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# On a conjecture by Palis

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## Introduction

Let $M$ be a smooth compact manifold without boundary and let $\mathrm{Diff}^1(M)$ be the set of $C^1$ diffeomorphisms with the $C^1$ topology. Aiming at the understanding of the complement of hyperbolic systems, Palis raised the following conjecture ([3]):

$C^1$ **Palis Conjecture.** Every diffeomorphism in $\mathrm{Diff}^1(M)$ can be approximated by an Axiom A diffeomorphism or else by one exhibiting a homoclinic bifurcation involving a homoclinic tangency or a cycle of hyperbolic periodic saddles with different indices.

This conjecture is true for surfaces by the work of Pujals and Sambarino ([4]). Since Axiom A diffeomorphisms which are not Morse-Smale ones have a transversal homoclinic point, one can state a weak version of this conjecture as follows:

**Conjecture (Palis).** The set of Morse-Smale diffeomorphisms together with the ones having a transversal homoclinic point forms a dense subset in $\mathrm{Diff}^1(M)$.

Sambarino announced that this is true in general at International Conference on Dynamical Systems held at IMPA (July, 2000). The theorem below is in the direction of these conjectures, which is not enough to prove even the weak version, but I believe that this is a promising step to the $C^1$ Palis Conjecture.

In the complement of the closure of the set of diffeomorphisms exhibiting a homoclinic bifurcation, we make a sort of hyperbolicity at least in an important part of $M$ for a dense subset of the complement. Here "a sort of hyperbolicity" means a dominated splitting with an additional property, "an important part" is the support of an ergodic invariant measure, and "dense subset" is $C^2$ Kupka-Smale diffeomorphisms. A dominated splitting on a compact invariant set $\Lambda$ is a continuous, $f$-invariant (i.e., invariant under the derivative of $f$) splitting

$$TM|\Lambda = E \oplus F$$

such that there exist $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ satisfying

$$\|(Df^m)|E(x)\| \cdot \|(Df^{-m})|F(f^m(x))\| < \lambda \tag{1}$$

for all $x \in \Lambda$. We show first the domination property of the Lyapunov splitting (which is originally just measurable) from the Oseledec's theorem and extend this property continuously to the support of an ergodic measure supported on infinitely many points by using an extended version of the Ergodic Closing Lemma. If $f \in \mathrm{Diff}^1(M)$, denote by $\Lambda(f)$ the set of points satisfying the following properties: there exists a splitting $T_xM = \bigoplus_{j=1}^{l} E_j(x)$ (the Lyapunov splitting at $x$) and numbers $\lambda_1(x) > \cdots > \lambda_l(x)$ (the Lyapunov exponents at $x$) such that $\lim_{n\to\pm\infty} \frac{1}{n} \log \|(D_xf^n)v\| = \lambda_j(x)$ for every $1 \le j \le l$ and $0 \ne v \in E_j(x)$. By Oseledec's theorem, $\mu(\Lambda(f)) = 1$ for every $f$-invariant probability measure $\mu$ on the Borel $\sigma$-algebra of $M$. Here $E_j(x)$ $(1 \le j \le l)$ are just measurable functions of $x$. Let $\mathcal{H}^1(M)$ be the set of $C^1$ diffeomorphisms exhibiting a homoclinic tangency and let $\mathcal{M}_e(f)$ be the set of ergodic probability measures of $f$.

**Theorem.** There is a dense subset $\mathcal{D}$ in the complement of $\overline{\mathcal{H}^1(M)}$ such that if $f \in \mathcal{D}$, for every $\mu \in \mathcal{M}_e(f)$ supported on infinitely many points, there exist dominated splittings

$$TM|\mathrm{supp}(\mu) = E^- \oplus F,$$

$$TM|\mathrm{supp}(\mu) = E \oplus F^+$$

such that

$$E^-(x) = \bigoplus_{\lambda_j(x)<0} E_j(x), \quad F(x) = \bigoplus_{\lambda_j(x)\ge 0} E_j(x),$$

$$E(x) = \bigoplus_{\lambda_j(x)\leq 0} E_j(x), \quad F^+(x) = \bigoplus_{\lambda_j(x)>0} E_j(x)$$

at $\mu$-a.e. $x$ of $\mathrm{supp}(\mu)$. Moreover, setting $E^0(x) = \bigoplus_{\lambda_j(x)=0} E_j(x)$, $\dim E^0(x) = 1$ when $E^0(x) \neq \{0\}$.

