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Lyapunov Analysis on a Geometric Method

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概要

A geometric method is set up for Lyapunov analysis of natural Hamiltonian systems with N degrees of freedom. For a geodesic on a Riemannian manifold, one considers geodesic deviations through the Jacobi equation, a linearization of the geodesic equation, which is a second-order differential equation. However, one needs first-order linearized equations for Lyapunov analysis of deviations. Hence a question arises as to how one obtains a kind of Hamiltonian equations corresponding to Jacobi equations. A geometric answer to this question is given through the geometry of the cotangent bundle of the Riemannian manifold; first a geodesic equation is lifted up to a first-order equation in the cotangent bundle and then the linearization procedure is performed to obtain a first-order differential equation which corresponds to the Jacobi equation in question. Through this procedure, one can obtain Lyapunov vectors that satisfy the so-required properties that (i) the N-th Lyapunov vector is tangent to the trajectory in question for all time, (ii) the $(N+1)$-th Lyapunov vector points to the direction of the gradient of the Hamiltonian for all time, and (iii) the other Lyapunov vectors are orthogonal to the plane spanned by the above-mentioned two vectors for all time, while these properties are not satisfied in linearized Newton's equations of motion.

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1 Introduction

Lyapunov analysis in terms of Lyapunov exponents is a useful tool to investigate long-term behaviours of nearby trajectories of dynamical systems. Since Lyapunov exponents are defined so as to measure averaged exponential stability/instability of a trajectory during infinitely long time evolution, they are suitable to study averaged properties of the dynamical systems accordingly. In Hamiltonian systems, phase portrait in the phase space [MB93, YO87] and phase transition in a model of condensed matter [BC87] are discussed in terms of Lyapunov exponents. Moreover, Lyapunov spectra have a universal characteristic independent of individual dynamical systems; it is shown that according to whether the system is a fully developed chaotic system or moderately chaotic system, the dependence of the Lyapunov spectra $\{\lambda_i\}$ on indices $\{i/N\}$ is straight [LPRV87] or curved [Yam98], if the system is subject to nearest neighbour interaction. Here N stands for the degrees of freedom. In contrast with this, local Lyapunov exponents, which are defined for finite time intervals, are suitable to study dynamic properties during short periods. For example, wondering motions between tori and a chaotic sea in a Hamiltonian system with many degrees of freedom are studied through the largest local Lyapunov exponents [Yam96].

Further, Lyapunov analysis in terms of Lyapunov vectors is of considerable use in studying temporal stability/instability of trajectories in detail. A cluster motion is discussed in relation to the Lyapunov vector corresponding to the smallest positive Lyapunov exponent in a symplectic mapping [KK92].

The Lyapunov exponents and the Lyapunov vectors are obtained from linearized equations of motion. Let us consider a dynamical system whose equation of motion is

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = f^i(x), \qquad (i = 1, \cdots, \ell) \tag{1}$$

then the linearized equation of motion is, using the Einstein's sum rule,

$$\frac{\mathrm{d}X^i}{\mathrm{d}t} = \frac{\partial f^i}{\partial x^j}(x(t))X^j, \qquad (i = 1, \cdots, \ell) \tag{2}$$

where $x(t)$ is a trajectory of the dynamics (1), and X^i is i-th element of the vector $\boldsymbol{X} = X^i \dfrac{\partial}{\partial x^i}$. We introduce tangent vectors $\{\boldsymbol{X}_1(t), \boldsymbol{X}_2(t), \cdots, \boldsymbol{X}_\ell(t)\}$ each of which follows the linearized equation of motion (2). Lyapunov vectors $\{\boldsymbol{V}_1(t), \boldsymbol{V}_2(t), \cdots, \boldsymbol{V}_\ell(t)\}$ are obtained by orthogonalizing the tangent vectors with the Gram-Schmidt method from $\boldsymbol{X}_1(t)$ to $\boldsymbol{X}_\ell(t)$ at each time. That is,

$$\boldsymbol{V}_i(t) = \boldsymbol{X}_i(t) - \sum_{j=1}^{i-1} \frac{\langle \boldsymbol{X}_i(t), \boldsymbol{V}_j(t)\rangle}{\langle \boldsymbol{V}_j(t), \boldsymbol{V}_j(t)\rangle} \boldsymbol{V}_j(t), \qquad (i = 1, \cdots, \ell)$$

where $\langle \boldsymbol{X}, \boldsymbol{V}\rangle$ stands for an inner product between $\boldsymbol{X}$ and $\boldsymbol{V}$. The Lyapunov exponents $\{\lambda_1, \lambda_2, \cdots, \lambda_\ell\}$ and the local Lyapunov exponents $\{\lambda_1(n,\tau), \lambda_2(n,\tau), \cdots, \lambda_\ell(n,\tau)\}$ are calculated from the Lyapunov vectors as

