Numerical integration of stochastic differential equations by composition methods

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Abstract

In this paper, "composition methods (or operator splitting methods) "for autonomous stochastic differential equations (SDEs) are formulated to make numerical approximation schemes for the equations. In the methods, the exponential map, which is given by solution of a stochastic differential equation, is approximated by composition of the stochastic flows derived from simpler and exact integrable vector field operators having stochastic coefficients. The local errors of the numerical schemes derived from the stochastic composition methods are investigated in detail. The new schemes are advantageous to preserve the special character of SDEs numerically and are useful for approximations of the solutions to stochastic non-linear equations. To examine the superiority, several numerical simulations on the basis of the schemes are carried out for stochastic differential equations which are treated in the mathematical finance and stochastic Hamilton dynamical systems.

Key words. stochastic differential equations, Lie algebra, composition methods (operator splitting method), mathematical finance, stochastic Hamilton dynamical systems, stochastic non-linear system

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1   Introduction

The theory of stochastic differential equations is understood as a fundamental tool for the description of random-phenomena treated in physics, engineering, economics and mathe- matical finance. However, it is often difficult to obtain the solutions of stochastic differential equations explicitly. Hence, there has been increasing interest in numerical analysis of stochastic differential equations, and many numerical schemes for stochastic differential equations has been formulated (*e.g.* Gard (1988), Kloeden and Platen (1992), Saito and Mitsui (1993)).

The purpose of the present paper is to propose some new numerical schemes for autonomous stochastic differential equations on the basis of "composition methods". The reasons why we address such a topic are as follows:

In numerical analysis for deterministic ordinary differential equations, whether or not some special character or structure of the equations is preserved precisely is an important point in performing reliable numerical calculations. From this point of view, various numerical methods to realize the characters of differential equations have been investigated. Indeed, we can find out such examples as energy conservative methods (Greenspan (1984), Ishimori (1994)), symplectic integrators for Hamilton dynamical systems (Suzuki (1990), Forest and Ruth (1990), Yoshida (1990)), and composition methods (McLachlan (1995), Moreau and Vandewalle (1996)). Particularly, the composition methods are useful to make numerical schemes which leave some structure or character of general differential equations numerically invariant. Hence, it seems to be quite natural that we investigate the methods for stochastic differen- tial equations to make numerical schemes having the conservation properties; this is the first reason for setting up the purpose mentioned above.

Moreover, in the theory of differential equations, composition methods are also known as operator-splitting methods, and they are often utilized for approximations of non-linear equation of which solutions are not obtained explicitly (Yanenko (1959), Iserles (1984)). In consideration of this, we may expect that the methods bring us a convenient and powerful way of approximations for "stochastic non-linear" differential equations, and this is another reason for our purpose.

Here, we will outline the original composition methods for ordinary differential equations. Let X denote vector fields on some space with coordinates x, with flows $\exp(tX)$, that is, the solutions of differential equations of the form $\dot{x}(t) = X(x)$ are given by the form $x(t) = \exp(tX)(x(0))$. Then, the vector field X is to be integrated numerically with fixed time step t. In the framework, we can apply composition methods to the differential equation, if one can write $X = A + B$ in such a way that $\exp(tA)$ and $\exp(tB)$ can both be calculated *explicitly* (more generally, this can be relaxed by approximations of the exponential maps). In the most elementary case, the method gives the approximation for $x(t)$ through

$$\tilde{x}(t) = \exp(tA)\exp(tB)(x) = x(t) + O(t^2);$$

the last equality is shown by using Baker-Campbell-Hausdorff (BCH) formula in Lie algebraic theory.

Thus, in composition methods, we use the exponential representation of solutions to differential equations as an important tool. To formulate the methods for stochastic differential equations, therefore, one needs the same notion for the equations. In Kunita (1980), such a topic has been investigated in detail. Hence, Section 2 is devoted to review his work and to

set up some notations of stochastic differential equations and vector fields. In addition, to estimate approximation errors of our new numerical schemes, we will prove some propositions with respect to time asymptotics of multiple stochastic integrals.

On the basis of the results in Section 2, composition methods are formulated for autonomous stochastic differential equations in Section 3. Through the methods, we will obtain some numerical schemes for the stochastic equations. Then, the approximation-error of numerical solutions derived from the schemes must be estimated. In this paper, as the first step, we apply the local error estimation in mean-square sense, which has established by Saito and Mitsui (1993), to the obtainable numerical solutions. BCH formula will be useful for calculating the errors as in case of composition methods for deterministic differential equations.

In Section 4, to examine the superiority of the new schemes, we investigate some examples of the numerical simulations on the basis of the schemes. In the first example, the following non-linear scalar stochastic differential equation is treated, which is often adopted as a model of an asset price process in mathematical finance (Geman and Yor (1993)):

$$dS(t) = S(t)dt + 2\sqrt{S(t)} \circ dW(t), \quad S(0) = s(> 0). \tag{1.1}$$

It is known that a solution $S(t)$ to this equation takes always a "non-negative "value for any $t \in [0, T]$. Through the standard stochastic numerical schemes, however, such a character of this equation is not always preserved numerically. In contrast with this, we will see that our new schemes by composition methods leave the structure invariant numerically, and thereby we can examine the first advantage of composition methods. In the second and third examples, we investigate the second advantage of composition methods as a tool of approximation for stochastic non-linear systems. Particularly, in case of deterministic differential equations, such a superiority is often found out in Hamilton dynamical systems as dimensional splitting methods. Therefore, in the third example, we treat the composition methods for "stochastic "Hamilton systems (Misawa (1999)). In the end of the section, we further touch upon a way to make numerical schemes which realize the numerical preservation of conserved quantities for stochastic systems (Misawa (2000)) .

Finally, some concluding remarks and future problems are given in Section 5.

2 Representation of solutions of SDEs

Now, we start with a review of representation of solutions of stochastic differential equations (SDEs) in the framework of Kunita's work (Kunita (1980)). Let us consider an autonomous SDE of Stratonovich type (*e.g.* Ikeda and Watanabe (1989), Arnold (1973)) under the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

$$dS(t) = b(S(t))dt + \sum_{j=1}^{r} g_j(S(t)) \circ dW^j(t) \tag{2.1}$$

defined on a connected C^∞-manifold M of dimension d, where $b = (b^i)_{i=1}^d$ and $g_j = (g_j^i)_{i=1}^d$ $(j = 1, \cdots, r)$ are d-dimensional C^∞ functions on M, respectively, and $W(t) = (W^1(t), \cdots W^r(t))$ is a standard Wiener Process. Here $S(t)$ is assumed to be adapted with a non-decreasing family of sigma-algebra $(\mathcal{F}_t)_{t \geq 0} \subset \mathcal{F}$. Note that Eq.(2.1) is rewritten in the form of SDE of Itô type as follows:

$$dS_t = \{b(S(t)) + \frac{1}{2} \sum_{k=1}^r \sum_{i=1}^d g_k^i \partial_i g_k(S(t))\}dt + \sum_{j=1}^r g_j(S(t))dW^j(t), \qquad (2.2)$$

where $\partial_i = \partial/\partial S^i$. Using the coefficient-functions in (2.1), we define C^∞-vector fields $X_0, X_1, \cdots, X_r$ as follows:

$$X_0 = \sum_{i=1}^d b^i \partial_i, \quad X_j = \sum_{i=1}^d g_j^i \partial_i \quad (j = 1, \cdots, r). \qquad (2.3)$$

The proof of Kunita's lemma (lemma 2.1 in Kunita (1980)) suggests that the solution of SDE (2.1) with an initial value $S(0) = s$ is *formally* represented as

$$S(t) = (\exp Y_t)(s), \qquad (2.4)$$

where $Y_t(\omega)$ is the vector field for each t and a.s. $\omega \in \Omega$ given by

$$
\begin{aligned}
Y_t &= \sum_{i=0}^r W^i(t)X_i + \frac{1}{2}\sum_{i<j}^r [W^i, W^j](t)[X_i, X_j] + \frac{1}{36}\sum_{i=0}^r \sum_{j=1}^r tW^i(t)[[X_i, X_j], X_j] \\
&+ \frac{1}{18}\sum_{i<j,k}^r [[W^i, W^j], W^k](t)[[X_i, X_j], X_k] \\
&+ \sum_{J;4\leq|J|} \{\sum_{\Delta J}^* c_{\Delta J} W^{\Delta J}(t)\}X^J, \quad (i = 0, 1, \cdots, r; j, k = 1, \cdots, r).
\end{aligned} \qquad (2.5)
$$

In (2.5), $[X_i, X_j]$ is the Lie bracket defined by $X_i X_j - X_j X_i$, and $X^J = [\cdots[X_{j_1}, X_{j_2}]\cdots]X_{j_m}]$ $(J = (j_1, \cdots, j_m))$. Moreover, we set $W^0(t) = t$ and define $[W^i, W^j](t)$ and $[[W^i, W^j], W^k](t)$ as multiple Wiener-Stratonovich integrals of degrees equal to 2 and 3 given by

$$[W^i, W^j](t) = \int_0^t W^i(\tau) \circ dW^j(\tau) - \int_0^t W^j(\tau) \circ dW^i(\tau),$$

and

$$[[W^i, W^j], W^k](t) = \int_0^t [W^i, W^j](\tau) \circ dW^k(\tau) - \int_0^t W^k(\tau) \circ d[W^i, W^j](\tau),$$

respectively. Moreover, $|J|$ denotes the length of a multi-index J; that is, if $J = (j_1, \cdots, j_m)$, then $|J| = m$. $W^{\Delta J}(t)\}$ and $c_{\Delta J}$ are the multiple Wiener-Stratonovich integrals with respect to $(W^0(t), W^1(t), \cdots, W^r(t))$ and the constant coefficients which are determined from a single or double divided index ΔJ of J, respectively, and $\sum_{\Delta J}^*$ denotes the sum for all single and double

divided indices of J; on the details of the definitions of ΔJ, $W^{\Delta J}(t)$ and $c_{\Delta J}$, see pp.285-289 in Kunita's paper (1980).