## I. Three lemmas

For the proof of the Theorem, we need three lemmas. The first lemma is an extended version of Mañé's Ergodic Closing Lemma ([2]). The proof is essentially the same as the original one. For any Borel subset $B$ of $M$, a neighborhood U of $f$ and $\varepsilon > 0$, let $\Sigma_B(\mathcal{U}, \varepsilon)$ be the set of points $x \in M$ satsisfying the following property: $\exists g \in \mathcal{U}, \exists y \in \mathrm{Per}(g) \cap B$ such that $d(f^i(x), g^i(y)) \leq \varepsilon$ for all $0 \leq i \leq k\}$, where $k$ is the $g$-period of $y$. Then, define $\Sigma_B(f) = \bigcap_{n\geq 1} \Sigma(\mathcal{U}_n, \varepsilon_n)$, where $\mathcal{U}_n$ is a bases of neighborhoods of $f$ and $\varepsilon_n > 0$ is a sequence of positive numbers converging to 0.

**Lemma 1 (An extended Ergodic Closing Lemma).** Given a Borel subset $B$ of $M$,

$$\mu(\Sigma_B(f)) \geq \mu(B)$$

for every $\mu \in \mathcal{M}_e(f)$.

The second is $C^1$ local perturbation lemma by Franks ([1]):

**Lemma 2 (Franks' Lemma).** For any neighborhood $\mathcal{U}$ of $f$, there exist $\varepsilon > 0$ and a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of $f$ such that given $g \in \mathcal{U}_0$, a finite set $\{x_1 \ldots, x_N\} \subset M$, a neighborhood $U$ of $\{x_1 \ldots, x_N\}$ and linear maps $L_l : T_{x_l}M \to T_{g(x_l)}M$ such that $\|L_l - D_{x_l}g)\| \leq \varepsilon$ for all $1 \leq l \leq N$, then there exists $\bar{g} \in \mathcal{U}$ such that $\bar{g}(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M - U)$ and $D_{x_l}\bar{g} = L_l$ for all $1 \leq l \leq N$.

The following lemma has been proved in [4] for the case of surfaces, and there is no essential difference between surfaces and the general case. Let $\alpha(E(p), F(p))$ be the angle between two subspaces $E(p)$ and $F(p)$ in $T_pM$.

**Lemma 3.** If $f \in \mathrm{Diff}^1(M) - \overline{\mathcal{H}^1(M)}$, then there exist a neighborhood $\mathcal{U}$ of $f$ and $c > 0$ such that

$$\alpha(E_g^s(p), E_g^u(p)) > c$$

for any hyperbolic periodic saddle $p$ of $g$ in $\mathcal{U}$.

## II. Outline of the proof of the Theorem

In this section we shall attempt to give an outline of the proof of the Theorem briefly. The proof will appear elsewhere. Let $f$ be a $C^2$ Kupka-Smale diffeomorphism and take $x \in \Lambda(f)$ such that $\mu_x = \mu$. Then, by hypothesis, $\mathcal{O}(x)$ is not a finite orbit and is recurrent. Since $f$ is $C^2$, it is known that both $E^-(x)$ and $E^+(x)$ cannot be the whole $T_xM$. So the remaining cases are:

*Case 1.* $T_xM = E^-(x) \oplus F(x)$;

*Case 2.* $T_xM = E(x) \oplus F^+(x)$;

*Case 3.* $T_xM = E^0(x)$.