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\boldsymbol{V}_i(t)\|}{\|\boldsymbol{V}_i(0)\|}, \qquad (i = 1, \cdots, \ell)$$

and

$$\lambda_i(n,\tau) = \frac{1}{\tau} \ln \frac{\|\boldsymbol{V}_i(n\tau)\|}{\|\boldsymbol{V}_i((n-1)\tau)\|}, \qquad (i = 1, \cdots, \ell)$$

respectively. Here τ is an arbitrary finite constant. Values of the Lyapunov exponents do not depend on the choice of initial set of Lyapunov vectors with unit probability [Ose68, BGS76].

However, values of the Lyapunov exponents and the Lyapunov vectors may depend on the choice of representation of dynamics, since we can get various linearized equations of motion from one equation of motion when we transform the independent parameter t to s as

$$\mathrm{d}s = \phi(x)\mathrm{d}t. \tag{3}$$

Applying the transformation (3), the equation of motion (1) becomes

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = \frac{f^i(x)}{\phi(x)}. \qquad (i = 1, \cdots, \ell) \tag{4}$$

The equations (1) and (4) are essentially same since the latter goes back to the former by the inverse transformation of (3). However, the linearized equation of (4) is

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} = \frac{1}{\phi(x)} \left[\frac{\partial f^i(x)}{\partial x^j} - \frac{f^i(x)}{\phi(x)} \frac{\partial \phi(x)}{\partial x^j} \right] X^j, \qquad (i = 1, \cdots, \ell) \tag{5}$$

and the inverse transformation of (3) no longer gives the equation (2) unless $\phi(x)$ is a constant function, since the transformed equation is

$$\frac{\mathrm{d}X^i}{\mathrm{d}t} = \left[\frac{\partial f^i(x)}{\partial x^j} - \frac{f^i(x)}{\phi(x)} \frac{\partial \phi(x)}{\partial x^j} \right] X^j. \qquad (i = 1, \cdots, \ell) \tag{6}$$

Accordingly, the Lyapunov exponents and the Lyapunov vectors may take different values for different representations of a dynamics, for instance, for the representations (1) and (4).

The above consideration leads to a question: What is a "good" representation of dynamics? To answer this question, we must introduce some natural requirements which are satisfied in the "good" representation. In Hamiltonian systems with N degrees of freedom, λ_N and λ_{N+1} always vanish since they correspond to tangential direction of trajectories and gradient direction of Hamiltonian functions respectively, and no global stability/instability appears for the two directions. However, for naive representation, local instability appears even for the two directions. We therefore require that at any time (i) N-th Lyapunov vector points to tangential direction of trajectories, (ii) $(N+1)$-th Lyapunov vector points to gradient direction of Hamiltonian functions, and (iii) the other Lyapunov vectors are orthogonal to the two directions. From the "good" representation, we obtain purely stable/unstable directions without influence of the two marginal directions corresponding to the zero Lyapunov exponents.

The three requirements are satisfied by a geometric method developed in this paper. A geometric method is introduced to analytically estimate the largest Lyapunov exponent λ_1 with the aid of statistical mechanics [Pet93, CP93, CPC99], and it regards trajectories as geodesics on a Riemannian manifold. Stability/Instability of geodesics is given by Jacobi(-Levi-Civita) equation, but it is a second order differential equation while Lyapunov analysis needs a first order differential equation as seen in equations (2) and (5). We therefore lift the Jacobi equation to the cotangent bundle of the Riemannian manifold.

In this paper, we consider natural Hamiltonian systems with N degrees of freedom

$$H(q,p) = \frac{1}{2}\delta^{ij} p_i p_j + V(q). \tag{7}$$

Equation of motion of the system (7) is usually represented as

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} + \frac{\partial V}{\partial q^i} = 0, \qquad (i = 1, \cdots, N) \tag{8}$$

whose linearized equation of motion is

$$\frac{\mathrm{d}^2 X^i}{\mathrm{d}t^2} + \frac{\partial^2 V}{\partial q^i \partial q^j} X^j = 0. \qquad (i, j = 1, \cdots, N) \tag{9}$$

We numerically compare this usual method with the geometric method through a model system.

