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A General Criterion for the Eckhaus Instability in Gradient/Skew-Gradient Systems

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Abstract

A stability problem is considered for a family of spatially periodic stationary solutions in 1-D gradient/skew-gradient systems. In these systems, a first integral can be found for the equation for stationary solutions. Regard this integral as a functional on the set of stationary solutions, it is shown that a stability-instability transition occurs at extremal points of the functional, i.e., a first integral is an index of instability property of stationary solutions. The result gives a simple but general criterion for the Eckhaus instability in various model equations for pattern formation such as the Ginzburg-Landau equation, Swift-Hohenberg equation and activator-inhibitor reaction-diffusion systems.

keywords: gradient/skew-gradient systems, Eckhaus instability

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1 Introduction

It is well known that various interesting patterns with spatially periodic structure are observed in the world of nature, for example, thermal convection phenomena, biological morphogenesis, and vortices in the superconductivity. In order to describe the process of spatially periodic pattern formation, various model equations are proposed. The purpose of this paper is to study a stability problem for a family of spatially periodic stationary solutions within the framework of gradient/skew-gradient systems. Our main result gives a simple but very general criterion for the instability of spatially periodic stationary solutions. In fact, the criterion gives a transparent and unified viewpoint to understand the Eckhaus instability of spatially periodic stationary solutions in various model equations.

Let us consider the n-component system

$$Tu_t = Du_{xx} + f(u) \quad -\infty < x < \infty, \tag{1.1}$$

where $u(x,t) = {}^t(u_1, u_2, \cdots, u_n) \in \mathbf{R}^n$. We assume that T is a nonnegative diagonal matrix and D is a regular matrix such that (1.1) is well posed in an appropriate sense. In addition, we assume that D satisfies the condition

$$ {}^tDQ = QD \tag{1.2}$$

for some diagonal matrix

$$Q = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}, \qquad 0 \le k \le n, \tag{1.3}$$

where I_k denotes the identity matrix of order k. As for the nonlinear term, we assume that $f(u)$ is written as

$$f(u) = Q\frac{\partial H}{\partial u}$$

with some smooth function $H = H(u) : \mathbf{R}^n \to \mathbf{R}$. Under these assumptions, we notice that the Jacobian matrix f_u of f satisfies

$$ {}^tf_u(u)Q = Qf_u(u), \tag{1.4}$$

and there is an energy-like functional

$$E[u] = \int \left\{ \frac{1}{2}\langle Du_x, Qu_x\rangle - H(u) \right\} dx,$$

where $\langle\ ,\ \rangle$ stands for the usual inner product on $\mathbf{R}^n$. In fact, from $Q^2 = I_n$, ${}^tQ = Q$, ${}^tDQ = QD$ and ${}^tT = T$, it follows (under an appropriate asymptotic or boundary condition) that

$$\begin{aligned}
\frac{d}{dt}E[u(x,t)] &= \int \left\{ \frac{1}{2}\langle Du_{xt}, Qu_x\rangle + \frac{1}{2}\langle Du_x, Qu_{xt}\rangle - \langle \frac{\partial H}{\partial u}, u_t\rangle \right\} dx \\
&= \int \left\{ \frac{1}{2}\langle QDu_{xt}, u_x\rangle + \frac{1}{2}\langle Du_x, Qu_{xt}\rangle - \langle Q\frac{\partial H}{\partial u}, Qu_t\rangle \right\} dx \\
&= \int \left\{ \frac{1}{2}\langle {}^tDQu_{xt}, u_x\rangle + \frac{1}{2}\langle Du_x, Qu_{xt}\rangle - \langle f(u), Qu_t\rangle \right\} dx \\
&= \int \left\{ -\langle Du_{xx}, Qu_t\rangle - \langle f(u), Qu_t\rangle \right\} dx \\
&= -\int \langle u_t, TQu_t\rangle\, dx.
\end{aligned}$$

The system (1.1) is said to be a *gradient* system when TQ is nonnegative, and *skew-gradient* system otherwise [13].