The equation (2.4) with (2.5) mean that the solution $S(t, \omega)$ equals $\phi(1, s, \omega)$ a.s., where $\phi(\tau, s, \omega)$ is the solution of the *ordinary* differential equation

$$\frac{d\phi(\tau)}{d\tau} = Y_t(\omega)(\phi(\tau)), \quad \phi(0) = s \tag{2.6}$$

regarding t and ω as parameters.

Remark 2.1: If the Lie algebra generated by $X_0, X_1, \cdots, X_r$ is of finite dimension, Ben Arous has proved that the stochastic infinite series in Eq.(2.5) actually converges before a stopping time. Therefore, in the case, the representation of solution (2.4) with (2.5) is well-defined (Theorem 20 in Ben Arous (1989)). We will see such examples in 4.1 and 4.2 of Section 4.

Now, in what follows, we suppose that one may obtain the explicit representation of solution (2.4) with (2.5). Moreover, we restrict ourselves to the SDEs (2.1) with a one-dimensional Wiener process; that is, we consider

$$dS(t) = b(S(t))dt + g(S(t)) \circ dW(t), \tag{2.7}$$

where b and g are d-dimensional vector-valued C^∞ functions. Then, we may rewrite (2.5) in the more simpler form in terms of Kloeden-Platen's representation for multiple Wiener-Stratonovich integrals and multiple Wiener-Itô ones (Kloeden and Platen (1989)); they are defined by

$$J_{(\alpha)}(t, s) = \int_s^{s+t} \int_s^{\tau_k} \cdots \int_s^{\tau_2} \circ dY^{(j_1)}(\tau_1) \cdots \circ dY^{(j_{k-1})}(\tau_{k-1}) \circ dY^{(j_k)}(\tau_k) \tag{2.8}$$

$$I_{(\alpha)}(t, s) = \int_s^{s+t} \int_s^{\tau_k} \cdots \int_s^{\tau_2} dY^{(j_1)}(\tau_1) \cdots dY^{(j_{k-1})}(\tau_{k-1}) dY^{(j_k)}(\tau_k), \tag{2.9}$$

respectively, where $\alpha = (j_1, \cdots, j_k)$, $(j_i = 0, 1; i = 1, \cdots, k)$ and

$$dY^{(j)}(u) = \begin{cases} du & \text{for } j = 0 \\ dW(u) & \text{for } j = 1 \end{cases}$$

In what follows, we denote $J_{(\alpha)}(t, s)$ and $I_\alpha(t, s)$ by $J_{(\alpha)}(t)$ and $I_\alpha(t)$, respectively, if $s = 0$

Remark 2.2: Note that $J_{(\alpha)}(t, s)$ can be rewritten by $I_{(\alpha)}(t, s)$ (see pp.174-175 in Kloeden and Platen's book (1992)). For example, we have

$$J_{(j_1)} = I_{(j_1)} \quad (j_1 = 0, 1) \tag{2.10}$$

$$J_{(j_1, j_2)} = I_{(j_1, j_2)} + \frac{1}{2} 1_{\{j_1 = j_2 = 1\}} I_{(0)} \quad (j_1, j_2 = 0, 1) \tag{2.11}$$

$$J_{(j_1,j_2,j_3)} = I_{(j_1,j_2,j_3)} + \frac{1}{2}(1_{\{j_1=j_2=1\}}I_{(0,j_3)} + 1_{\{j_2=j_3=1\}}I_{(j_1,0)}) \quad (j_1,j_2,j_3 = 0,1), \qquad (2.12)$$

where $1_{\{.\}}$ denotes the defining function. In general, any multiple Stratonovich integral $J_{(\alpha)}$ can be written as a multiple Itô integral $I_{(\alpha)}$ or a finite sum of $I_{(\alpha)}$ and multiple Itô integrals $I_{(\beta)}$ satisfying

$$\ell(\alpha) + n(\alpha) \le \ell(\beta) + n(\beta), \qquad (2.13)$$

where $\ell(\alpha) = \{$the number of elements of $\alpha\}$ and $n(\alpha) = \{$the number of 0 in the elements of $\alpha\}$ (Remark 5.2.8 in Kloeden and Platen (1992)).

In terms of (2.8) and stochastic Stratonovich integration by parts formula for $W^i(t)$ $(i = 0,1)$ and $[W^0, W^1](t)$, we can rewrite the equation (2.5) into the following form:

$$
\begin{aligned}
Y_t &= J_{(0)}(t)X_0 + J_{(1)}(t)X_1 + \frac{1}{2}(J_{(0,1)}(t) - J_{(1,0)}(t))[X_0, X_1] \\
&+ \frac{1}{18}\{2J_{(0,1,0)}(t) - 2J_{(1,0,0)}(t) + J_{(0)}(t)J_{(1,0)}(t) - J_{(0)}(t)J_{(0,1)}(t)\}[[X_0, X_1], X_0] \\
&+ \frac{1}{18}\{2J_{(0,1,1)}(t) - 2J_{(1,0,1)}(t) + J_{(1)}(t)J_{(1,0)}(t) - J_{(1)}(t)J_{(0,1)}(t)\}[[X_0, X_1], X_1] \\
&+ \frac{1}{36}\{J_{(0)}(t)\}^2[[X_0, X_1], X_1] + \sum_{J;4\le|J|} K^J(t)X^J.
\end{aligned}
\qquad (2.14)
$$

Here in the last term of the right-hand side of the above equation, $K^J(t) = \{\sum_{\Delta J}{}^* c_{\Delta J}W^{\Delta J}(t)\}$, and $J = (j_1, \cdots, j_\ell)$ $(j_i = 0,1; i = 1, \cdots, \ell; \ell \ge 4)$. In terms of Remark 2.2, this can be described by multiple Wiener-Itô integrals as follows:

$$
\begin{aligned}
Y_t &= I_{(0)}(t)X_0 + I_{(1)}(t)X_1 + \frac{1}{2}(I_{(0,1)}(t) - I_{(1,0)}(t))[X_0, X_1] \\
&+ \frac{1}{18}\{(2I_{(0,1,0)}(t) - 2I_{(1,0,0)}(t) + I_{(0)}(t)I_{(1,0)}(t) - I_{(0)}(t)I_{(0,1)}(t))[[X_0, X_1], X_0] \\
&+ \frac{1}{18}\{(2I_{(0,1,1)}(t) - 2I_{(1,0,1)}(t) + I_{(1)}(t)I_{(1,0)}(t) - I_{(1)}(t)I_{(0,1)}(t))[[X_0, X_1], X_1] \\
&+ \frac{1}{36}\{I_{(0,0)}(t) + \{I_{(0)}(t)\}^2\}[[X_0, X_1], X_1] + \sum_{J;4\le|J|} H^J(t)X^J,
\end{aligned}
\qquad (2.15)
$$

where $H^J(t)$ is another version of $K^J(t)$ under each multi-index J, which is derived by transforming multiple Stratonovich integrals in $K^J(t)$ into Itô ones through Remark 2.2. Therefore, for each multi-index J, $H^J(t)$ is described as a polynomial function of multiple-Itô integrals. In the next section, we will formulate the numerical schemes of SDE (2.7) on the basis of (2.4) with (2.15).

Now, in the remainder of this section, as a preparation for the error estimation to the new schemes in the next section, we will prove a proposition concerning with the coefficients $H^J(t)$ of X^J for multiple indeces J in (2.15). For this purpose, we first give a lemma for multiple Itô integrals (2.9) proved by Kloeden and Platen (lemma 5.7.5 in Kloeden and Platen (1992); Gard

(1988)). Let $E[\cdot|\mathcal{F}_s]$ be the conditional expectation with respect to a non-decreasing family of σ-subalgebra $\mathcal{F}_s$.