This article is organized as follows. In section 2 we review Riemannian geometry and Riemannian manifolds on which geodesics become trajectories of the natural Hamiltonian dynamics. In section 3 we lift vector field, metric, Christoffel symbol and connection from Riemannian manifolds to their cotangent bundles. Dynamics on the cotangent bundles are described in section 4, and we construct a set of Lyapunov vectors satisfying the three requirements mentioned above. Section 5 is for numerical calculations of the lifted Jacobi equation. The final section 6 is devoted to summary and discussions.

2 Riemannian Geometry

2.1 Jacobi equations

Let $(M, \boldsymbol{g})$ be an m-dimensional Riemannian manifold with metric $\boldsymbol{g}$. The metric induces the Levi-Civita connection ∇ on M; for vector fields $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathscr{X}(M)$, $\mathscr{X}(M)$ denoting the set of vector fields on M, the covariant derivative $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ is defined, in terms of local coordinates $(x^1, \cdots, x^m)$, to be

$$\nabla_{\boldsymbol{X}}\boldsymbol{Y} = X^j \left[\frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk} Y^k\right] \frac{\partial}{\partial x^i},$$

where (X^i) and (Y^i) are components of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, the Christoffel symbols Γ^i_{jk} are defined as

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{\ell k}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell}\right),$$

with components of the metric

$$g_{ij} = \boldsymbol{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \qquad g_{ij} g^{jk} = \delta_i^k.$$

For a geodesic $c(s)$ with s the arc length parameter, the tangent vector $\boldsymbol{\xi}$ to the geodesic satisfies the geodesic equation

$$\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi} = \left[\frac{\mathrm{d}\xi^i}{\mathrm{d}s} + \Gamma^i_{jk}\xi^j\xi^k\right] \frac{\partial}{\partial x^i} = 0, \tag{10}$$

where

$$\boldsymbol{\xi} = \frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\partial}{\partial x^i}\bigg|_{c(s)}$$

with

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{11}$$

We are interested in stability/instability of geodesics. To this end, we consider a family of geodesic which looks like a fluid whose stream curves are geodesics. Then we may consider that the tangent vector $\boldsymbol{\xi}$ is extended to be a vector field defined in the neighbourhood of the original geodesic $c(s)$. We may also assume that a vector field $\boldsymbol{X}$ is defined in such a manner that $[\boldsymbol{\xi}, \boldsymbol{X}] = 0$ in the same domain as $\boldsymbol{\xi}$. The vector field $\boldsymbol{\xi}$ may have singularity at which $\boldsymbol{\xi}$ is not defined uniquely and $\boldsymbol{X}$ vanishes there. With this in mind, we operate equation (10) with $\nabla_{\boldsymbol{X}}$ and use the definition of the Riemannian curvature to obtain the Jacobi equation

$$\nabla_{\boldsymbol{\xi}}\nabla_{\boldsymbol{\xi}}\boldsymbol{X} + \boldsymbol{R}(\boldsymbol{X}, \boldsymbol{\xi})\boldsymbol{\xi} = 0. \tag{12}$$

The Jacobi equation, a linearization of the geodesic equation, is used to analyze stability/instability of geodesics. As is well known, the Riemannian curvature tensor is defined as $\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}$ for $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z} \in \mathfrak{X}(M)$, and their local components are put in the form

$$R_{ijk\ell} = R_{ijk}{}^{m} g_{m\ell} = \boldsymbol{g}\left(\boldsymbol{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right),$$
$$R_{ijk}{}^{\ell} = \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} + \Gamma^\ell_{im}\Gamma^m_{jk} - \Gamma^\ell_{jm}\Gamma^m_{ik}.$$

The Riemannian curvature tensor has the symmetries such as

$$\begin{aligned}\boldsymbol{g}(\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z},\boldsymbol{W}) &= -\boldsymbol{g}(\boldsymbol{R}(\boldsymbol{Y},\boldsymbol{X})\boldsymbol{Z},\boldsymbol{W}) \\ &= -\boldsymbol{g}(\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{W},\boldsymbol{Z}) = \boldsymbol{g}(\boldsymbol{R}(\boldsymbol{Z},\boldsymbol{W})\boldsymbol{X},\boldsymbol{Y}).\end{aligned} \tag{13}$$

In the next two subsections, we show two examples of Riemannian manifolds whose geodesics or projection of geodesics coincide with trajectories of the natural Hamiltonian system (7). The whole configuration space with local coordinate $(q^1, \cdots, q^N)$ is referred as M_C.

