

# On the topological orbit equivalence in a class of substitution minimal systems

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In this note, a partial answer to the problem to characterize the topological orbit equivalence class of substitution minimal systems. The characterization is given in terms of the Perron-Frobenius eigenvalue of a matrix associated with a substitution.

## 1. TOPOLOGICAL ORBIT EQUIVALENCE IN CANTOR SYSTEMS

A topological dynamical system  $(X, \phi)$  is called a Cantor system if  $X$  is a Cantor set and  $\phi$  is a minimal homeomorphism on  $X$ .

**Definition 1.1.** Let  $(X, \phi)$  be a Cantor system. We put

$$\begin{aligned} \tilde{K}^0(\phi) &= C(X, \mathbb{Z})/Z_\phi \\ \tilde{K}_+^0(\phi) &= C(X, \mathbb{Z}_+)/Z_\phi \\ \tilde{u}_\phi &= [1] \end{aligned}$$

where  $C(X, \mathbb{Z})$  is the abelian group of continuous functions on  $X$  with integer values,  $[1]$  is the equivalence class of the constant function 1 by the subgroup  $Z_\phi$  and

$$Z_\phi = \{f \in C(X, \mathbb{Z}) \mid \int_X f d\mu = 0 \text{ for every } \phi\text{-inv. prob. meas. on } X\}.$$

**Definition 1.2.** Let  $(X_i, \phi_i)$  be Cantor systems for  $i = 1, 2$ .  $\phi_1$  and  $\phi_2$  are said to be topologically orbit equivalent if there exists a homeomorphism  $F : X_1 \rightarrow X_2$  such that  $F(\text{Orb}_{\phi_1}(x)) = \text{Orb}_{\phi_2}(F(x))$  for every  $x \in X_1$  where  $\text{Orb}_{\phi_i}(y)$  is the orbit of  $y$  by  $\phi_i$ .

**Theorem 1.3** ([GPS]). *The triple  $(\tilde{K}^0(\phi), \tilde{K}_+^0(\phi), \tilde{u}_\phi)$  is a complete invariant of the topological orbit equivalence in the class of Cantor systems.*

Put

$$\begin{aligned} X &= \prod_{i=1}^{\infty} \{0, 1, \dots, n_i\}, \quad n_i \geq 2, \\ \phi : X &\rightarrow X, \quad \text{the addition of } (1, 0, 0, \dots) \text{ with carries.} \end{aligned}$$

Then,  $(X, \phi)$  is a Cantor systems and called the odometer system with base  $(n_1, n_2, n_3, \dots)$ . The invariant  $\tilde{K}^0$  of  $\phi$  is the group  $\{l/m \mid l \in \mathbb{Z}, m \text{ divides some } \prod_{i=1}^k n_i\}$  and we denote the group of this form by  $\mathbb{Z}_{(q)}$  where  $q = \prod_{i=1}^{\infty} n_i$  as a formal product. The invariant  $\tilde{K}^0$  of  $\phi$  is  $(\mathbb{Z}_{(q)}, \mathbb{Z}_{(q)} \cap \mathbb{R}_+, 1)$  as the triple.

## 2. DEFINITION OF SUBSTITUTION SYSTEMS

Let  $A$  be an alphabet, i.e. a finite set, and  $A^+$  be the set of words on  $A$ .

**Definition 2.1.** A map  $\sigma : A \rightarrow A^+$  is called a substitution on  $A$ .

Let  $\sigma$  be a substitution on  $A$ . A substitution  $\sigma$  is naturally extended on  $A^+$  and  $A^{\mathbb{Z}}$ . We put  $\mathcal{L}(\sigma) = \{u \in A^+ | u \text{ occurs in some } \sigma^k(a), k \geq 1, a \in A\}$  and denote by  $M(\sigma)$  the  $A \times A$  matrix whose  $(a, b)$ -entry is the number of occurrences of  $b$  in  $\sigma(a)$  and call it the composition matrix of  $\sigma$ . A substitution  $\sigma$  is said to be of constant length if the length of  $\sigma(a)$  does not depend on the choice of  $a$  and to be primitive if there exists an integer  $k \geq 1$  such that for every  $a, b \in A$ ,  $a$  occurs in  $\sigma^k(b)$ , equivalently  $M(\sigma)$  is a primitive matrix.

**Remark 2.2.** As the alphabet  $A$  is a finite set, there exist an integer  $k \geq 1$  and letters  $a, b$  such that

1.  $a$  is a prefix of  $\sigma^k(a)$ ;
2.  $b$  is a suffix of  $\sigma^k(b)$ ;
3.  $ba \in \mathcal{L}(\sigma)$ .

Then  $x = \lim_{n \rightarrow \infty} \sigma^{kn}(b) \cdot \sigma^{kn}(a)$  converges in  $A^{\mathbb{Z}}$  where the dot means the separation between the  $-1$ -st coordinate and the  $0$ -th one.

**Remark 2.3.** We always assume that every substitution  $\sigma$  in this note satisfies the following conditions:

1. there exists a letter  $a$  such that  $\lim_{n \rightarrow \infty} \sigma^n(a) = \infty$ ;
2. a point  $x$  given as above is aperiodic.

