepsilon \in (0,1]$. This is a contradiction. Therefore, SP_ε has a solution on $[0,T]$ for each $\varepsilon \in (0,1]$. Moreover, the above estimates hold for $t \in [0,T]$. Particularly, $\ell_\varepsilon(t) L_1$ for $0 \leq t \leq \hat{T}_\varepsilon$ and $\varepsilon \in (0,1]$.

Hence, we can take a subsequence $\{\varepsilon_n\} \subset \{\varepsilon\}$ with $\varepsilon_n \to \infty$ such that

$$\ell_{\varepsilon_n} \to \ell \text{ weakly in } W^{1,3}(0,T) \text{ and in } C([0,T]),$$

$$u_{\varepsilon_n} \to u \text{ weakly in } W^{1,2}(0,T;L^2(0,\infty)),\ \text{weakly* in } L^\infty(0,T;H^1(0,\infty)),$$
$$\text{and in } C([0,T]\times[0,L_1]).$$

It is clear a $\{u,\ell\}$ is the solution of $\mathrm{SP}(f,u_0,\ell_0)$ on $[0,T]$. □

The following inequality is a direct consequence of the proof of Theorem 3.1.

Lemma 3.3. *Suppose all assumptions in Theorem 3.1 hold. Let $\{u,\ell\}$ be a solution of SP(f,u_0,ℓ_0) on $[0,\infty)$. Then, it holds that*

$$|u_t(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}|\ell'(t)|^3 \leq \frac{1}{2+\alpha}\frac{d}{dt}\int_0^{\ell(t)} u^{2+\alpha}(t)dx \text{ for a.e. } t \in [0,\infty).$$

The next proposition is concerned with the convergence of solutions.

Proposition 3.1. *Let $T > 0$. Assume that $\ell_{0n} > 0$ and $u_{0n} \in H^1(0,\infty)$ satisfy the condition in Theorem 3.1 for each $n = 1,2,\cdots$. Moreover, suppose that $\ell_{0n} \to \ell_0$ as $n \to \infty$ where $\ell_0 > 0$ and $u_{0n} \to u_0$ weakly in $H^1(0,M_0)$ and in $C([0,M_0])$ where $M_0 = \sup_{n=1,2,\dots} \ell_{0n}$ and $u_0 \in H^1(0,M_0)$. Let $\{u_n,\ell_n\}$ be a solution of SP(f,u_{0n},ℓ_{0n}) on $[0,T]$. Then, there exists a subsequence $\{n_j\}$ such that*

$$\ell_{n_j} \to \ell \text{ weakly in } W^{1,3}(0,T) \text{ and } C([0,T]),$$

$$u_{n_j} \to u \text{ weakly in } W^{1,2}(0,T;L^2(0,M)),\ \text{weakly* in } L^\infty(0,T;H^1(0,M))$$
$$\text{and in } C([0,T]\times[0,M]),$$

where $M = \sup\{\ell_n(t); n = 1,2,\cdots \text{ and } t \in [0,T]\}$. Moreover, $\{u,\ell\}$ is a solution of SP(f,u_0,ℓ_0) on $[0,T]$.

Proof. By using the same argument in the proof of Theorem 2.1 it follows from Lemma 3.3 that $\{\ell_n\}$ is bounded in $W^{1,3}(0,T)$. Hence, $M = \sup\{\ell_n(t); n = 1,2,\cdots \text{ and } t \in [0,T]\} < \infty$. Moreover, $\{u_n\}$ is bounded in $W^{1,2}(0,T;L^2(0,M))$ and $L^\infty(0,T;H^1(0,M))$. Similarly to the proof of Theorem 3.1 we obtain the assertion of this proposition. □

4 Large-time behavior

The purpose of this section is to prove Theorem 4.1 concerned with large-time behaviors. In order to give the statement of the theorem we introduce the following elliptic problem. For each $l > 0$ we denote by $(\mathrm{P})_\infty(l)$ the following problem:

$$\begin{cases} w_{xx} + w^{1+\alpha} = 0 & \text{in } (0,l), \\ w(0) = w(l) = 0. \end{cases}$$

By Brezis and Oswald [9; Theorem 1] the problem $(\mathrm{P})_\infty(l)$ has one and only one non-negative non-zero solution w for each $l > 0$. Moreover, $w > 0$ on $(0,l)$ and $\int_0^l w^\alpha dx < \infty$.