Note that $F^+(x)$ in $F(x)$ and $E^-(x)$ in $E(x)$ might be trivial. Since $\mathcal{O}(x)$ is dense in $\mathrm{supp}(\mu)$, it is enough to show the domination property (1) of the splittings in the cases above over $\mathcal{O}(x)$, and Case 3 does not occur. Let us first consider Case 1. If $E^-(x) \oplus F(x)$ does not have the property (1), there exist $x_0 \in \mathcal{O}(x)$ and arbitrarily large $n_0 \in \mathbf{Z}^+$ such that

$$\|(Df^n)|E^-(x_0)\| \cdot \|(Df^{-n})|F(f^n(x_0))\| \geq \frac{1}{2}$$

for all $1 \leq n \leq n_0$. As Mañé's argument in [2], using Franks' Lemma (Lemma 2), we can push $E$ gradually toward $F$ until they have a small angle at some point in $\gamma = \{f^j(x_0) : 0 \leq j \leq n_0\}$ for a diffeomorphism $C^1$ close to $f$ and coinciding with $f$ on $\gamma$ just by changing $f$ in an arbitrarily small neighborhood of $\gamma$. Since expansion on $F$ by the iteration of $Df$ is stronger than that on $E^-$, the small angle is recovered by sufficiently large number of the positive iterates. Then, by using Lemma 2 again, we can move the subspace to the original position at some point, and make expansion on $F$ after the point (if necessary) so that some large number of positive iterates by the derivative of a diffeomorphism $C^1$ close to $f$ have enough contraction on $E^-(x_0)$ and expansion on $F(x_0)$, to have a hyperbolic periodic saddle with $E^s(x_0) = E^-(x_0)$ and $E^u(x_0) = F(x_0)$ when a finite part of $\mathcal{O}(x_0)$ is close up preserving the splittings. However, by the lack of continuity of the Lyapunov splittings, it is not always possible to close $E^-$ and $F$ as well as a finite part of $\mathcal{O}(x_0)$. In order to overcome this, we need an extended version of the Ergodic Closing Lemma (Lemma 1). In fact, we can take a Borel set $B_0$ with $\mu(B_0)$ arbitrarily close to 1 such that the Lyapunov splitting is continuous on $B_0$. Therefore, taking $x$ above in advance from $\Lambda(f) \cap B_0 \cap \Sigma_{B_0}(f)$, we obtain the required periodic orbit, which contradicts to Lemma 3 provided that the small angle has been made sufficiently small. Case 2 can be argued similarly by $f^{-1}$ instead of $f$.

Now let us consider Case 3. Apply Lemma 2 to make a root of unity as an eigenvalue of a periodic orbit after closing by the Ergodic Closing Lemma, which brings uncountable number of nonhyperbolic periodic points near $x$ for a diffeomorphism $C^1$ close to $f$. By changing one of them to a hyperbolic saddle, we have a sequence $\{f_n \in \mathrm{Diff}^1(M) : n \geq 1\}$ converging to $f$ such that each $f_n$ has a hyperbolic periodic saddle $p_n$ with $\lim_{n\to+\infty} p_n = x$. This gives invariant subbundles $\bar{E}$ and $\bar{F}$ over $\mathcal{O}(x)$ on $T_xM$ by accumulation of the stable and unstable subspaces of $\mathcal{O}(p_n)$. Note that, by Lemma 3, their angles are uniformly bounded away from 0 and therefore $\bar{E}(x) \oplus \bar{F}(x) = T_xM$. Since $T_xM = E^0(x)$, this splitting cannot have a domination property. Moreover, it is possible to attach $\bar{E}(x)$ and $\bar{F}(x)$ at $\mu$-a.e. $x$ so that the splitting becomes measurable and invariant. Then, similarly to the argument for Case 1, a small angle between the stable and unstable subspaces of a hyperbolic periodic saddle is created, contradicting Lemma 3. Thus it turned out that Case 3 never occurs.

Finally, let us mension the case when dim $E^0(x) > 1$. We have already obtained the both types of dominated splittings in the Theorem, so $E^-(x) \oplus F(x)$ and $E(x) \oplus F^+(x)$ have always good angles at $\mu$-a.e. $x$, which enable us to concentrate our attention only to $E^0$ and forget the other kind of subbundles. Hence, by the same argument as in Case 3, creating a small angle between stable and unstable subspaces in $E^0$ leads to a contradiction and shows that dim $E^0(x) = 1$.

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