Let $u = \phi(x; s)$ be a family of spatially periodic stationary solutions of (1.1) parametrized by s with its minimal spatial period $\ell(s)$, that is, $\phi(x; s)$ satisfies

$$\begin{aligned}
&D\phi_{xx}(x; s) + f(\phi(x; s)) = 0, \\
&\phi(x; s) = \phi(x + \ell(s); s).
\end{aligned} \tag{1.5}$$

The aim of this paper is to investigate the stability of $\phi(x; s)$ in the space of uniformly bounded functions on $\mathbf{R}$, which is denoted by $BC(\mathbf{R})^n$. Namely, we consider the linearized eigenvalue problem

$$\lambda TW = DW_{xx} + f_u(\phi(x; s))W, \quad -\infty < x < \infty, \tag{1.6}$$

which is a system of linear ODEs with $\ell(s)$-periodic coefficients $f_u(\phi(x; s))$. We denote the spectrum of (1.6) by $\Lambda(s)$. Differentiating (1.5) with respect to x, we immediately find that $\lambda = 0$ is an eigenvalue of (1.6) with an eigenfunction $\phi_x(x; s)$. As is well-known, the spectrum near zero often determines the stability/instability of stationary solutions in dissipative systems.

Setting

$$Y = \begin{pmatrix} W \\ W_x \end{pmatrix}, \tag{1.7}$$

we can rewrite (1.6) as a system of first order linear ODEs

$$\frac{d}{dx}Y = (B(x;s) + \lambda K)Y, \tag{1.8}$$

where $B(x;s)$ and K are $2n \times 2n$-matrices given by

$$B(x;s) = \begin{pmatrix} 0 & I_n \\ -D^{-1}f_u(\phi(x;s)) & 0 \end{pmatrix} \tag{1.9}$$

and

$$K = \begin{pmatrix} 0 & 0 \\ D^{-1}T & 0 \end{pmatrix}, \tag{1.10}$$

respectively. Clearly, $B(x;s)$ is an $\ell(s)$-periodic function of x. We will consider conditions so that (1.8) has a uniformly bounded solution for some $\Re\{\lambda\} > 0$. To do so, it suffices to consider the monodromy matrix $\Phi(\ell(s);\lambda,s) : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ of (1.8), where $\Phi(x;\lambda,s)$ is the fundamental matrix of (1.8) defined by

$$\frac{d}{dx}\Phi(x;\lambda,s) = (B(x;s) + \lambda K)\Phi(x;\lambda,s), \quad \Phi(0;\lambda,s) = I_{2n}. \tag{1.11}$$

Then, λ becomes an eigenvalue of (1.8) if and only if $\Phi(\ell(s);\lambda,s)$ has an eigenvalue whose absolute value is equal to one.

Differentiating (1.5) with respect to x and s, we find that $\Phi(\ell(s);0,s)$ has a degenerate eigenvalue 1 (see [5, Lemma 3.1]). Moreover, we can show that if μ is an eigenvalue of $\Phi(\ell(s);\lambda,s)$, then $1/\mu$ is also an eigenvalue of $\Phi(\ell(s);\lambda,s)$ (see [5, Lemma 3.3]). Noting these facts, we consider conditions such that the degenerate eigenvalue 1 of $\Phi(\ell(s);\lambda,s)$ splits into two simple eigenvalues with the absolute value equal to one when λ moves from the origin into the right-half plane. In this case, the stationary solution is unstable with some spatially modulating unstable mode, and such instability is called the *Eckhaus instability* [2].