Theorem 4.1. *Suppose that the same assumptions as in Theorem 3.1 hold and $u_0 \geq c_0 v_*$ on $[0,\ell_0]$ where $c_0 \in (0,1]$ and v_* is a non-negative non-zero solution of $(P)_\infty(\ell_0)$. Let $\{u,\ell\}$ be a solution of $SP(f,u_0,\ell_0)$ on $[0,\infty)$. Then, u and ℓ grow up as $t \to \infty$, that is,*

$$\ell(t) \to \infty \text{ and } |u(t,x)| \to \infty \text{ for } x > 0 \text{ as } t \to \infty.$$

The following proposition will be used in the proof of Theorem 4.1.

Proposition 4.1. *For $l > 0$ let $w^{(l)}$ be a none-negative solution of $(P)_\infty(l)$ with $w^{(l)} \neq 0$. For each $x > 0$ $w^{(l)}(x) \to \infty$ as $l \to \infty$.*

In order to prove Proposition 4.1 we deal with the following initial boundary value problem $\mathrm{P}(l;v_0)$ for $l > 0$:

$$v_t = v_{xx} + v^{1+\alpha} \quad \text{in } (0,T)\times(0,l), \tag{4.1}$$

$$v(t,0) = v(t,l) = 0 \quad \text{for } t \in (0,T), \tag{4.2}$$

$$v(0,x) = v_0(x) \quad \text{for } x \in (0,l). \tag{4.3}$$

The following lemma guarantees the global existence and the large-time behavior of solutions of $\mathrm{P}(l;v_0)$.

Lemma 4.1. *Let $l > 0$ and $v_0 \in H_0^1(0,l)$ with $v_0 \geq 0$. Then the following properties hold.*

(1) Let $T > 0$. Then there exists a function $v \in W^{1,2}(0,T;L^2(0,l)) \cap L^\infty(0,T;H_0^1(0,l))$ satisfying $v \geq 0$ on $(0,T)\times(0,l)$, (4.1) and (4.3). Moreover,

$$\int_0^l |v_t(t)|^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^l |v_x(t)|^2 dx \leq \frac{1}{2+\alpha}\frac{d}{dt}\int_0^l v^{2+\alpha}(t)dx \text{ for a.e. } t \in [0,T]. \tag{4.4}$$

This means that $P(l;v_0)$ has a non-negative solution on $[0,\infty)$.

(2) Assume that $v_0 \geq c_0 v_\infty$ on $(0,l)$ where $c_0 \in (0,1]$ and v_∞ is a non-negative solution of $(P)_\infty(l)$ with $v_\infty \neq 0$. Let v be a solution of $P(l;v_0)$ on $[0,\infty)$. Then

$$v(t) \to v_\infty \quad \text{in } C([0,1]) \text{ as } t \to \infty.$$

Proof. (1) We can prove this assertion in a similar way to that of Theorem 3.1.
(2) It follows from (4.4) that

$$\int_0^t |v_\tau(\tau)|^2_{L^2(0,l)}d\tau + |v_x(t)|^2_{L^2(0,l)} \leq |v_{0x}|^2_{L^2(0,l)} + \frac{1}{2+\alpha}\int_0^l |v(\tau)|^{2+\alpha}dx \text{ for } t \geq 0.$$

Since $2+\alpha < 2$, there exists a positive constant C_1 such that

$$\int_0^t |v_\tau(\tau)|^2_{L^2(0,l)}d\tau \leq C_1 \text{ and } |v_x(t)|^2_{L^2(0,l)} \leq C_1 \text{ for } t \geq 0.$$

Therefore, we can take a subsequence $\{t_n\}$ with $t_n \to \infty$ such that

$$v(t_n) \to \hat{v}_\infty \text{ weakly in } H^1_0(0,l) \text{ and in } C([0,l]) \text{ as } n \to \infty,$$

where $\hat{v}_\infty$ is a solution of $(\mathrm{P})_\infty(l)$. In order to accomplish the proof it is sufficient to show that $\hat{v} \neq 0$ because $(\mathrm{P})_\infty(l)$ admits a unique non-negative non-zero solution. Immediately, we have

$$c_0 v_{\infty t} = c_0 v_{\infty xx} + c_0 v^{1+\alpha} \leq c_0 v_{\infty xx} + (c_0 v)^{1+\alpha} \text{ on } (0,l).$$