Lemma 2.1 : For any $\alpha = (j_1, \cdots, j_k), (j_i = 0, 1; i = 1, \cdots, k)$ and $q = 1, 2, \cdots$,

$$E[|I_{(\alpha)}(\Delta t, s)|^{2q}|\mathcal{F}_s] = O((\Delta t)^{q(\ell(\alpha)+n(\alpha))}) \quad (\Delta t \downarrow 0), \tag{2.16}$$

where $\ell(\alpha)$ and $n(\alpha)$ are the indices defined in Remark 2.2; that is, $\ell(\alpha)$ ={the number of elements of α} and $n(\alpha)$ ={the number of 0 in the elements of α}

Let $F(t, s)$ be a function of multiple stochastic integrals $I_{(\alpha)}(t, s)$. Suppose that

$$E[\{F(\Delta t, s)\}^2|\mathcal{F}_s] = O((\Delta t)^m) \quad (\Delta t \downarrow 0) \tag{2.17}$$

holds. Then, in what follows, we call the real number m "*mean-square order (MSO)* "of $F(t, s)$.

From Lemma 2.1 we see that MSO of a multiple Itô integral $I_{(\alpha)}(t, s)$ is equal to $\ell(\alpha) + n(\alpha)$. Moreover, the following lemma shows MSO of $I_{(\alpha)}(t, s)I_{(\beta)}(t, s)$ is given by $\ell(\alpha) + n(\alpha) + \ell(\beta) + n(\beta)$; that is, MSO of a product of multiple Itô integrals equals to the sum of MSOs of the each stochastic integral.

Lemma 2.2 : For any multi-indeces α and β,

$$E[|I_{(\alpha)}(\Delta t, s)I_{(\beta)}(\Delta t, s)|^2|\mathcal{F}_s] = O((\Delta t)^{(\ell(\alpha)+n(\alpha)+\ell(\beta)+n(\beta))}) \quad (\Delta t \downarrow 0). \tag{2.18}$$

holds.

Proof: Through Schwartz inequality, we obtain

$$E[|I_{(\alpha)}(\Delta t, s)I_{(\beta)}(\Delta t, s)|^2|\mathcal{F}_s] \leq \sqrt{E[\{I_{(\alpha)}(\Delta t, s)\}^4|\mathcal{F}_s]E[\{I_{(\beta)}(\Delta t, s)\}^4|\mathcal{F}_s]}.$$

The lemma is straightforwardly proved by this inequality and Lemma 2.1.

Remark 2.3: By Lemma 2.1 and Lemma 2.2 together with Remark 2.2, we may verify that multiple Stratonovich integrals $J_{(\alpha)}(t, s)$ also satisfy the results in Lemma 2.1 and Lemma 2.2. Hence, for a given α, we find that MSO of a multiple Stratonovich integral $J_{(\alpha)}(t, s)$ is also equal to $\ell(\alpha) + n(\alpha)$; that is, MSO of $J_{(\alpha)}(t, s)$ agrees with that of $I_{(\alpha)}(t, s)$ for the same multi-index α.

We are now to proceed to our purpose. Using Lemma 2.1 and Lemma 2.2, we can estimate MSO of the each coefficient for X^J in (2.15) which is given by a polynomial function of multiple Itô integrals on $[0, t]$. For example, in case of $|J| = 1$, MSOs of the coefficients for X_0 and X_1, that is, MSOs of $I_0(t)$ and $I_1(t)$ are equal to 2 and 1, respectively. In case of $|J| = 2$, MSO of $I_{(0,1)}(t)$ or $I_{(1,0)}(t)$ in the coefficient for $[X_0, X_1]$ is given by 3, and thereby we can easily

prove that MSO of the coefficient $(I_{(0,1)}(t) - I_{(1,0)}(t))/2$ itself is also equal to 3. In the same manner, we find MSOs of the coefficients for X^J when $|J| = 3$; that is, MSOs of the coefficients for $[X_0, X_1], X_0]$ and $[X_0, X_1], X_1]$ are given by 5 and 4, respectively. Hence, in this case, *the least value* of MSOs of the coefficients for X^J equals 4. These facts suggest we may verify the following proposition, and it is just one we want.

Proposition 2.1: Suppose that k is a given integer more than or equal to 2, and that the multi-indices J satisfy $|J| = k$; that is, $J = (j_1, \cdots, j_k)$ ($j_i = 0$ *or* $1; i = 1, \cdots, k$). Then the least value of MSOs of the coefficients $H^J(t)$ for X^J in (2.15) is equal to $k + 1$.

Proof: We may prove this by induction. From the above examples, this proposition is obvious in case of $|J| = 2, 3$. We assume that the assertion of this proposition holds for the case of $|J| = \ell(\geq 3)$. That is, the least value of MSOs of the coefficients $H^J(t)$ for X^J under $J = (j_1, \cdots, j_\ell)$ in (2.15) equals $\ell + 1$. Then, note that under for the same J, the least value of MSOs of $K^J(t)$ in (2.14) agrees with that of $H^J(t)$, since $H^J(t)$ is only another version of $K^J(t)$ in terms of multiple Itô integrals.

Now, let us consider the coefficients $K^{\tilde{J}}(t)$ for $X^{\tilde{J}}$ under $|\tilde{J}| = \ell + 1$. According to Kunita (1980) pp.285-289, one can obtain them by adding Stratonovich integral with respect to dw or dt to the multiple Stratonovich integrals in the coefficients $K^J(t)$ for $|J| = \ell$. From the assumption, the least value of MSOs of $K^J(t)$ for $|J| = \ell$ is equal to $\ell + 1$. On the other hand, the following equations show that the MSOs for the integrals by increments dw and dt correspond to 1 and 2, respectively:

$$E[|\int_s^{s+\Delta t} dw(\tau)|^2|\mathcal{F}_s] = O(\Delta t), \quad E[|\int_s^{s+\Delta t} dt|^2|\mathcal{F}_s] = O((\Delta t)^2) \ (\Delta t \downarrow 0).$$

Hence, in consideration these facts, one see that the least value of MSOs of $K^{\tilde{J}}(t)$ is given by $\ell + 1 + 1 = \ell + 2$, and thereby, the least value of MSOs of $H^{\tilde{J}}(t)$ is also so. Thus, the assertion in our proposition is proved.

3   Composition methods for numerical integration of SDEs

Now, we proceed to the new stochastic numerical schemes of SDEs on the basis of composition methods. We start with a numerical integration of the stochastic equation (2.7) on the discretized time series in the framework of the previous results on representation of solutions to SDEs. It adopts an equidistant discretization of the time interval $[0, T]$ with stepsize

$$\Delta t = \frac{T}{N}$$

for fixed natural number N. Let $t_n = n\Delta t (n = 0, 1, 2, \cdots, N)$ be the n-th step-point. Then, for all $n \in \{0, \cdots, N\}$, we abbreviate $S_n = S(t_n)$. Moreover, we use ΔW_n for $n = 0, 1, \cdots, N$ to

denote the increments $W(t_{n+1}) - W(t_n)$; they are independent Gaussian random variables with mean 0 and variance Δt, that is, $N(0, \Delta t)$-distributed random variables.

On account of (2.4), we may find the numerical solutions $S_n(n = 0, 1, \cdots, N)$ to SDE (2.7) by

$$S_{n+1} = \exp{(Y_{n\Delta t})}(S_n) \ (n = 0, 1, 2, \cdots N - 1), \tag{3.1}$$

formally, where $Y_{n\Delta t}$ is a vector field derived by replacing all the multiple Wiener integrals $I_{(\alpha)}(t) = I_{(\alpha)}(t, 0)$ in (2.15) by $I_{(\alpha)}(\Delta t, n\Delta t)$. In the followings, $I_{(\alpha)}(\Delta t, n\Delta t)$ is denoted by $I_{(\alpha), n}(\Delta t)$. Moreover, we set $S_0 = S(0) = s_0$. According to the theory of ordinary differential equations, $\exp{(Y_{n\Delta t})}(\cdot)$ is often called the time-Δt map or exponential map. However, it is usually difficult to find out the explicit form of the exponential map, and hence, we need to build an approximation for (3.1).

To carry out this, we formulate a new stochastic numerical scheme as the following two procedures, which are composed of the truncation of the vector field (2.15) and a composition method (or operator splitting method) to the exponential map derived from the truncated vector field:

Procedure 1: For the vector field Y_t described by (2.15), we define a "truncated" vector field $\hat{Y}_t$ which is given by a truncation of the higher-order terms with respect to MSO of the coefficients for X^J in (2.15). Then, we define a numerical sequence $(\hat{S}_n)_{n=0}^N$ through

$$\hat{S}_{n+1} = \exp(\hat{Y}_{n\Delta t})(\hat{S}_n) \ (n = 0, 1, \cdots . N - 1), \tag{3.2}$$

where $\hat{S}_0 = S(0) = s_0$.