Here we introduce a functional that will play an important role in our stability analysis. Multiplying (1.5) by $Q\phi_x$ and integrating it, we see that

$$J[\phi] := \frac{1}{2}\langle D\phi_x, Q\phi_x\rangle + H(\phi)$$

is constant in x. In other words, $J[\phi] \equiv Const.$ is a first integral for the equation in (1.5). Thus we can define a functional $J(s)$ by

$$J(s) := J[\phi(\cdot; s)] = \frac{1}{2}\langle D\phi_x(x; s), Q\phi_x(x; s)\rangle + H(\phi(x; s)) \tag{1.12}$$

on the one-parameter family of stationary solutions $\phi(x; s)$ of (1.1). We note that $dJ(s)/ds$ is computed as

$$\begin{aligned} \frac{d}{ds}J(s) &= \langle D\phi_x(x; s),\ Q\phi_{xs}(x; s)\rangle - \langle D\phi_{xx}(x; s),\ Q\phi_s(x; s)\rangle \\ &= \langle D\phi_x(0; s),\ Q\phi_{xs}(0; s)\rangle - \langle D\phi_{xx}(0; s),\ Q\phi_s(0; s)\rangle. \end{aligned} \tag{1.13}$$

Now, our main result is roughly stated as follows (a precise statement of the result will be given in [5, Theorem 3.11]).

Theorem 1.1 *The Eckhaus instability occurs if and only if*

$$dJ/d\ell := \frac{dJ(s)}{ds} \Big/ \frac{d\ell(s)}{ds}$$

and

$$I(s) := \int_0^{\ell(s)} \langle T\phi_x(x; s), Q\phi_x(x; s)\rangle\, dx$$

have the same sign.

It should be noted that $\phi(x; s)$ is not required to be small in this theorem. For gradient systems, $I(s)$ is necessarily positive so that the positivity of $dJ/d\ell$ implies the instability. Therefore, a stability-instability transition must occur at extremal points of $J(s)$ when $\ell(s)$ is strictly monotone in s. Thus, a first integral $J(s)$ is an index of instability property of stationary solutions.

Remark 1.1 In [5], we find that *the phase diffusion constant,* which determine the Eckhaus instability, in the theoretical physics[4, 6] is given by

$$D_{\|} = -\ell(s)^2 \, \frac{dJ}{d\ell} \, I(s)^{-1}.$$

Therefore, we see that the sign of phase diffusion constant is determined by the sign of $dJ/d\ell$. The above formula is also obtained by the perturbative renormalization group approach to the study of the phase diffusion equation [11]. Recalling the definition of $dJ/d\ell$, this formula also implies that a first integral $J(s)$ is an index of instability property of stationary solutions when $\ell(s)$ is strictly monotone in s.

Our method for the proof of Theorem 1.1 is based on a careful analysis of the linearized eigenvalue problem (1.6) when λ varies around $\lambda = 0$. One of the advantages to introduce the gradient/skew-gradient structure is that the analysis of an adjoint system for (1.6) becomes easier than that for systems without gradient/skew-gradient structure. In fact, we can express solutions of the adjoint system for (1.8) by using solutions of (1.8). This advantage enables us to derive a rather explicit description for the behavior of eigenvalues of the monodromy matrix $\Phi(\ell(s); \lambda, s)$. For more details, see [5].

In the next section, we apply Theorem 1.1 to various systems such as a generalized Ginzburg-Landau equation, Swift-Hohenberg equation, and reaction-diffusion systems of activator-inhibitor type. They should be helpful for us to understand a mathematical implication of Theorem 1.1.

2 Applications

In this section, we give several applications of Theorem 1.1 to demonstrate its usefulness.

Application 1. (Ginzburg-Landau equation)

Consider the 1-D Ginzburg-Landau equation

$$w_t = w_{xx} + \mu w - |w|^2 w, \quad -\infty < x < \infty, \tag{2.1}$$

or equivalently,

$$u_t = u_{xx} + u(\mu - u^2 - v^2),$$
$$(2.2) \qquad v_t = v_{xx} + v(\mu - u^2 - v^2),$$

where $w = u + iv \in \mathbf{C}$ and $\mu > 0$ is a real parameter. It is known that (2.1) has a family of spatially periodic stationary solutions given by

$$(2.3) \qquad \phi(x; s) = \sqrt{\mu - s^2} \begin{pmatrix} \cos sx \\ \sin sx \end{pmatrix}, \quad 0 < s^2 < \mu.$$

It was shown in [4, 6] by using the formal perturbation method that the stationary solution is unstable if $\mu/3 < s^2 < \mu$. To prove this fact in a mathematically rigorous manner, routine and tiresome arguments [2] based on usual functional analysis approach were needed.