This inequality together with (4.1) implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^l |[c_0 v_\infty - v(t)]^+|^2 dx + \int_0^l |[c_0 v_\infty - v(t)]^+_x|^2 dx \\
\leq\ & \int_0^l ((c_0 v_\infty)^{1+\alpha} - v^{1+\alpha}(t))[c_0 v_\infty - v(t)]^+ dx \\
\leq\ & \frac{1}{1+\alpha}\int_0^l (c_0 v_\infty)^{1+\alpha}|[c_0 v_\infty - v(t)]^+|^2 dx \\
\leq\ & \frac{1}{1+\alpha}|[c_0 v_\infty - v(t)]^+|^2_{L^\infty(0,l)}\int_0^l (c_0 v_\infty)^{1+\alpha} dx \\
\leq\ & \frac{1}{2}\int_0^l |[c_0 v_\infty - v(t)]^+_x|^2 dx + C_2\int_0^l |[c_0 v_\infty - v(t)]^+|^2 dx \text{ for a. e. } t \geq 0,
\end{aligned}$$

where C_2 is a positive constant independent of $t \geq 0$. By applying Gronwall's inequality we get

$$[c_0 v_\infty - v]^+ = 0 \quad \text{on } (0,\infty) \times (0,l),$$

that is, $c_0 v_\infty \leq v$ on $(0,\infty) \times (0,l)$. Thus we infer that $\hat{v}_\infty \neq 0$, that is, $\hat{v}_\infty = v_\infty$. □

The following lemma is concerned with the comparison principle for solutions of $\mathrm{P}(l; v_0)$.

Lemma 4.2. *For $l > 0$ let v_∞ be a non-negative solution of $(P)_\infty(l)$ with $v_\infty \neq 0$. Suppose that for $i = 1,2$, $v_{0i} \in H^1_0(0,l)$ satisfies $v_{0i} \geq c_0 v_\infty$ on $(0,l)$ where $c_0 \in (0,1]$, and $v_{01} \leq v_{02}$ on $(0,l)$. Let v_1 be a solution of $P(l; v_{01})$ on $[0,\infty)$ and $v_2 \in W^{1,2}_{loc}([0,\infty); L^2(0,l)) \cap L^\infty([0,\infty); H^1(0,l))$ satisfy (4.1) with $v_2(0) = v_{02}$ and $v_2(t,0) \geq 0$ and $v_2(t,l) \geq 0$ for $t \geq 0$. Then, $v_1 \leq v_2$ on $(0,\infty) \times (0,l)$.*

Proof. Similarly to Lemma 4.1, we obtain $v_i \geq c_0 v_\infty$ on $[0,\infty)\times(0,l)$. By using this inequality we observe that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^l |[v_1(t)-v_2(t)]^+|^2dx + \int_0^l |[v_1(t)-v_2(t)]_x^+|^2dx \\
\leq\ & \int_0^l ((v_1)^{1+\alpha}(t) - v_2^{1+\alpha}(t))[v_1(t)-v_2(t)]^+dx \\
\leq\ & \frac{1}{1+\alpha}\int_0^l (v_1)^{1+\alpha}(t)|[v_1(t)-v_2(t)]^+|^2dx \\
\leq\ & \frac{1}{1+\alpha}|[v_1(t)-v_2(t)]^+|^2_{L^\infty(0,l)}\int_0^l (c_0 v_\infty)^{1+\alpha}dx \\
\leq\ & \frac{1}{2}\int_0^l |[v_1(t)-v_2(t)]_x^+|^2dx + C_3\int_0^l |[v_1(t)-v_2(t)]^+|^2dx \text{ for a.e. } t\geq 0,
\end{aligned}$$

where C_3 is a positive constant independent of t. Immediately, we can prove that $v_1 \leq v_2$ on $[0,\infty)\times(0,l)$. □

Lemma 4.3. *For $l \geq \pi$ let $w^{(l)}$ be a nonnegative solution of $P_\infty(l)$ with $w^{(l)} \neq 0$. Then, it holds:*

$$w^{(l)}(x) \geq \sin\frac{\pi x}{l} \quad \textit{for } x \in (0,l).$$

Proof. Set $z(x) = \sin\frac{\pi x}{l}$ and let v be a solution of $P(l;z)$ on $[0,\infty)$. We observe that

$$0 = z_t = z_{xx} + \frac{\pi^2}{l^2}z \leq z_{xx} + z^{1+\alpha}.$$

Similarly to the proof of Lemma 4.2 we obtain $z \leq v$ on $[0,\infty)\times(0,l)$. Lemma 4.1 (2) implies that $v(t) \to w^{(l)}$ in $C([0,1])$ as $t\to\infty$. Therefore, $z \leq w^{(l)}$ on $(0,l)$. □