Procedure 2: For $\hat{S}_{n+1} = \exp(\hat{Y}_{n\Delta t})(\hat{S}_n)$, we apply a "composition method" in a way analogous to that in the theory of ordinary differential equations. Suppose that the vector field $\hat{Y}_{n\Delta t}$ is of the form

$$\hat{Y}_{n\Delta t} = A_{n\Delta t} + B_{n\Delta t}, \tag{3.3}$$

where $\exp(A_{n\Delta t})$ and $\exp(B_{n\Delta t})$ can both be explicitly calculated through (2.6). Then an approximation to the exponential map $\hat{Y}_{n\Delta t}$ is given by $\exp(A_{n\Delta t}) \exp(B_{n\Delta t})$. Hence, the sequence of $(\hat{S}_n)_{n=0}^N$ in Procedure 1 is approximated by

$$\tilde{S}_{n+1} = \exp(A_{n\Delta t}) \exp(B_{n\Delta t})(\tilde{S}_n), \ (n = 0, 1, \cdots N - 1), \tag{3.4}$$

where $\tilde{S}_0 = S(0) = s_0$.

After all, we regard $(\tilde{S}_n)_{n=0}^N$ as a numerical approximation to the exact discretized solutions $(S_n)_{n=0}^N$.

Next, we will turn to estimate local errors of mean-square sense for the numerical approximation scheme mentioned above. For this purpose, in this paper, we use the notion of "local error order" defined by Saito and Mitsui (1993).

Definition 3.1: Suppose that $S(t)$ and $\bar{S}_n$ are an exact solution and the numerical approximation to SDE (2.7), respectively. Moreover, let $E_{\tau,\xi}$ be the expectation conditioned on starting at ξ at time τ. Then *the local error order* α is defined by

$$E_{t_n,s}[|\bar{S}_{n+1} - S(t_{n+1})|^2] = O((\Delta t)^{\alpha+1}), \quad (\Delta t \downarrow 0), \tag{3.5}$$

where $t_k = k\Delta t$ $(k = n, n+1)$, $|\cdot|$ denotes the Euclidean norm on the space R^d, and we set $S(t_n) = \bar{S}_n = s$.

We note that the accuracy of a numerical scheme improves with increasing the local order.

Remark 3.1: In the framework of another definition of local error order by Kloeden and Platen (1992), the local order of $\bar{S}_n$ satisfying (3.5) is given by $(\alpha+1)/2$, since the difference of $\bar{S}_{n+1}$ and $S(t_{n+1})$ is squared. Hence, if the above-mentioned local order for a certain numerical scheme is equal to α, one can calculate Kloeden and Platen's local error order β through $\beta = (\alpha+1)/2$.

Now, we will to apply thus local error estimation to our approximation procedures. In what follows, we set $S_n = s$, where s is a given value.

Local error estimation for the truncation error in Procedure 1:

First, we investigate a truncation error in Procedure 1. Let $H_n{}^J(\Delta t)$ be the coefficient for X^J in $Y_{n\Delta t}$ which is represented by a polynomial function of multiple-Itô integrals for a given multi-index J as in (2.15).

Proposition 3.1: Suppose that a truncation vector field $\hat{Y}_{n\Delta t}$ is given in the following form:

$$\hat{Y}_{n\Delta t} = \sum_{J; 1 \le |J| \le \gamma} H_n{}^J(\Delta t) X^J. \tag{3.6}$$

That is, we assume the terms in $Y_{n\Delta t}$ satisfying $|J| \ge \gamma + 1$ are neglected. Then,

$$E_{t_n,s}[|\hat{S}_{n+1} - S_{n+1}|^2] = O((\Delta t)^{\gamma+2}), \quad (\Delta t \downarrow 0). \tag{3.7}$$

Proof: In terms of Proposition 2.1, we can easily show that the least value of MSOs of $H_n{}^J(\Delta t)$ in the neglected terms equals $\gamma + 2$, since $|J| \ge \gamma + 1$. This fact together with the definition of exponential map (2.15) straightforwardly indicate (3.7).

Thus, we obtain the local order $\gamma + 1$ for numerical approximation solutions $(\hat{S}_n)_{n=0}^N$ in the sense of Definition 3.1.

Remark 3.2: If the Lie algebra generated by X_0 and X_1 is of finite dimension, our error estimation mentioned above agree with that on truncation of stochastic exponential maps by

Ben Arous (1989), since the convergence of Eq.(2.14) (or (2.15)) is actually guaranteed (*cf.* Remark 2.1). That is, under the assumption that such a convergence holds, the local error estimation mentioned above exactly holds. In general, however, our result may give only a formal error estimation. Indeed, according to Castell (1993), when the stochastic series does not always converge, the asymptotic expansion of stochastic exponential maps is estimated only in a "probability "sense. We will investigate this problem in future works.

Local error estimation for the composition scheme in Procedure 2:

Next, we will proceed to the local error estimation for Procedure 2. We can carry out it by the Baker-Campbell-Hausdorff (BCH) formula (Bourbaki (1989)) together with Lemma 2.2; the formula is given by the following form:

$$
\exp\left(\epsilon(\Delta t)X\right)\exp\left(\delta(\Delta t)Y\right) = \exp(\epsilon(\Delta t)X + \delta(\Delta t)Y + \frac{1}{2}\epsilon(\Delta t)\delta(\Delta t)[X,Y]
$$
$$
+\frac{1}{12}(\epsilon(\Delta t)^2\delta(\Delta t)[X,[X,Y]] + \epsilon(\Delta t)\delta(\Delta t)^2[Y,[Y,X]]) + \cdots), \tag{3.8}
$$

where X and Y are C^∞ vector fields, and $\epsilon(\Delta t)$ and $\delta(\Delta t)$ are any functions with respect to Δt; in our case, they are corresponding to polynomial functions of multiple Itô stochastic integrals $I_{(\alpha),n}(\Delta t)$.

Proposition 3.2: Let $\hat{Y}_{n\Delta t}$ be a truncated vector field given by (3.6). Suppose that the vector fields $A_{n\Delta t}$ and $B_{n\Delta t}$ in a decomposition (3.3) for $\hat{Y}_{n\Delta t}$ are described by

$$
A_{n\Delta t} = \sum_{J;1\leq|J|\leq\gamma} F_n{}^J(\Delta t)X^J, \quad B_{n\Delta t} = \sum_{J;1\leq|J|\leq\gamma} G_n{}^J(\Delta t)X^J, \tag{3.9}
$$

respectively, and that the least values of MSOs of $F_n{}^J(\Delta t)$ and $G_n{}^J(\Delta t)$ in (3.9) are given by α and β, respectively. If $X_A^{J\alpha}$ and $X_B^{J\beta}$, which are vector fields corresponding to the coefficients with α and β as MSO, respectively, satisfy $[X_A^{J\alpha}, X_B^{J\beta}] \neq 0$, then

$$
E_{t_n,s}[|\tilde{S}_{n+1} - \hat{S}_{n+1}|^2] = O((\Delta t)^{\alpha+\beta})), \quad (\Delta t \downarrow 0). \tag{3.10}
$$

Proof: Let $F_n{}^{J\alpha}(\Delta t)$ and $G_n{}^{J\beta}(\Delta t)$ be the coefficients in (3.9) of which MSOs are equal to α and β, respectively. Then, in an analogous way to that in Lemma 2.2, we can prove that

$$
E_{t_n,s}[|F_n{}^{J\alpha}(\Delta t)G_n{}^{J\beta}(\Delta t)|^2] = O((\Delta t)^{\alpha+\beta}) \quad (\Delta t \downarrow 0). \tag{3.11}
$$

Therefore, on account of BCH formula (3.8), (3.11) and the assumption of α and β, one may find that

$$
\begin{aligned}
E_{t_n,s}[|\tilde{S}_{n+1} - \hat{S}_{n+1}|^2] &= E_{t_n,s}[|\exp\left(A_{n\Delta t} + B_{n\Delta t}\right)(s) - \exp\left(A_{n\Delta t}\right)\exp\left(B_{n\Delta t}\right)(s)|^2] \\
&= E_{t_n,s}[|\exp\left(A_{n\Delta t} + B_{n\Delta t}\right)(s) - \exp\left(A_{n\Delta t} + B_{n\Delta t} + \frac{1}{2}[A_{n\Delta t}, B_{n\Delta t}] + \cdots)(s)|^2] \\
&= O((\Delta t)^{\alpha+\beta})).
\end{aligned} \tag{3.12}
$$

Thus, the local order between $(\tilde{S}_n)_{n=0}^N$ and $(\hat{S}_n)_{n=0}^N$ is given by $\alpha + \beta - 1$.