2.2 The Eisenhart metric

Let M_E be $M_C \times \boldsymbol{R}^2$ with local coordinate $(q^0 = t, q^1, \cdots, q^N, q^{N+1})$. The coordinate q^{N+1} is determined as

$$q^{N+1} = \frac{c_1^2}{2}t + c_2^2 - \int_0^t L\left(q, \frac{\mathrm{d}q}{\mathrm{d}t}\right)\mathrm{d}t,$$

where the Lagrangian L is $L = \sum_{i=1}^n (\dot{q}^i)^2/2 - V(q)$, and c_1 and c_2 are real arbitrary constants. The Eisenhart metric $\boldsymbol{g}_E$ is

$$(\boldsymbol{g}_E)_{ij} = \begin{pmatrix} -2V(q) & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and hence $\mathrm{d}s^2 = c_1^2\mathrm{d}t^2$. We set $c_1^2 = 1$ since the constant c_1 is arbitrary.

The non-vanishing Christoffel symbols are

$$\Gamma^i_{00} = -\Gamma^{n+1}_{0i} = \frac{\partial V}{\partial q^i}, \quad (i = 1, \cdots, N)$$

and the geodesic equation is, using $\mathrm{d}s = \mathrm{d}t$,

$$\frac{\mathrm{d}^2 q^0}{\mathrm{d}t^2} = 0,$$
$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} + \frac{\partial V}{\partial q^i} = 0, \quad (i = 1, \cdots, N)$$
$$\frac{\mathrm{d}^2 q^{n+1}}{\mathrm{d}t^2} + \frac{\mathrm{d}L}{\mathrm{d}t} = 0.$$

The second equation coincide with Newton's equation (8) , and hence trajectories of a Hamiltonian system are obtained as the canonical projection π of geodesics on the configuration space-time

$$\pi : M_C \times \boldsymbol{R}^2 \to M_C \times \boldsymbol{R} : (q^0, q^1, \cdots, q^N, q^{N+1}) \mapsto (q^0, q^1, \cdots, q^N).$$

The non-vanishing elements of the Riemannian curvature tensor is

$$R_{0ij0} = \frac{\partial^2 V}{\partial q^i \partial q^j}, \qquad (i, j = 1, \cdots, N)$$

and the Jacobi equation (12) becomes

$$\frac{\mathrm{d}^2 Y^i}{\mathrm{d}t^2} + \frac{\partial^2 V}{\partial q^i \partial q^j} Y^j = 0, \qquad (i = 1, \cdots, N)$$

with the assumption $Y^0 = 0$. Note this equation is the same as linearized Newton's equation (9) .

2.3 The Jacobi metric

Let M_J be an open submanifold of the configuration space $\boldsymbol{R}^N$ defined by

$$M_J = \{q \in M_C | V(q) < E\},$$

on which is defined the Jacobi metric $\boldsymbol{g}_J$ to be

$$(g_J)_{ij} = 2[E - V(q)]\delta_{ij}. \tag{14}$$

From the equation (11), the arc length parameter s is shown to be related to the time parameter t by

$$\mathrm{d}s^2 = 4[E - V(q)]^2 \mathrm{d}t^2, \tag{15}$$

and the tangent vector to a geodesic is always unity accordingly:

$$\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = g_{ij}\xi^i\xi^j = 2(E - V)\delta_{ij}\frac{\mathrm{d}q^i}{\mathrm{d}s}\frac{\mathrm{d}q^j}{\mathrm{d}s} = [2(E - V)]^2 \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 1.$$