Let  $T$  be a bilateral shift on  $A^{\mathbb{Z}}$  and  $X_\sigma$  be the closure of the orbit of  $x$  by  $T$ . Put  $T_\sigma = T|_{X_\sigma}$ .

**Definition 2.4.** The substitution system arising from a substitution  $\sigma$  is  $(X_\sigma, T_\sigma)$ .

**Proposition 2.5** ([Qu]). *If a substitution  $\sigma$  is primitive, then  $T_\sigma$  is uniquely ergodic and minimal.*

We always assume that every substitution in this note is primitive.

3. THE INVARIANT  $\tilde{K}^0(T_\sigma)$ .

**Definition 3.1.** A substitution  $\sigma$  is said to be proper if there exist an integer  $k \geq 1$  and letters  $a, b$  such that for every letter  $c$ ,  $a$  is a prefix of  $\sigma^k(c)$  and  $b$  is a suffix of  $\sigma^k(c)$ .

**Remark 3.2** ([DHS]). A proper substitution is not a special one from the view point of dynamical systems because for every substitution  $\sigma$  there exists a proper substitution  $\zeta$  such  $T_\zeta$  is topologically conjugate to  $T_\sigma$ .

We first consider the case where a substitution  $\sigma$  is proper.

**Definition 3.3.** We put

$$\begin{aligned} K^0(T_\sigma) &= \varinjlim (M(\sigma) : \mathbb{Z}^s \rightarrow \mathbb{Z}^s) \quad \text{where } s = |A|, \\ K_+^0(T_\sigma) &= \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}_+^s), \\ u_{T_\sigma} &= {}^t(1, \dots, 1), \end{aligned}$$

where  $\varphi_n$  is a natural homomorphism, which satisfies that  $\varphi_n = \varphi_{n+1}M(\sigma)$  and  $K^0(T_\sigma) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^s)$ . Define  $p_\sigma : K^0(T_\sigma) \rightarrow \mathbb{R}$  by  $p_\sigma(\varphi_n(a)) = \lambda^{-(n-1)}\alpha(a)$  for  $a \in \mathbb{Z}^s$  where  $\lambda$  is the Perron-Frobenius eigenvalue of  $M(\sigma)$  and  $\alpha$  is the left eigenvector corresponding to  $\lambda$  such that  $\sum_i \alpha_i = 1$ .

**Theorem 3.4** (From a result of [DHS]). *The invariant  $(\tilde{K}^0(T_\sigma), \tilde{K}_+^0(T_\sigma), \tilde{u}_{T_\sigma})$  defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system  $(X_\sigma, T_\sigma)$  is  $(K^0(T_\sigma)/\ker(p_\sigma), K_+^0(T_\sigma)/\ker(p_\sigma), p_\sigma(u_{T_\sigma})) = (\text{Im}(p_\sigma), \text{Im}(p_\sigma) \cap \mathbb{R}_+, 1)$ .*

Therefore, if  $\lambda$  is rational, i.e. integral, then  $\tilde{K}^0(T_\sigma) = \mathbb{Z}_{(d, \lambda^\infty)}$  for some integer  $d \geq 1$ .

Next, we consider the case where a substitution  $\sigma$  is not proper.

**Definition 3.5.** A word  $u \in \mathcal{L}(\sigma)$  is a return word to  $ba$ , where  $a$  and  $b$  are letters, if

1.  $a$  is a prefix of  $u$ .
2.  $b$  is a suffix of  $u$ .
3.  $bua \in \mathcal{L}(\sigma)$ .
4.  $ba$  occurs in  $bua$  only twice.

**Remark 3.6.** The number of return words is finite because of the minimality of  $T_\sigma$ . The length of a return word  $u$  to  $ba$  is the first return time to the cylinder set  $[b.a]$  of the points in the cylinder set  $[b.ua]$  where  $[u.v] = \{y \in X_\sigma | y_{[-|u|, |v|]} = uv\}$  for words  $u, v$ .

Fix an integer  $k \geq 1$  and letters  $a, b$  such that the conditions of Remark 2.2 hold. Put  $W = \{w_1, \dots, w_r\}$  indexed in order of occurrence without multiplicities in  $x_{[0, +\infty)}$ . Define a substitution  $\tau$  on the alphabet  $R = \{1, \dots, r\}$  by

$$\tau(i) = i_1 \dots i_l \quad \text{if } \sigma^k(w_i) = w_{i_1} \dots w_{i_l}.$$

**Proposition 3.7** ([DHS]). *The substitution  $\tau$  defined as above is primitive and proper. The substitution system arising from  $\tau$  is topologically conjugate to the induced transformation on  $[b.a]$  by  $T_\sigma$ .*

**Definition 3.8.** We put

$$\begin{aligned} K^0(T_\sigma) &= \varinjlim (M(\tau) : \mathbb{Z}^r \rightarrow \mathbb{Z}^r), \\ K_+^0(T_\sigma) &= \bigcup_{n=1}^{\infty} \psi_n(\mathbb{Z}_+^r), \\ u_{T_\sigma} &= {}^t(|w_1|, \dots, |w_r|), \end{aligned}$$

where  $\psi_n$  is a natural homomorphism as in Definition 3.3. Define  $p_\sigma : K^0(T_\sigma) \rightarrow \mathbb{R}$  by  $p_\sigma(\psi_n(a)) = \mu^{-(n-1)}\beta(a)$ ,  $a \in \mathbb{Z}^r$ , where  $\mu$  is the Perron-Frobenius eigenvalue of  $M(\tau)$  and  $\beta$  is the left eigenvector corresponding to  $\beta$  such that  $\sum_i \beta_i |w_i| = 1$ .