Remark 3.3: By further manipulating the BCH formula to eliminate higher order terms, we can obtain the schemes which give higher-order approximations to the exponential map. For example, the scheme corresponding to "leapfrog ", which is well-known in deterministic numerical analysis, is given by

$$\exp\left((\Delta t)(X+Y)\right) = \exp\left(\frac{\Delta t Y}{2}\right) \exp\left(\Delta t X\right) \exp\left(\frac{\Delta t Y}{2}\right) + O((\Delta t)^3). \tag{3.13}$$

In a way analogous to that in (3.4), we define a stochastic leapfrog scheme as follows:

$$\tilde{S}_{n+1} = \exp\left(\frac{B_n \Delta t}{2}\right) \exp\left(A_n \Delta t\right) \exp\left(\frac{B_n \Delta t}{2}\right)(\tilde{S}_n), \quad (n = 0, 1, \cdots, N-1). \tag{3.14}$$

Then, using BCH formula (3.8) and (3.11) repeatedly, we can estimate the local error for this scheme as follows:

$$E_{t_n,s}[|\tilde{S}_{n+1} - \hat{S}_{n+1}|^2] = O((\Delta t)^{\alpha + 2\beta}). \tag{3.15}$$

Thus, we can make another numerical scheme having the better local order than that of (3.4). Moreover, using this scheme as a basis element for further leapfrog schemes, we may also produce an approximation to exponential map up to any order in a similar way to that in ordinary numerical analysis. This will be investigated in the future work.

Total local error estimation for the numerical scheme by Procedure 1 and 2:

Finally, we estimate the local error order between the exact discretized solutions $(S_n)_{n=0}^N$ and the numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$. This is easily carried out by using Proposition 3.1 and Proposition 3.2 (or Remark 3.3) for the local orders in the above two procedures together with

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \leq E_{t_n,s}[|S_{n+1} - \hat{S}_{n+1}|^2] + E_{t_n,s}[|\hat{S}_{n+1} - \tilde{S}_{n+1}|^2], \tag{3.16}$$

and thereby, we obtain the following theorem:

Theorem 3.1: Under the conditions of Proposition 3.1 and Proposition 3.2,

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \leq O((\Delta t)^\delta) \ (\Delta t \downarrow 0) \tag{3.17}$$

holds, where $\delta = \min(\alpha + \beta, \gamma + 2)$ in case of (3.4) and $\delta = \min(\alpha, \beta + 2\gamma)$ in case of (3.14).

We call a value of $\delta - 1$ the local order "*of weak sense* "for the scheme giving the numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$, since the estimation of error order is indirectly derived from the inequality (3.16).

In the followings, we will investigate some examples of new numerical schemes for (2.7) which are derived from the procedures mentioned above, and estimate the local error on the basis of Theorem 3.1.

Example 3.1: Suppose that a truncated vector field $\hat{Y}_{n\Delta t}$ in Procedure 1 is given by

$$\hat{Y}_{n\Delta t} = I_{(0),n}(\Delta t)X_0 + I_{(1),n}(\Delta t)X_1,$$
$$= \Delta t X_0 + \Delta W_n X_1. \tag{3.18}$$

On account of (2.15), we see that γ in Proposition 3.1 for this truncated vector field equals 1. We further set $A_{n\Delta t} = \Delta t X_0$ and $B_{n\Delta t} = \Delta W_n X_1$ in the decomposition (3.3), and assume that the explicit forms of both exponential maps for them are obtained through (2.6). In this case, α and β in Proposition 3.2 become 2 and 1, respectively, because of Lemma 2.1. Then, the scheme (3.4) is put into the following form:

Scheme 3.1:

$$\tilde{S}_{n+1} = \exp(\Delta t X_0)\exp(\Delta W_n X_1)(\tilde{S}_n). \tag{3.19}$$

Assume that $[X_0, X_1] \neq 0$. Then, Theorem 3.1 indicates

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \leq O((\Delta t)^3). \tag{3.20}$$

Thus, we find that the local order of weak sense for Scheme 3.1 equals 2, and this result shows that the accuracy of this scheme corresponds to that of the stochastic Taylor schemes of local order 2 (Saito and Mitsui (1993)).

Example 3.2: For $\hat{Y}_n(\Delta t)$ in Example 3.1, we set $A_{n\Delta t} = \Delta t X^A_0 + \Delta W_n X^A_1$ and $B_{n\Delta t} = \Delta t X^B_0 + \Delta W_n X^B_1$ in (3.3), where $X_0 = X^A_0 + X^B_0$ and $X_1 = X^A_1 + X^B_1$. We assume that $[X^A_1, X^B_1] \neq 0$ and that the explicit forms of both exponential maps for them are obtained. In this case, α and β in Proposition 3.2 become 1 and 1, respectively, and hence the local error order of (3.4) becomes 1; the accuracy for the scheme corresponds to that of Euler-Maruyama scheme (Saito and Mitsui 1993). In order to make a scheme having better accuracy than that of it, we use (3.14) instead of (3.4):

Scheme 3.2:

$$\tilde{S}_{n+1} = \exp\left(\frac{B_{n\Delta t}}{2}\right)\exp(A_{n\Delta t})\exp\left(\frac{B_{n\Delta t}}{2}\right), \tag{3.21}$$

where $A_{n\Delta t} = \Delta t X^A_0 + \Delta W_n X^A_1$ and $B_{n\Delta t} = \Delta t X^B_0 + \Delta W_n X^B_1$ under $X_0 = X^A_0 + X^B_0$ and $X_1 = X^A_1 + X^B_1$.

Then, Theorem 3.1 indicates (3.20) also holds in this case; that is, the local error order of this scheme is equal to 2. This means that the accuracy of Scheme 3.2 corresponds to that of Taylor schemes of local order 2.

Example 3.3: We will make a scheme with more better accuracy than that of the schemes mentioned above. For this purpose, we choose the following vector field as $\hat{Y}_{n\Delta t}$ in (3.6):

$$\hat{Y}_{n\Delta t} = \Delta t X_0 + \Delta W_n X_1 + \frac{1}{2}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))[X_0, X_1]. \tag{3.22}$$

Then, from (2.15), we see that γ in Proposition 3.1 for this truncated vector field becomes 2. Moreover, we set

$$\exp(A_{n\Delta t}) = \exp(\Delta t X_0) \tag{3.23}$$

and

$$\exp(B_{n\Delta t}) = \exp(\Delta W_n X_1 + \frac{1}{2}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))[X_0, X_1]). \tag{3.24}$$

Assume that the explicit forms of both exponential maps for them are obtained through (2.6), respectively. In this case, α and β in Proposition 3.2 become 2 and 1, respectively, because of Lemma 2.1. Moreover, we adopt the scheme (3.14) for these vector fields:

Scheme 3.3:
$$\tilde{S}_{n+1} = \exp(\frac{B_{n\Delta t}}{2})\exp(A_{n\Delta t})\exp(\frac{B_{n\Delta t}}{2})(\tilde{S}_n), \tag{3.25}$$

where $\exp(A_{n\Delta t})$ is given by (3.23) and $\exp(B_{n\Delta t}/2)$ is derived from replacing $B_{n\Delta t}$ by $B_{n\Delta t}/2$ in (3.24).

Then, because of Theorem 3.1, we find that

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \le O((\Delta t)^4), \tag{3.26}$$

and hence that the local order of weak sense for Scheme 3.3 equals 3.

4 Examples

In this section, we will give several examples of applying our new stochastic numerical schemes to stochastic differential equations concretely.

4.1 Numerical simulation to a non-linear asset price process in mathematical finance

As was mentioned in Section 1, we first work with the following non-linear scalar SDE which is often treated as a model of an asset price process of Bessel type in mathematical finance (Geman and Yor (1993); *cf.* Remark 4.1):

$$dS(t) = S(t)dt + 2\sqrt{S(t)} \circ dW(t), \quad S(0) = s(> 0). \tag{4.1}$$

This system has a structure that the value of solution becomes to be "non-negative" for any $t \in [0,T]$. In standard stochastic numerical schemes, however, this is not always preserved numerical- ly; especially, if an initial value s is close to zero, the numerical solutions often go into the domain of negative values in the midst of numerical simulations. Such a trouble will be observed in the numerical results mentioned later. In contrast with this, through the results in

previous section, we may obtain the scheme which leaves the structure of the stochastic system (4.1) invariant numerically. We will examine it.

First, from the equations (2.3) and (4.1), we see that the vector fields X_0 and X_1 become

$$X_0 = S\frac{d}{dS}, \quad X_1 = 2\sqrt{S}\frac{d}{dS}, \tag{4.2}$$

respectively. Here, we note that $[X_0, X_1] = -X_1/2$, and the Lie algebra generated by X_0 and X_1 is of finite dimension. Hence, Remark 2.1 indicates that Eq.(2.14) and (2.15) actually converge in this case.