The geodesic equation for the Jacobi metric is expressed as

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}s^2} + \frac{1}{2(E-V)}\left[2\frac{\partial(E-V)}{\partial q^j}\frac{\mathrm{d}q^j}{\mathrm{d}s}\frac{\mathrm{d}q^i}{\mathrm{d}s} - g^{ij}\frac{\partial(E-V)}{\partial q^j}\right] = 0,$$

which is equivalent to Newton's equation of motion (8) on account of (15). However, the Jacobi equation with the curvature tensor fot the Jacobi metric is not brought into the same equation as (9), a linearization of Newton's equation of motion, in general. Components of the curvature tensor are indeed put in the form [Ong75, CPC99]

$$R_{ijk\ell} = \frac{1}{n-2}\left[C_{ij}\delta_{k\ell} - C_{jk}\delta_{i\ell} + C_{k\ell}\delta_{ij} - C_{i\ell}\delta_{jk}\right],$$

where

$$C_{ij} = \frac{n-2}{4(E-V)^2}\left[2(E-V)\frac{\partial^2 V}{\partial q^i \partial q^j} + 3\frac{\partial V}{\partial q^i}\frac{\partial V}{\partial q^j} - \frac{1}{2}\delta^{k\ell}\frac{\partial V}{\partial q^k}\frac{\partial V}{\partial q^\ell}\delta_{ij}\right].$$

3 Geometry on the Cotangent Bundle

In the previous section, we have found that trajectories of natural Hamiltonian systems can be regarded as geodesics on suitable Riemannian manifolds, and that stability/instability of the trajectories are analyzed through the Jacobi equation, a linearization of the geodesic equation. However, the Jacobi equation is a second-order differential equation, while Lyapunov analysis is applied to first-order differential equations. We hence need a first-order differential equation, associated with the Jacobi equation in order to apply Lyapunov analysis. To find such a first-order differential equation, we first study geodesic equations lifted to the cotangent bundle T^*M of the Riemannian manifold M, and then perform the linearization of the lifted geodesic equation to obtain a first-order differential equation, which turns out to be a lift of the Jacobi equation. To carry out this procedure, we need some geometric setting-up on the cotangent bundle. We discuss a way to lift vector fields on M to T^*M, define a lifted Riemannian metric on T^*M, and study the Levi-Civita connection formed on T^*M, which defines a connection on T^*M in the sense that every tangent space to T^*M is decomposed into a horizontal and a vertical subspace.

3.1 Lift of vector fields

Let (x^i) and (x^i, p_i) be local coordinates in M and T^*M, respectively. Let $\theta = p_i \mathrm{d}x^i$ be the standard one-form on T^*M. For vector fields on M, a way to lift them is not unique. The way we adopt here is the following: For $\boldsymbol{X} \in \mathscr{X}(M)$, the lifted vector field $\widetilde{\boldsymbol{X}}$ is defined through

$$\mathcal{L}_{\widetilde{\boldsymbol{X}}}\theta = 0, \tag{16}$$

where $\mathcal{L}$ denote the Lie derivation. Using Cartan's formula

$$\mathcal{L}_{\widetilde{\boldsymbol{X}}}\theta = \mathrm{d}(\iota(\widetilde{\boldsymbol{X}})\theta) + \iota(\widetilde{\boldsymbol{X}})\mathrm{d}\theta,$$

where $\iota(\widetilde{\boldsymbol{X}})\theta = \theta(\widetilde{\boldsymbol{X}})$, we verify that the vector field $\widetilde{\boldsymbol{X}}$ takes the form

$$\widetilde{\boldsymbol{X}} = X^i \frac{\partial}{\partial x^i} + \hat{X}^{\bar{i}} \frac{\partial}{\partial p_i}, \tag{17}$$

where $\bar{i} = i + m$ and

$$\hat{X}^{\bar{i}} = -p_j \frac{\partial X^j}{\partial x^i}.$$

3.2 Adapted frame

By the use of the connection ∇ on M, we can introduce the adapted frame on (an open subset of) T^*M by

$$D_i = \frac{\partial}{\partial x^i} + p_k \Gamma^k_{ij} \frac{\partial}{\partial p_j}, \qquad D_{\bar{i}} = \frac{\partial}{\partial p_i}, \tag{18}$$

and the adapted coframe

$$\theta^i = \mathrm{d}x^i, \qquad \theta^{\bar{i}} = \mathrm{d}p_i - p_k \Gamma^k_{ij} \mathrm{d}x^j, \tag{19}$$

which are dual, that is, satisfy

$$\theta^i(D_j) = \delta^i_j, \qquad \theta^i(D_{\bar{j}}) = 0, \qquad \theta^{\bar{i}}(D_j) = 0, \qquad \theta^{\bar{i}}(D_{\bar{j}}) = \delta^i_j.$$

See reference [YI73] for adapted frame in the tangent bundle TM.