Here we will apply Theorem 1.1 to a generalized Ginzburg-Landau equation

$$w_t = w_{xx} + g(|w|^2)w, \quad -\infty < x < \infty,$$

or equivalently,

$$u_t = u_{xx} + g(u^2 + v^2)u,$$
$$(2.4) \qquad v_t = v_{xx} + g(u^2 + v^2)v,$$

where $w = u + iv \in \mathbf{C}$ and g is a smooth function. We immediately find that (2.4) is a gradient system with respect to the energy

$$E[w] := \frac{1}{2} \int \left\{ u_x^2 + v_x^2 - G(u^2 + v^2) \right\} dx,$$

where

$$G(z) := \int g(z) dz.$$

We also see that

$$J[w] := \frac{1}{2} \left\{ u_x^2 + v_x^2 + G(u^2 + v^2) \right\}$$

is constant in x if $w = (u, v)$ is a stationary solution of (2.4).

Suppose that (2.4) has a stationary solution of the form

$$(2.5) \qquad w = \phi(x; s) = a(s) \begin{pmatrix} \cos sx \\ \sin sx \end{pmatrix},$$

where $a(s)$ must satisfy

(2.6) $$-s^2 + g(a^2(s)) = 0.$$

Then, we have

$$J(s) = J[\phi(x;s)] = \frac{1}{2}\left\{s^2 a(s)^2 + G(a(s)^2)\right\}.$$

By using (2.6) and its differentiation with respect to s, we have

$$\begin{aligned}\frac{d}{ds}J(s) &= sa(s)^2 + (s^2 + g(a(s)^2))a(s)a'(s) \\ &= s\left\{a(s)^2 + \frac{2g(a(s)^2)}{g'(a(s)^2)}\right\}.\end{aligned}$$

Hence, by $\ell(s) = 2\pi/s$, we have

$$dJ/d\ell = -\frac{s^3}{2\pi}\left\{a(s)^2 + \frac{2g(a(s)^2)}{g'(a(s)^2)}\right\}.$$

On the other hand, we have

$$\int_0^{\ell(s)} \langle T\phi_x(x;s), Q\phi_x(x;s)\rangle dx = 2\pi s a(s)^2.$$

Thus we obtain the following result by applying Theorem 1.1.

Theorem 2.1 *Let $w = \phi(x;s)$ be a stationary solution of* (2.4) *given by* (2.5) . *If*

$$a(s)^2 + \frac{2g(a(s)^2)}{g'(a(s)^2)} < 0,$$

then $\phi(x;s)$ is unstable.

In the 1-D Ginzburg-Landau equation (2.2), taking $a(s)^2 = \mu - s^2$ and $g(z) = \mu - z$, and applying the above theorem, we see easily that $\phi(x;s)$ is unstable provided $\mu/3 < s^2 < \mu$.

Application 2. (Swift-Hohenberg equation)

Let us consider the 1-D Swift-Hohenberg equation

(2.7) $$u_t = \alpha u - (1 + \partial_{xx})^2 u - u^3, \quad -\infty < x < \infty.$$

Putting

$$v = u + u_{xx}, \tag{2.8}$$

(2.7) is rewritten as

$$\begin{aligned} u_t &= -v_{xx} + \alpha u - u^3 - v, \\ 0 &= u_{xx} + u - v. \end{aligned} \tag{2.9}$$

It is easy to see that the Swift-Hohenberg equation (2.9) is a gradient system with

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$H = H(u,v) = \frac{1}{2}\alpha u^2 - \frac{1}{4}u^4 - uv + \frac{1}{2}v^2,$$

and TQ is positive semi-definite.