We proceed to investigate Scheme 3.1 in Example 3.1 to SDE (4.1). On account of (3.19), we suppose that $A_{n\Delta t}$ and $B_{n\Delta t}$ in (3.3) are given by

$$A_{n\Delta t} = \Delta t X_0 = \Delta t S\frac{d}{dS}, \quad B_{n\Delta t} = \Delta W_n X_1 = \Delta W_n 2\sqrt{S}\frac{d}{dS}. \tag{4.3}$$

Then, in consideration of (2.6), we obtain the exponential maps for $A_{n\Delta t}$ and $B_{n\Delta t}$ explicitly as follows:

$$\exp(A_{n\Delta t})(s) = s\exp(\Delta t), \quad \exp(B_{n\Delta t})(s) = \{\Delta W_n + \sqrt{s}\}^2. \tag{4.4}$$

Inserting them into (3.19), we find that Scheme 3.1 for SDE (4.1) is given by

$$\tilde{S}_{n+1} = \{\Delta W_n + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t), \tag{4.5}$$

where $\tilde{S}_0 = S(0) = s$. Evidently, the numerical solutions derived from our scheme "never" take negative values, and this is just a result we want.

Next we will obtain Scheme 3.3 for (4.1). On account of (3.23), in this case, we also obtain $s\exp(\Delta t)$ as $\exp(A_{n\Delta t})(s)$. In contrast with this, the equation (3.24) is put into

$$\exp(B_{n\Delta t}) = \exp\{(\Delta W_n - \frac{1}{4}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)))X_1\}, \tag{4.6}$$

since $[X_0, X_1] = -X_1/2$. In similar to way in that of Scheme 3.1, this is also calculated explicitly as follows:

$$\exp(B_{n\Delta t})(s) = \{\Delta W_n - \frac{1}{4}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{s}\}^2, \tag{4.7}$$

and thereby we obtain Scheme 3.3 for the SDE (4.1) as

$$\tilde{S}_{n+1} = \{\frac{\Delta W_n}{2} - \frac{1}{8}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{\hat{s}}\}^2 \tag{4.8}$$

together with

$$\hat{s} = \{\frac{\Delta W_n}{2} - \frac{1}{8}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t), \tag{4.9}$$

where $\tilde{S}_0 = S(0) = s$. This also indicates that the numerical solutions derived from this scheme take non-negative values.

Now, we proceed to the numerical simulations of (4.1) on the basis of our schemes. As mentioned in Section 3, Theorem 3.1 indicates the schemes (4.5) and (4.8) with (4.9) have the local order of weak sense 2 and 3, respectively. To examine these results, we will compare the numerical accuracy of our schemes with that of the standard numerical schemes. For this purpose, we adopt here Euler-Maruyama scheme (Taylor scheme of local order 1) and Kloeden's Taylor scheme of local order 3 (Kloeden and Platen (1992), Saito and Mitsui (1993)); they are given in the following forms for the SDE (4.1):

Euler-Maruyama scheme:

$$S_{n+1} = S_n + (S_n + 1)\Delta t + 2\sqrt{S_n}\Delta W_n. \tag{4.10}$$

Kloeden's Taylor scheme of local order 3:

$$
\begin{aligned}
S_{n+1} =\ & S_n + (S_n + 1)\Delta t + 2\sqrt{S_n}\Delta W_n \\
& + \{(\Delta W_n)^2 - \Delta t\} \\
& + 2\sqrt{S_n}I_{(1,0),n}(\Delta t) + \sqrt{S_n}I_{(0,1),n}(\Delta t) \\
& + \frac{1}{2}(S_n + 1)(\Delta t)^2.
\end{aligned}
\tag{4.11}
$$

In the schemes (4.5), (4.8) with (4.9), (4.10) and (4.11), ΔW_n, $I_{(1,0),n}(\Delta t)$ and $I_{(0,1),n}(\Delta t)$ are numerically realized by the independent $N(0,1)$ random numbers γ_n and $\hat{\gamma}_n$ ($n = 0, 1, \cdots$) as follows (Kloeden and Platen (1992)):

$$
\begin{aligned}
\Delta W_n &= \gamma_n\sqrt{\Delta t} \\
I_{(1,0),n}(\Delta t) &= \frac{1}{2}(\gamma_n + \frac{1}{\sqrt{3}}\hat{\gamma}_n)(\Delta t)^{3/2} \\
I_{(0,1,n)}(\Delta t) &= \frac{1}{2}(\gamma_n - \frac{1}{\sqrt{3}}\hat{\gamma}_n)(\Delta t)^{3/2}.
\end{aligned}
\tag{4.12}
$$

Moreover, we here choose $T = 1$ and $N = 1000$, and hence the stepsize $\Delta t = 10^{-3}$.

Table 4.1 and Table 4.2 indicate the examples of the numerical solutions from these schemes mentioned above (in case of Table 4.2, those schemes except (4.10)) with the initial value $s = 0.01$ and $s = 0.001$, respectively. Here we have used the same sequences of random numbers for each scheme together with (4.12). As was mentioned in the introductory part of this section, from these results we observe that the values of numerical solutions derived from the standard schemes become to be negative in midst of their simulations, if their initial values are close to zero; in contrast with these results, our each scheme is free from such a trouble. Thus, our scheme (4.5) and (4.8) with (4.9) have a superiority with respect to numerical realization of the character of (4.1), that is, of non-negativity of solutions than the standard schemes.

Table 4.3 indicates the results with the initial value $s = 1$. By the estimation of local order with respect to these schemes, we see that the accuracy for the scheme (4.5) corresponds to that for Taylor scheme of local order 2, and the accuracy for the scheme (4.8) with (4.9) corresponds

to Taylor scheme of local order 3. We may consider that Table 4.3 supports such a result (note that Euler scheme is just a Taylor scheme of local order 1).

Remark 4.1: Let us consider the following SDE:

$$dS(t) = S(t)dt + \frac{1}{1-\gamma}\{S(t)\}^\gamma \circ dW(t), \quad S(0) = s(> 0),$$

where $0 < \gamma < 1$. This is also often treated as a model of an asset price process in mathematical finance, which is a generalization of (4.1). For this process, we can also construct the numerical schemes as mentioned above in a similar way. Indeed, Scheme 3.1 for this SDE, of which local order equals 2, is given by

$$\tilde{S}_{n+1} = [\{\Delta W_n + \tilde{S}_n^{1-\gamma}\}^2]^{1/\{2(1-\gamma)\}} \exp(\Delta t),$$

where $\tilde{S}_0 = S(0) = s$. Note that the numerical solutions derived from this scheme also satisfy "non-negativity".

4.2 Example of Scheme 3.2 for a non-linear SDE

We now turn into the example of Scheme 3.2 given by (3.21). Let us consider the following non-linear scalar SDE:

$$dS(t) = S(t)dt + \{S(t) + 2\sqrt{S(t)}\} \circ dW(t), \quad S(0) = s(> 0). \tag{4.13}$$

In this case, the vector fields X_0 and X_1 are set by

$$X_0 = S\frac{d}{dS}, \quad X_1 = (S + 2\sqrt{S})\frac{d}{dS} \tag{4.14}$$

respectively. Then, we remark that $[X_0, X_1] = -\sqrt{S}(d/dS)$, $[X_0, [X_0, X_1]] = -[X_0, X_1]/2$ and $[X_1, [X_0, X_1]] = -[X_0, X_1]/2$ hold, respectively; hence the Lie algebra generated by X_0 and X_1 is of finite dimension. Hence, as in 4.1, Eq.(2.14) and (2.15) also actually converge in this case.

We may regard the SDE (4.13) as a linear SDE with the random perturbation $2\sqrt{S(t)} \circ dW(t)$. On account of this, as $A_{n\Delta t}$ and $B_{n\Delta t}$ in Example 3.2, we adopt

$$A_{n\Delta t} = \Delta t S\frac{d}{dS} + \Delta W_n S\frac{d}{dS}, \quad B_{n\Delta t} = \Delta W_n 2\sqrt{S}\frac{d}{dS}; \tag{4.15}$$

that is, we set $X_0^A = S(d/dS)$, $X_1^A = S(d/dS)$, $X_0^B = 0$ and $X_1^B = 2\sqrt{S}(d/dS)$. Then, in consideration of (2.6), we obtain the exponential maps for them explicitly as follows:

$$\exp(A_{n\Delta t})(s) = s\exp(\Delta t + \Delta W_n), \quad \exp(B_{n\Delta t})(s) = \{\Delta W_n + \sqrt{s}\}^2. \tag{4.16}$$

Inserting these equations into (3.21), we find Scheme 3.2 for the SDE (4.13); it is given by

$$\tilde{S}_{n+1} = \{\Delta W_n/2 + \sqrt{\{\Delta W_n/2 + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t + \Delta W_n)}\}^2, \tag{4.17}$$

where $\tilde{S}_0 = S(0) = s$.