The lifted vector field $\widetilde{\boldsymbol{X}}$ takes the form, with respect to the adapted frame,

$$\widetilde{\boldsymbol{X}} = X^i D_i + X^{\bar{i}} D_{\bar{i}}, \tag{20}$$

where

$$X^{\bar{i}} = -p_j \nabla_i X^j$$

and

$$\nabla_i X^j = \frac{\partial X^j}{\partial x^i} + \Gamma^j_{ik} X^k.$$

For instance, the tangent vector field $\boldsymbol{\xi}$ to a congruence of geodesics is lifted to

$$\widetilde{\boldsymbol{\xi}} = \xi^i D_i.$$

3.3 Lift of metric

In order to discuss orthogonality of vector fields on T^*M later, we here define a lifted metric $\widetilde{\boldsymbol{g}}$ on T^*M by

$$\widetilde{\boldsymbol{g}} = g_{ij}\theta^i \otimes \theta^j + g^{ij}\theta^{\bar{i}} \otimes \theta^{\bar{j}}. \tag{21}$$

We adopt this metric on T^*M to discuss orthogonality of Lyapunov vectors on T^*M.

3.4 Levi-Civita connection of T^*M

The Christoffel symbols for the Levi-Civita connection on T^*M are brought about from the lifted metric $\widetilde{\boldsymbol{g}}$. The Christoffel symbols with respect to the standard frame are given by

$$\hat{\Gamma}^A_{BC} = \frac{1}{2}\hat{g}^{AD}\left(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC}\right),$$

where the Roman capitals A, B, C, D run from 1 to $2m$,

$$\partial_i = \frac{\partial}{\partial q^i}, \qquad \partial_{\bar{i}} = \frac{\partial}{\partial p_i}, \qquad (i = 1, \cdots, m)$$

and $\hat{g}_{AB} = \widetilde{\boldsymbol{g}}(\partial_A, \partial_B)$. Let us denote by Δ^A_α the transformation matrix between the standard frame and the adapted frame

$$D_\alpha = \Delta^A_\alpha \partial_A,$$

where Greek letters also run from 1 to $2m$, indicating the indices for the adapted frame.

By using the transformation matrix Δ^A_α, the Christoffel symbol with respect to the adapted frame and with respect to the standard frame are shown to be related as

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} = \Delta^\alpha_A \Delta^B_\beta \Delta^C_\gamma \hat{\Gamma}^A_{BC} + \Delta^\alpha_A D_\beta \Delta^A_\gamma.$$

The Christoffel symbols, $\hat{\Gamma}^A_{BC}$, with respect to the standard frame are symmetric in lower indices, B and C, but are not those with respect to the adapted frame, $\widetilde{\Gamma}^\alpha_{\beta\gamma}$. We describe the non-commutativity as Ω,

$$[D_\beta, D_\gamma] = \Omega_{\beta\gamma}{}^\alpha D_\alpha,$$

and then obtain

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} - \widetilde{\Gamma}^\alpha_{\gamma\beta} = \Delta^\alpha_A \left(D_\beta \Delta^A_\gamma - D_\gamma \Delta^A_\beta\right) = \Omega_{\beta\gamma}{}^\alpha.$$

We are to determine $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}$ as components of the Levi-Civita connection $\widetilde{\nabla}$ on T^*M. The covariant derivative of $\widetilde{\boldsymbol{X}}$ with respect to $\widetilde{\boldsymbol{Y}}$ is defined to be

$$\widetilde{\nabla}_{\widetilde{\boldsymbol{Y}}}\widetilde{\boldsymbol{X}} = \left[\widetilde{Y}^{\beta} D_{\beta}\widetilde{X}^{\alpha} + \widetilde{\Gamma}^{\alpha}_{\beta\gamma}\widetilde{Y}^{\beta}\widetilde{X}^{\gamma}\right] D_{\alpha}. \tag{22}$$