The existence of a family of spatially periodic stationary solutions for (2.10) was studied in [2]. Suppose that $(1-s^2)^2 < \alpha$ and $2/5 < s^2 < 2$. Then, for sufficiently small ε, (2.7) has a solution of the form

$$u(x;s) = 2\varepsilon\cos(sx) + O(\varepsilon^3), \quad \varepsilon = \sqrt{\frac{\alpha-(1-s^2)^2}{3}}. \tag{2.10}$$

Noting (2.8) and (2.10), the system (2.9) has a periodic stationary solution of the form

$$\phi(x;s) = 2\sqrt{\frac{\alpha-(1-s^2)^2}{3}}\cos(sx)\begin{pmatrix} 1 \\ 1-s^2 \end{pmatrix} + h.o.t. \tag{2.11}$$

with the period $\ell(s) = 2\pi/s$. Then, neglecting the higher order terms and using the second equation of (1.13), we have

$$\frac{d}{ds}J(s) = \frac{8s^3}{3}(\alpha - 3(1-s^2)^2),$$

which yields

$$dJ/d\ell = -\frac{4s^5}{3\pi}(\alpha - 3(1-s^2)^2).$$

Moreover, direct calculation yields

$$\int_0^{\ell(s)} \langle T\phi_x(x;s), Q\phi_x(x;s)\rangle\, dx = \frac{2\pi s}{3}(\alpha - (1-s^2)^2) > 0.$$

Thus we have the following result by applying Theorem 1.1.

Theorem 2.2 *Let $(u,v) = \phi(x;s)$ be a stationary solution of* (2.9) *of the form* (2.11). *If $\alpha/3 < (1-s^2)^2 < \alpha$, then $\phi(x;s)$ is unstable.*

Figure 1 shows relations between the functional $J(s)$ and the existence and instability of stationary solutions from a view point of bifurcation theory. Notice that the Eckhaus instability criterion is given by $dJ(s)/ds = 0$.

The above approach may be applicable to the study of spatially periodic stationary solutions of the extended Fisher-Kolmogorov equation [10]

$$u_t = -\gamma u_{xxxx} + u_{xx} + u - u^3. \tag{2.12}$$

In fact, by putting

$$v = \gamma u_{xx} - \frac{1}{2}u,$$

(2.12) is rewritten as (1.1) with

$$T = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$H = H(u,v) = \frac{1}{8}(1+4\gamma)u^2 + \frac{1}{2}v^2 - \frac{1}{4}u^4 + \frac{1}{2}uv.$$

Notice that the extended Fisher-Kolmogorov equation is a gradient system.

Application 3. (Activator-inhibitor systems)

Let us consider the following reaction-diffusion system of activator-inhibitor type [8]

$$\begin{aligned} \tau_1 u_t &= d_1 u_{xx} + \alpha u - u^3 - v, \\ \tau_2 v_t &= d_2 v_{xx} + u - \gamma v, \end{aligned} \tag{2.13}$$

where $\tau_1, \tau_2, d_1, d_2, \alpha, \gamma > 0$. We find that the activator-inhibitor system (2.13) is a skew-gradient system with

$$T = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$H = H(u,v) = \frac{1}{2}\alpha u^2 - \frac{1}{4}u^4 - uv + \frac{1}{2}\gamma v^2.$$

Applying standard arguments based on the Liapunov-Schmidt method [7, 8], we can construct a family of spatially periodic stationary solutions of (2.13) when $0 < \alpha < \gamma$ and $\alpha\gamma < 1$ as follows: Let us define

$$\hat{d}_2(s^2) = \frac{1}{s^2(\alpha - d_1 s^2)} - \frac{\gamma}{s^2}. \tag{2.14}$$