Proof of Proposition 4.1. Let $l > 0$. By uniqueness of the problem $(P)_\infty(l)$ the nonnegative solution $w^{(l)}$ can be expressed as

$$w^{(l)}(x) = \frac{1}{l}\int_0^x y(l-x)(w^{(l)})^{1+\alpha}(y)dy + \frac{1}{l}\int_x^l x(l-y)(w^{(l)})^{1+\alpha}(y)dy \quad \text{for } x\in(0,l). \tag{4.5}$$

Now, we assume that $l \geq \pi$. Then, Lemma 4.3 implies $w^{(l)}(x) \geq \sin\frac{\pi x}{l}$ for $x \in (0,l)$. Let $x > 0$ and $l \geq \max\{\pi, 2x\}$. Hence, we have

$$\begin{aligned}
w^{(l)}(x) &\geq \frac{1}{l}\int_x^l x(l-y)(w^{(l)})^{1+\alpha}(y)dy \\
&\geq \frac{1}{l}\int_{l/2}^l x(l-y)(\sin\frac{\pi y}{l})^{1+\alpha}dy \\
&\geq \frac{1}{l}\int_{l/2}^l x(l-y)\sin\frac{\pi y}{l}dy \\
&\geq \frac{xl}{\pi}\int_{\pi/2}^\pi (1-\frac{\xi}{\pi})\sin\xi d\xi \to \infty \quad \text{as } l\to\infty.
\end{aligned}$$

Thus we have proved this lemma. □

Proof of Theorem 4.1. First, we assume that

$$\ell(t) \le M_1 \quad \text{for } t \ge 0, \tag{4.6}$$

where M_1 is a positive constant.
Lemma 3.3 implies

$$\begin{aligned} &|u_t(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}|\ell'(t)|^3 \\ \le\ & \frac{1}{2+\alpha}\frac{d}{dt}\int_0^{\ell(t)} u^{2+\alpha}(t)dx \qquad \text{for } t \in [0,\infty). \end{aligned}$$

Then we have

$$\begin{aligned} &\int_0^t |u_\tau(\tau)|^2_{L^2(0,\ell(\tau))}d\tau + \frac{1}{2}|u_x(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}\int_0^t |\ell'(\tau)|^3 d\tau \\ \le\ & \frac{1}{2+\alpha}\int_0^{\ell(t)} u^{2+\alpha}(t)dx + \int_0^{\ell_0} |u_{0x}|^2 dx \quad \text{for } t \in [0,\infty). \end{aligned}$$

Let $\varepsilon > 0$. Then, since $-1 < \alpha < 0$, by using Poincare's inequality we obtain

$$\begin{aligned} \int_0^{\ell(t)} u^{2+\alpha}(t)dx &\le \varepsilon \int_0^{\ell(t)} |u(t)|^2 dx + K_\varepsilon \ell(t) \\ &\le \varepsilon M_2 \int_0^{\ell(t)} |u_x(t)|^2 dx + K_\varepsilon \ell(t) \quad \text{for } t \ge 0, \end{aligned}$$

where K_ε and M_2 are positive constants depending only on ε and α, and M_1, respectively. By choosing ε as $\frac{1}{2M_2}$ we get

$$\int_0^t |u_\tau(\tau)|^2_{L^2(0,\ell(\tau))}d\tau + |u_x(t)|^2_{L^2(0,\ell(t))} + \int_0^t |\ell'(\tau)|^3 d\tau \le M_3 \text{ for } t \ge 0, \tag{4.7}$$

where M_3 is a positive constant independent of t.
Therefore, we can take a subsequence $\{t_n\}$ with $t_n \to \infty$ such that

$$u(t_n) \to u_\infty \text{ weakly in } H^1(0, M_1) \text{ and in } C([0, M_1]) \text{ as } n \to \infty.$$

From the assumption (4.6) it follows that $\ell(t) \to \ell_\infty \in \mathbb{R}$ as $t \to \infty$. On account of Proposition 3.1 we infer that

$$u(t_n + \cdot) \to u^* \text{ weakly in } W^{1,2}(0,1; L^2(0, M_1)), \ell(t_n + \cdot) \to \ell^* \text{ weakly in } W^{1,3}(0,1),$$

and $\{u^*, \ell^*\}$ is a solution of $\mathrm{SP}(f, u_\infty, \ell_\infty)$ on $[0,1]$. Moreover, by (4.7) we see that

$$u_t(t_n + \cdot) \to 0 \text{ in } L^2(0,1; L^2(0, M_1)) \text{ and } \ell'(t_n + \cdot) \to 0 \text{ in } L^3(0,1).$$