The covariant derivative of the lifted metric $\widetilde{\boldsymbol{g}}$ must vanish for all vector fields on T^*M,

$$\widetilde{\nabla}_{\widetilde{\boldsymbol{X}}}\widetilde{\boldsymbol{g}} = X^{\beta}\left[D_{\beta}\widetilde{g}_{\gamma\delta} - \widetilde{\Gamma}^{\varepsilon}_{\beta\gamma}\widetilde{g}_{\varepsilon\delta} - \widetilde{\Gamma}^{\varepsilon}_{\beta\delta}\widetilde{g}_{\gamma\varepsilon}\right]\theta^{\gamma}\otimes\theta^{\delta} = 0$$

which gives

$$D_{\beta}\widetilde{g}_{\gamma\delta} - \widetilde{\Gamma}^{\varepsilon}_{\beta\gamma}\widetilde{g}_{\varepsilon\delta} - \widetilde{\Gamma}^{\varepsilon}_{\beta\delta}\widetilde{g}_{\gamma\varepsilon} = 0,$$

and further

$$\begin{aligned} D_{\beta}\widetilde{g}_{\gamma\delta} + D_{\gamma}\widetilde{g}_{\delta\beta} - D_{\delta}\widetilde{g}_{\beta\gamma} &= \left(\widetilde{\Gamma}^{\epsilon}_{\beta\gamma} + \widetilde{\Gamma}^{\epsilon}_{\gamma\beta}\right)\widetilde{g}_{\epsilon\delta} + \left(\widetilde{\Gamma}^{\epsilon}_{\beta\delta} - \widetilde{\Gamma}^{\epsilon}_{\delta\beta}\right)\widetilde{g}_{\epsilon\gamma} + \left(\widetilde{\Gamma}^{\epsilon}_{\gamma\delta} - \widetilde{\Gamma}^{\epsilon}_{\delta\gamma}\right)\widetilde{g}_{\epsilon\beta} \\ &= \left(2\widetilde{\Gamma}^{\epsilon}_{\beta\gamma} - \Omega_{\beta\gamma}{}^{\epsilon}\right)\widetilde{g}_{\epsilon\delta} + \Omega_{\beta\delta}{}^{\epsilon}\widetilde{g}_{\epsilon\gamma} + \Omega_{\gamma\delta}{}^{\epsilon}\widetilde{g}_{\epsilon\beta}. \end{aligned}$$

Consequently, we obtain

$$\widetilde{\Gamma}^{\alpha}_{\beta\gamma} = \frac{1}{2}\widetilde{g}^{\alpha\delta}\left(D_{\beta}\widetilde{g}_{\gamma\delta} + D_{\gamma}\widetilde{g}_{\delta\beta} - D_{\delta}\widetilde{g}_{\beta\gamma}\right) + \frac{1}{2}\left(\Omega_{\beta\gamma}{}^{\alpha} + \Omega^{\alpha}{}_{\beta\gamma} + \Omega^{\alpha}{}_{\gamma\beta}\right),$$

where

$$\Omega^{\alpha}{}_{\beta\gamma} = \widetilde{g}^{\alpha\delta}\Omega_{\delta\beta}{}^{\epsilon}\widetilde{g}_{\epsilon\gamma}.$$

These components have the explicit form

$$\begin{aligned} &\widetilde{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk}, \quad \widetilde{\Gamma}^{i}_{j\bar{k}} = -\frac{1}{2}R_{j}{}^{ik\ell}p_{\ell}, \quad \widetilde{\Gamma}^{i}_{\bar{j}k} = -\frac{1}{2}R_{k}{}^{ij\ell}p_{\ell}, \quad \widetilde{\Gamma}^{i}_{\bar{j}\bar{k}} = 0, \\ &\widetilde{\Gamma}^{\bar{i}}_{jk} = \frac{1}{2}R_{jki}{}^{\ell}p_{\ell}, \quad \widetilde{\Gamma}^{\bar{i}}_{j\bar{k}} = -\Gamma^{k}_{ij}, \quad \widetilde{\Gamma}^{\bar{i}}_{\bar{j}k} = 0, \quad \widetilde{\Gamma}^{\bar{i}}_{\bar{j}\bar{k}} = 0. \end{aligned} \tag{23}$$