Let $0 < s^2 < \alpha/d_1$ and $\hat{d}_2(s^2) < d_2$. Then, for (s^2, d_2) near $(s^2, \hat{d}_2(s^2))$, we have

$$\phi(x; s) = 2\sqrt{\frac{(d_2 - \hat{d}_2(s^2))s^2}{3}} \cos(sx) \begin{pmatrix} c(s) \\ c(s)^2 \end{pmatrix} + h.o.t. \tag{2.15}$$

where $c(s) := \alpha - d_1 s^2$. Neglecting higher order terms and noting $\ell(s) = 2\pi/s$, direct calculation yields

$$\int_0^{\ell(s)} \langle T\phi_x(x; s), Q\phi_x(x; s)\rangle dx = \frac{4\pi s^3}{3}(d_2 - \hat{d}_2)c(s)^2(\tau_1 - \tau_2 c(s)^2).$$

On the other hand, we can calculate dJ/ds in a manner similar to the Swift-Hohenberg equation, but it is extremely complicated. According to the *computer algebra*, the result turns out to be

$$\frac{d}{ds}J(s) = -\frac{4}{3}s^3 K(s^2, d_2),$$

where $K(s^2, d_2)$ is a polynomial of degree four in s^2 and quadratic degree in d_2. Hence we obtain

$$dJ/d\ell = \frac{2s^5}{3\pi}K(s^2, d_2).$$

Thus we have the following result.

Theorem 2.3 *Let $(u, v) = \phi(x; s)$ be a stationary solution of* (2.13) *of the form* (2.15). *Suppose that $K(s^2, d_2) > 0$ (resp. $K(s^2, d_2) < 0$). If $\tau_1 > \tau_2 c(s)^2$ (resp. $\tau_1 < \tau_2 c(s)^2$), then $(u, v) = \phi(x; s)$ is unstable.*

We can draw the null cline defined by $K(s^2, d_2) = 0$ on the (s^2, d_2) -plane by the help of computer algebra, and obtain a diagram as in Figure 1 when $\tau_1 > c(s)^2\tau_2$. Although our results does not necessarily guarantee the stability of bifurcating stationary solutions in the region $\{(s^2, d_2) \mid K(s^2, d_2) < 0\}$, numerical simulations suggest that these stationary solutions are stable when $\tau_1 > c(s)^2\tau_2$. Since the sign of $K(s^2, d_2)$ is independent of τ_1 and τ_2, this result implies the following by the aid of numerical simulations: the bifurcating stationary solutions are stable in the region $\{(s^2, d_2) \mid K(s^2, d_2) < 0\}$ when the ratio of time constant coefficients of activator and inhibitor τ_1/τ_2 is large, whereas these stationary solutions become unstable and various complicated behavior of solutions can be observed when τ_1/τ_2 is small. For example, when d_1 is small, the stationary solutions lose their stability, and there appear metastable patterns which can be constructed by the singular perturbation method [1, 9]. On the other hand, when d_1 is large, there appear oscillatory patterns which cannot be observed in gradient systems.

Finally, we briefly mention another example known as the Gierer-Meinhardt system [3]

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} - \alpha u + \frac{u^p}{v^q} + \sigma, \\ \tau v_t &= d v_{xx} - v + \frac{u^r}{v^s}, \end{aligned} \tag{2.16}$$

where the parameters are assumed to satisfy $p > 1,\ q, r > 0,\ s, \sigma \geq 0$ and

$$\frac{p-1}{q} < \frac{r}{s+1}.$$

Then there exists a unique positive spatially homogeneous stationary solution.

We immediately find that the Gierer-Meinhardt system is a skew-gradient system when $p+1 = r$ and $q+1 = s$. In fact, (2.16) is rewritten as (1.1) with

$$T = \begin{pmatrix} r & 0 \\ 0 & q\tau \end{pmatrix}, \quad D = \begin{pmatrix} r\varepsilon^2 & 0 \\ 0 & qd \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$H = H(u, v) = -\frac{\alpha r}{2}u^2 + \frac{q}{2}v^2 + \frac{u^r}{v^q} + r\sigma u.$$

Recalling [12], in a manner similar to (2.13), we can construct bifurcating stationary solutions with spatially periodic structure around the unique positive spatially homogeneous stationary solution. In a similar manner to the above, we can also obtain a criterion for the Eckhaus instability as in Theorem 2.3.