Obviously, $u^*(t) = u_\infty$ and $\ell^*(t) = \ell_\infty$ for $t \in [0,1]$ so that u_∞ is a solution of $(\mathrm{P})_\infty(\ell_\infty)$ and $u_{\infty x}(\ell_\infty) = \ell'_*(t) = 0$. By using Lemmas 4.1 and 4.2 we have

$$u \geq c_0 v^* \quad \text{on } (0,\infty) \times (0,\ell_0), \tag{4.8}$$

because $u(t,0) = 0$ and $u(t,\ell_0) \geq 0$ for $t \geq 0$. Accordingly, $u_\infty \neq 0$. Here, maximum principle or the expression (4.5) together with the above fact implies $u_{\infty x}(\ell_\infty) < 0$. This is a contradiction. Hence, we obtain

$$\ell(t) \to \infty \text{ as } t \to \infty, \tag{4.9}$$

since ℓ is the increasing function.

Next, we observe that $u(t) \in H^2(0,\ell(t))$ for a.e. $t \in [0,\infty)$; so that $u_x(t) \in C^{1/2}([0,\ell(t)])$ for a.e. $t \in [0,\infty)$. Also, $\ell \in C^{2/3}([0,T])$ for each $T > 0$ since $\ell' \in L^3(0,T)$. By using the classical theory (cf. [10; Chapter 19]) for parabolic equations we see that u_{xx} and u_t are continuous on $Q(s_0,T;\ell)$ for some $s_0 > 0$ and each $T > 0$. Hence, the strong maximum principle together with (4.8) shows that $u(t) > 0$ on $(0,\ell(t))$ for $t > s_0$. Moreover, Hopf Lemma (cf. [10; Theorem 15.4.1]) guarantees that $u_x(t,\ell(t)) < 0$ and $u_x(t,0) > 0$ for $t > s_0$. Therefore, for $s > s_0$ we can take a positive constant c (which may depend on s) satisfying $u(s,x) \geq cw^{(s)}$ on $(0,\ell(s)$ where $w^{(s)}$ is a non-negative solution of $(\mathrm{P})_\infty(\ell(s))$ with $w^{(s)} \neq 0$.

Here, for $s \geq s_0$ we denote by $(\mathrm{P})(s)$ the following problem:

$$\begin{cases} v_t^{(s)} = v_{xx}^{(s)} + (v^{(s)})^{1+\alpha} & \text{in } (s,\infty) \times (0,\ell(s)), \\ v^{(s)}(t,0) = v^{(s)}(t,\ell(s)) = 0 & \text{for } t \geq s, \\ v^{(s)}(s,x) = u(s,x) & \text{for } 0 < x < \ell(s). \end{cases}$$

Lemmas 4.1 and 4.2 show that there exists one and only one non-negative solution $v^{(s)}$ of $(\mathrm{P})(s)$ for each $s \geq s_0$. Moreover, by using Lemma 4.2, again, we have

$$u(t,x) \geq v^{(s)}(t,x) \quad \text{for } t \geq s \text{ and } 0 \leq x \leq \ell(s).$$

Lemma 4.1 (2) implies $v^{(s)}(t) \to v_\infty^{(s)}$ in $C([0,\ell(s)])$ as $t \to \infty$ where $v_\infty^{(s)}$ is a non-negative solution of $(\mathrm{P})_\infty(\ell(s))$. Proposition 4.1 and (4.9) guarantee that

$$v_\infty^{(s)}(x) \to \infty \quad \text{as } s \to \infty \text{ for each } x > 0.$$

Let $x > 0$ and $K > 0$. Then, there exists a positive number T_1 such that $v^{(s)}(x) \geq K$ for $s \geq T_1$. We fix $s \geq T_1$. Also, we have $|v^{(s)}(t,x) - v_\infty^{(s)}(x)| \leq 1$ for $t \geq T_2$ where T_2 is some positive constant. Therefore,

$$u(t,x) \geq v^{(s)}(t,x) \geq v_\infty^{(s)}(x) - 1 \geq K - 1 \quad \text{for } t \geq T_2.$$

Thus we have proved Theorem 4.1. □

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