Now, in a similar way to that in Subsection 4.1, we will compare the numerical accuracy of this scheme with that of the standard numerical schemes for (4.13). In this case, Euler-Maruyama scheme and Kloeden's Taylor scheme of local order 3 are written in the following forms:

Euler-Maruyama scheme:

$$S_{n+1} = S_n + \{\frac{3}{2}(S_n + \sqrt{S_n}) + 1\}\Delta t + (S_n + 2\sqrt{S_n})\Delta W_n. \tag{4.18}$$

Taylor scheme of local order 3:

$$
\begin{aligned}
S_{n+1} &= S_n + \{\frac{3}{2}(S_n + \sqrt{S_n}) + 1\}\Delta t + (S_n + 2\sqrt{S_n})\Delta W_n \\
&+ \frac{1}{2}(S_n + 3\sqrt{S_n} + 2)\{(\Delta W_n)^2 - \Delta t\} \\
&+ \frac{3}{2}(S_n + \frac{5}{2}\sqrt{S_n} + 1)I_{(1,0),n}(\Delta t) + \frac{3}{2}(S_n + \frac{11}{6}\sqrt{S_n} + 1)I_{(0,1),n}(\Delta t) \\
&+ \frac{1}{6}(S_n + \frac{7}{2}\sqrt{S_n} + 3)\{(\Delta W_n)^3 - 3\Delta t\Delta W_n\} \\
&+ \frac{9}{8}(S_n + \frac{17}{12}\sqrt{S_n} + \frac{5}{6})(\Delta t)^2.
\end{aligned} \tag{4.19}
$$

Therefore, inserting (4.12) into the schemes (4.17)-(4.19), we obtain numerical solutions through the schemes.

Table 4.4 shows the results with the initial value $s = 1$, $T = 1$, $N = 1000$ and $\Delta t = 10^{-3}$. According to Theorem 3.1 and Example 3.2, we may expect the accuracy for our scheme (4.17) is corresponding to that for Taylor scheme of local order 2. Table 4.4 indicates that such an expectation is practically valid.

4.3 Composition method to stochastic Hamilton dynamical systems

As mentioned in Section 1, composition methods (or operator splitting methods) are not only a superior integrating method for differential equations in preserving the special character or structure of the equations, but also often useful for approximations of non-linear equations of which solutions are not obtained explicitly. The examples of 4.1 and 4.2 mentioned above show that these facts are also true in case of stochastic systems. As also mentioned, such a advantage is remarkable in case of dynamical systems with multiple space dimensions or Hamilton dynamical systems as standing for dimensional splitting methods (*e.g.* Yanenko (1959); Iserles (1984)). In consideration of this, we will investigate numerical schemes by composition methods for stochastic dynamical systems with "Hamiltonian structure" (Misawa (1999),(2000)).

First we review stochastic Hamilton dynamical systems (Misawa (1999)). Let us consider the following 2ℓ-dimensional stochastic dynamical systems:

$$d\begin{pmatrix} x^i(t) \\ x^{\ell+i}(t) \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i}H_0(x(t)) \\ -\partial_i H_0(x(t)) \end{pmatrix} dt + \begin{pmatrix} \partial_{\ell+i}H_1(x(t)) \\ -\partial_i H_1(x(t)) \end{pmatrix} \circ dW(t), \quad (i = 1, \cdots \ell) \tag{4.20}$$

where $x = (x^k)_{k=1}^{2\ell}$ and $\partial_j = \partial/\partial x^j$ $(j = 1, 2, \cdots 2\ell)$, respectively. In (4.20), $H_\alpha(x)$ $(\alpha = 0, 1)$ are smooth scalar functions on $R^{2\ell}$. Formally, one may regard this as a Hamilton dynamical system

$$\frac{d}{dt} \begin{pmatrix} x^i \\ x^{\ell+i} \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i} \hat{H}(x) \\ -\partial_i \hat{H}(x), \end{pmatrix} \quad (i = 1, \cdots, \ell)$$

with a "randomized" Hamiltonian $\hat{H}$ given by

$$\hat{H} = H_0 + H_1 \gamma_t,$$

where γ_t is a one-dimensional Gaussian white noise. On account of this fact, we call (4.20) and H_α $(\alpha = 0, 1)$ an (ℓ-dimensional) *stochastic Hamilton dynamical system* and the *Hamiltonian*.

Now, we proceed to an example of our new scheme for stochastic Hamilton systems. For simplicity, we set $\ell = 1$ and denote $x^1(t)$ and $x^2(t)$ by $q(t)$ and $p(t)$, respectively. Let us consider the class of Hamilton systems with the typical Hamiltonian $H_0 = p^2/2 + V_0(q)$ and $H_1 = p^2/2 + V_1(q)$, where $V_0(q)$ and $V_1(q)$ are any potential functions. Then the equation (4.20) turns to be

$$d \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ -V_0'(q(t)) \end{pmatrix} dt + \begin{pmatrix} p(t) \\ -V_1'(q(t)) \end{pmatrix} \circ dW(t). \tag{4.21}$$

In general, this is a stochastic "non-linear" system. For this system, the vector fields X_0 and X_1 become

$$X_0 = p\partial_q - V_0'(q)\partial_p, \quad X_1 = p\partial_q - V_1'(q)\partial_p, \tag{4.22}$$

respectively.

We are to apply our scheme to this system. As an important example of the decomposition of (3.3) for the above system, we choose the following splitting of Scheme 3.2 type:

$$A_{n\Delta t} = p(\Delta t + \Delta W_n)\partial_q, \quad B_{n\Delta t} = -(V_0'(q)\Delta t + V_1'(q)\Delta W_n)\partial_p. \tag{4.23}$$

This corresponds to the decomposition mentioned in Example 3.2; that is, X_0^A, X_1^A, X_0^B and X_1^B in Example 3.2 are given by $p\partial_q$, $p\partial_q$, $-V_0'(q)\partial_p$ and $-V_1'(q)\partial_p$, respectively. Then we note that $\exp(A_{n\Delta t})$ and $\exp(B_{n\Delta t})$ are exponential maps which correspond to the flows of solutions to the following SDEs, respectively:

$$d \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} p(t) \\ 0 \end{pmatrix} \circ dW(t).$$

$$d \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -V_0'(q(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ -V_1'(q(t)) \end{pmatrix} \circ dW(t).$$

Therefore, we can obtain the explicit forms of them; this may be regarded as an example of dimensional splitting. The results are given by

$$\exp(A_{n\Delta t}) \begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} p_n(\Delta t + \Delta W_n) + q_n \\ p_n \end{pmatrix}$$

$$\exp(B_{n\Delta t})\begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} q_n \\ -(\Delta t V_0'(q_n) + \Delta W_n V_1'(q_n)) + p_n \end{pmatrix}.$$

Inserting these equations into (3.21), we finally find the numerical scheme 3.2 for this system as follows:

$$\begin{pmatrix} \tilde{q}_{n+1} \\ \tilde{p}_{n+1} \end{pmatrix} = \begin{pmatrix} \hat{q}_n \\ -\frac{1}{2}(\Delta t V_0'(\hat{q}_n) + \Delta W_n V_1'(\hat{q}_n)) + \hat{p}_n \end{pmatrix} \tag{4.24}$$

where

$$\begin{pmatrix} \hat{q}_n \\ \hat{p}_n \end{pmatrix} = \begin{pmatrix} \tilde{p}_n(\Delta t + \Delta W_n) + \tilde{q}_n \\ -\frac{1}{2}(\Delta t V_0'(\tilde{q}_n) + \Delta W_n V_1'(\tilde{q}_n)) + \tilde{p}_n \end{pmatrix}. \tag{4.25}$$

As in the example in Subsection 4.2, this scheme has the local order of weak sense 2. Thus, for the class of stochastic Hamilton dynamical systems with typical Hamiltonians mentioned above, we can numerically approximate them through our scheme (4.24) with (4.25) having the accuracy corresponding to Taylor scheme of local order 2.

4.4 Remark on composition methods and conserved quantities in stochastic dynamical systems

Finally, we remark on numerical schemes for stochastic dynamical systems which preserve "conserved quantities "of the systems. It is well-known that conserved quantities play an essential role to determine the structure of dynamical systems; hence, it is important to find a numerical scheme which has the conservation properties on the quantities for stochastic systems. On the other hands, composition methods often give such schemes in deterministic systems. Therefore, we may expect that one may obtain the schemes through our results, which have the advantageous to preserve them for stochastic systems, and in the remainder of this section, we will briefly examine it.

Let us consider d-dimensional stochastic dynamical systems (2.7). Suppose that a smooth function $I = I(S)$ satisfies

$$X_0 I = 0, \quad X_1 I = 0, \tag{4.26}$$

where X_0 and X_1 are the vector fields given by (2.3). According to Misawa (1999), $I(S)$ becomes a constant quantity; that is, $I(S(t)) = constant$ holds on the diffusion process $S(t)$ governed by (2.7).

Under some conditions, we may straightforwardly make a stochastic scheme satisfying numerical preservation of conserved quantities. Assume that the exponential maps of $A_{n\Delta t} = \Delta t X_0$ and $B_{n\Delta t} = \Delta W_n X_1$ are explicitly calculated. Then, it is obvious that Scheme 3.1 preserves the conserved quantity I numerically, because of the definition of exponential map and (4.26).