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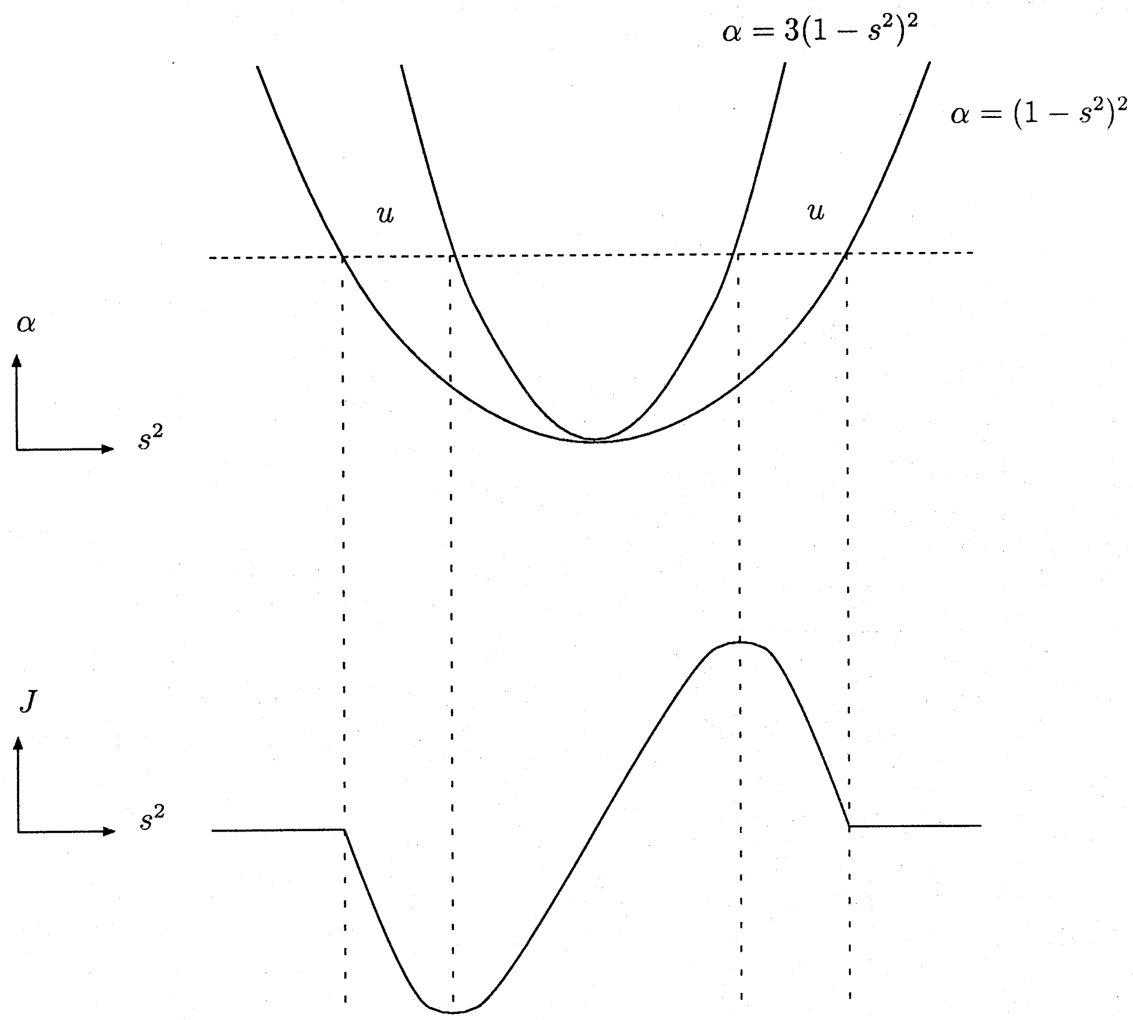

Figure 1