**Theorem 3.9** (From a result of [DHS]). *The invariant  $(\tilde{K}^0(T_\sigma), \tilde{K}_+^0(T_\sigma), \tilde{u}_{T_\sigma})$  defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system  $(X_\sigma, T_\sigma)$  is  $(K^0(T_\sigma)/\ker(p_\sigma), K_+^0(T_\sigma)/\ker(p_\sigma), p_\sigma(u_{T_\sigma})) = (\text{Im}(p_\sigma), \text{Im}(p_\sigma) \cap \mathbb{R}_+, 1)$ .*

Therefore, if  $\mu$  is integral, then  $\tilde{K}^0(T_\sigma) = \mathbb{Z}_{(d', \mu^\infty)}$  for some integer  $d' \geq 1$ .

**Remark 3.10.** Given a substitution  $\sigma$ , there exist an infinite graph and a partial order on the edge set of the graph which induces a minimal homeomorphism on the infinite path space which is topologically conjugate to  $T_\sigma$ . If  $\sigma$  is proper, then the connection rule between vertices in the corresponding graph is given by  $M(\sigma)$ . If  $\sigma$  is not proper, then the connection rule is given by  $M(\tau)$ . This is the reason why the way to compute the invariant  $\tilde{K}^0(T_\sigma)$  is different between the case where  $\sigma$  is proper and the case where  $\sigma$  is not proper. See [DHS] for more details.

**Theorem 3.11** ([Yu]). *Let  $\sigma$  be a substitution whose  $M(\sigma)$  has an integral Perron-Frobenius eigenvalue  $\lambda$ . Then, the substitution system arising from the substitution  $\sigma$  is topologically orbit equivalent to the odometer system with base  $(d, \lambda, \lambda, \dots)$  (called a stationary odometer system) for some integer  $d \geq 1$ . In particular, every substitution system arising from a substitution of constant length is topologically orbit equivalent to a stationary odometer system.*

Key lemma for the proof is the following.

**Lemma 3.12.**  $\mu = \lambda^k$ .

*Proof.* Let  $S$  be an  $R \times A$  matrix whose  $(a, i)$ -entry is the number of occurrences of  $a$  in  $w_i$ . Then  $SM(\sigma)^k = M(\tau)S$ . Therefore,  $\mu = \lambda^k$  because of the Perron-Frobenius Theorem.  $\square$

**Remark 3.13.** When  $\sigma$  is proper,  $d = \sum_i \alpha_i$  where  $\alpha = (\alpha_1, \dots, \alpha_s)$  is the left Perron-Frobenius eigenvector of  $M(\sigma)$  such that every  $\alpha_i$  is integral and  $(\alpha_i, \alpha_j) = 1$  if  $i \neq j$ . When  $\sigma$  is not proper,  $d' = \sum_i \beta_i |w_i|$  where  $\beta = (\beta_1, \dots, \beta_r)$  is the left Perron-Frobenius eigenvector of  $M(\tau)$  such that each  $\beta_i$  is integral and  $(\beta_i, \beta_j) = 1$  if  $i \neq j$ .

The converse of Theorem 3.11:

**Theorem 3.14** ([Yu]). *Let  $(X, \phi)$  be an arbitrary stationary odometer system and its base be  $(d, \lambda, \lambda, \dots)$ . Then, there exists a proper and primitive substitution  $\sigma$  of constant length such that  $T_\sigma$  is topologically orbit equivalent to  $\phi$ .*

*Proof.* We may assume that  $d > 1$  and  $\lambda > 1$ . It is enough to show that there exists a proper and primitive substitution  $\sigma$  of constant length  $\lambda^n$  on the alphabet  $\{1, \dots, d\}$  for some integer  $n \geq 1$ . Take  $n \geq 1$  such that  $\lambda^n > 3 \vee d$ . Put  $v = {}^t(\lambda^n, \lambda^n, \dots, \lambda^n, (\lambda^n - d + 1)\lambda^n)$ . Let  $M$  be the integral  $d \times d$  matrix whose  $(i, j)$ -entry is the  $\kappa^{j-1}(i)$ -th entry of  $v$  for  $1 \leq i, j \leq d$  where  $\kappa$  is the permutation on  $\{1, 2, \dots, d\}$  defined by  $\kappa(d) = 1$  and  $\kappa(i) = i + 1$  if  $1 \leq i < d$ . We can find a proper and primitive substitution  $\sigma$  such that  $M(\sigma) = M$ .  $\square$

## REFERENCES

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