Now, we investigate a trivial example of a stochastic dynamical system with a conserved quantity and the numerical scheme through composition methods. Let us consider

$$d\begin{pmatrix} S^1(t) \\ S^2(t) \end{pmatrix} = \begin{pmatrix} S^2(t) \\ -S^1(t) \end{pmatrix} dt + \begin{pmatrix} S^2(t) \\ -S^1(t) \end{pmatrix} \circ dW(t); \tag{4.27}$$

this is a stochastic system with the conserved quantity $I(S) = \frac{1}{2}((S^1)^2 + (S^2)^2)$, since (4.26) holds. However, as mentioned in Misawa (2000), the ordinary schemes do not conserve $I(S)$ numerically. On the other hand, for this system, we adopt Scheme 3.1 with $A_{n\Delta t} = \Delta t X_0 = \Delta t(S^2 \partial_1 - S^1 \partial_2)$ and $B_{n\Delta t} = \Delta W_n X_1 = \Delta W_n(S^2 \partial_1 - S^1 \partial_2)$; then through (2.6), the numerical scheme is explicitly given by

$$
\begin{pmatrix} \tilde{S}^1_{n+1} \\ \tilde{S}^2_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(\Delta t) & \sin(\Delta t) \\ -\sin(\Delta t) & \cos(\Delta t) \end{pmatrix} \begin{pmatrix} \cos(\Delta W_n) & \sin(\Delta W_n) \\ -\sin(\Delta W_n) & \cos(\Delta W_n) \end{pmatrix} \begin{pmatrix} \tilde{S}^1_n \\ \tilde{S}^2_n \end{pmatrix}. \tag{4.28}
$$

Therefore, for any n, the numerical solutions (4.28) satisfy $I(\tilde{S}^1_n, \tilde{S}^2_n) = constant$. Thus, our scheme numerically preserve a conserved quantity I of the stochastic system (4.27), and this fact also shows the superiority of the scheme derived through composition methods.

5 Concluding remarks

In this paper, we have formulated composition methods for stochastic differential equations (SDEs), and thereby we have made some stochastic numerical schemes. Then, the several examples have indicated that the new schemes have a superiority in conservation properties on character of SDEs and they are useful for approximations of the solutions to SDEs. Moreover, we have estimated local error orders for our schemes within the framework of Saito and Mitsui's definition. Finally, we give some remarks and future problems concerning with this work.

(i) As mentioned in Remark 3.2, we should carry out a more analytical error estimation for our schemes through the result on time asymptotics of exponential maps for SDEs by Castell (1993), since the stochastic series (2.14) is only a formal representation.

(ii) In our error estimation, we have addressed "local error order "for our schemes. In general, it is more important to estimate "global error order "for numerical schemes. In Burrage and Burrage (1999), the relationship between local order and global one has been discussed in some detail. Using the results, we may carry out global error estimation for our new schemes.

(iii) In this paper, we have treated the SDEs with one-dimensional Wiener process. On the other hand, it often happens that the error order of a numerical method collapses, if there is more than 1 Wiener process. Hence, we should investigate the error orders of our schemes in the case of SDEs with multi-dimensional Wiener process.

(iv) Moreover, Li and Liu (1997) and Kunita (2000) have studied on stochastic exponential maps for a more general class of stochastic processes, e.g. Lévy processes. Hence, through the works, we may formulate stochastic composition methods for such general stochastic processes.

The author will report these subjects in future works.

Acknowledgments

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n	Euler-Maruyama	Taylor of order 3	Scheme (4.5)	Scheme (4.8)
179	.0273725	.0105996	.0105927	.0105912
180	.0391646	.0183733	.0183659	.0183624
181	.0576639	.0323048	.0323008	.0322903
182	.0506609	.0265831	.0265762	.0265694
183	.0512129	.0262494	.0262425	.0262353
184	.0596316	.0318198	.0318131	.031804
185	.0561177	.0285982	.0285888	.0285826
186	.0478919	.0223767	.02237	.022363
187	.0507206	.0236332	.0236273	.0236186
188	.0462954	.0200634	.0200576	.0200496
189	.0373054	.014018	.0140095	.0140059
190	.0459159	.0190609	.0190539	.0190461
191	.0428106	.0164982	.01649	.0164839
192	.0524778	.0223075	.0222992	.0222907
193	.0365604	.0126323	.0126217	.0126192
194	.0190258	4.07906E-03	.0040724	4.07239E-03
195	.0120115	1.20801E-03	1.20498E-03	1.20484E-03
196	7.86052E-04	4.42925E-04	4.46361E-04	4.45822E-04
197	2.07421E-03	6.85707E-04	6.89868E-04	6.8875E-04
198	-9.47943E-05	7.45417E-05	7.31484E-05	7.32705E-05

Table 4.1: An example of numerical solutions from Euler-Maruyama scheme
(4.10), Taylor scheme of local order 3 (4.11), Scheme (4.5) and
Scheme (4.8) with (4.9) for (4.1) with an initial value s=0.01.

n	Taylor of order 3	Scheme (4.5)	Scheme (4.8)
348	.012517	.0126881	.0124984
349	.0128065	.012981	.0127872
350	8.45901E-03	8.60197E-03	8.44348E-03
351	4.89666E-03	5.00384E-03	4.88441E-03
352	.006811	6.93784E-03	6.79605E-03
353	.0119832	.0121529	.0119632
354	.0100548	.0102105	.010036
355	7.71649E-03	7.85315E-03	7.69978E-03
356	5.17094E-03	5.28222E-03	5.15692E-03
357	5.46758E-03	5.58249E-03	5.45263E-03
358	4.28647E-03	4.38672E-03	4.27267E-03
359	7.06959E-03	7.19804E-03	7.05171E-03
360	2.92672E-04	3.21066E-04	.0002912
361	5.76257E-04	6.15458E-04	5.7371E-04
362	2.57058E-03	2.65334E-03	2.56494E-03
363	3.09359E-03	3.18363E-03	3.08695E-03
364	8.20704E-04	8.66758E-04	8.17091E-04
365	2.12819E-04	1.90873E-04	2.14876E-04
366	5.3766E-04	5.01956E-04	5.40526E-04
367	-2.97776E-07	7.52643E-07	7.81725E-10

Table 4.2: An example of numerical solutions from Taylor scheme of local
order 3 (4.11), Scheme (4.5) and Scheme (4.8) with (4.9) for
(4.1) with an initial value s=0.001.

n	Euler-Maruyama	Taylor of order 3	Scheme (4.5)	Scheme (4.8)
980	3.72261	3.68752	3.68641	3.68752
981	3.7145	3.67843	3.67732	3.67843
982	3.73492	3.69778	3.69667	3.69778
983	3.92852	3.89192	3.89084	3.89192
984	4.10458	4.06816	4.06707	4.06816
985	4.21954	4.18242	4.18134	4.18242
986	4.19385	4.15585	4.15479	4.15586
987	4.30738	4.26862	4.26757	4.26862
988	4.4013	4.36165	4.3606	4.36165
989	4.46073	4.42002	4.41896	4.42002
990	4.46056	4.41882	4.41777	4.41882
991	4.33234	4.29109	4.29002	4.29109
992	4.25444	4.21288	4.2118	4.21288
993	3.97593	3.9392	3.93813	3.9392
994	3.72468	3.69204	3.69095	3.69204
995	3.48053	3.4519	3.45081	3.4519
996	3.6467	3.61833	3.61729	3.61834
997	3.79678	3.76838	3.76736	3.76838
998	3.74771	3.71863	3.7176	3.71863
999	3.72407	3.69411	3.69307	3.69412
1000	3.62125	3.5914	3.59035	3.59141

Table 4.3: An example of numerical solutions from Euler-Maruyama scheme (4.10), Taylor scheme of local order 3 (4.11), Scheme (4.5) and Scheme (4.8) with (4.9) for (4.1) with an initial value s=1.

n	Euler-Maruyama	Taylor of order 3	Scheme (4.17)
980	8.8134	9.51844	9.52098
981	9.09664	9.813	9.81558
982	9.33128	10.0542	10.0568
983	10.5972	11.4613	11.4644
984	11.3247	12.2459	12.2492
985	11.4263	12.3423	12.3456
986	10.7926	11.6704	11.6735
987	10.4671	11.3161	11.3191
988	10.1213	10.9412	10.9441
989	9.46724	10.2515	10.2542
990	9.70748	10.4991	10.502
991	9.07412	9.83114	9.83382
992	9.77413	10.5897	10.5926
993	9.35542	10.1398	10.1426
994	9.79685	10.6085	10.6114
995	9.86074	10.6656	10.6686
996	9.91146	10.7086	10.7115
997	9.57355	10.3428	10.3456
998	9.27501	10.0183	10.021
999	9.64218	10.4042	10.407
1000	9.47959	10.2216	10.2244

Table 4.4: An example of numerical solutions from Euler-Maruyama scheme (4.18), Taylor scheme of local order 3 (4.19), Scheme (4.17) for (4.13) with an initial value s=1.