The covariant derivative of $\widetilde{\boldsymbol{X}}$ with respect to the lifted vector field $\widetilde{\boldsymbol{\xi}}$ is expressed as

$$\begin{aligned} (\widetilde{\nabla}_{\widetilde{\boldsymbol{\xi}}}\widetilde{\boldsymbol{X}})^{i} &= \frac{\mathrm{d}X^{i}}{\mathrm{d}s} + \Gamma^{i}_{kj}\xi^{k}X^{j} - \frac{1}{2}R_{k}{}^{ij\ell}p_{\ell}\xi^{k}X^{\bar{j}}, \\ (\widetilde{\nabla}_{\widetilde{\boldsymbol{\xi}}}\widetilde{\boldsymbol{X}})^{\bar{i}} &= \frac{\mathrm{d}X^{\bar{i}}}{\mathrm{d}s} - \Gamma^{j}_{ik}\xi^{k}X^{\bar{j}} + \frac{1}{2}R_{jki}{}^{\ell}p_{\ell}\xi^{k}X^{j}. \end{aligned} \tag{24}$$

In particular, we have

$$\widetilde{\nabla}_{\widetilde{\boldsymbol{\xi}}}\widetilde{\boldsymbol{\xi}} = 0,$$

which implies that the lift of a geodesic on M is also a geodesic on T^*M with respect to the lifted metric $\widetilde{\boldsymbol{g}}$.

4 Dynamics on the Cotangent Bundle

4.1 Linearization of Hamilton's equation of motion

For a Hamiltonian H given on the cotangent bundle T^*M, Hamilton's equation of motion is given by

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial H}{\partial x^i}. \tag{25}$$

The linearization of Hamilton's equation of motion is expressed, for a vector field $\widetilde{\boldsymbol{X}} = X^i\partial_i + \hat{X}^{\bar{i}}\partial_{\bar{i}}$ on T^*M, as

$$\begin{aligned}
\frac{\mathrm{d}X^i}{\mathrm{d}s} &= \frac{\partial^2 H}{\partial p_i \partial x^j}X^j + \frac{\partial^2 H}{\partial p_i \partial p_j}\hat{X}^{\bar{j}},\\
\frac{\mathrm{d}\hat{X}^{\bar{i}}}{\mathrm{d}s} &= -\frac{\partial^2 H}{\partial x^i \partial x^j}X^j - \frac{\partial^2 H}{\partial x^i \partial p_j}\hat{X}^{\bar{j}},
\end{aligned}$$

where $\widetilde{\boldsymbol{X}}$ stands for a deviation of Hamiltonian flows. With respect to the adapted frame, the equation of deviation for $\widetilde{\boldsymbol{X}} = X^iD_i + X^{\bar{i}}D_{\bar{i}}$ takes the form

$$\begin{aligned}
\frac{\mathrm{d}X^i}{\mathrm{d}s} &= \left[\frac{\partial^2 H}{\partial p_i \partial x^j} + \frac{\partial^2 H}{\partial p_i \partial p_\ell}p_k\Gamma^k_{\ell j}\right]X^j + \frac{\partial^2 H}{\partial p_i \partial p_j}X^{\bar{j}},\\
\frac{\mathrm{d}X^{\bar{i}}}{\mathrm{d}s} &= -\left[\frac{\partial^2 H}{\partial x^i \partial x^j} + \frac{\partial^2 H}{\partial x^i \partial p_\ell}p_k\Gamma^k_{j\ell} + \left(\frac{\partial^2 H}{\partial p_\ell \partial x^j} + \frac{\partial^2 H}{\partial p_\ell \partial p_m}p_n\Gamma^n_{mj}\right)p_k\Gamma^k_{i\ell}\right.\\
&\qquad \left. -\frac{\partial H}{\partial x^k}\Gamma^k_{ij} + p_k\frac{\partial \Gamma^k_{ij}}{\partial x^m}\frac{\partial H}{\partial p_m}\right]X^j - \left[\frac{\partial^2 H}{\partial x^i \partial p_j} + \frac{\partial^2 H}{\partial p_\ell \partial p_j}p_k\Gamma^k_{i\ell}\right]X^{\bar{j}}.
\end{aligned} \tag{26}$$

The geodesic flow on the cotangent bundle T^*M has the Hamiltonian

$$H^{(\mathrm{geo})}(x,p) = \frac{1}{2}g^{ij}(x)p_ip_j. \tag{27}$$

In fact, on account of the relation

$$-\frac{\partial g^{k\ell}}{\partial x^i} = g^{km}\Gamma^\ell_{mi} + g^{m\ell}\Gamma^k_{mi},$$

the Hamiltonian vector field associated with (27) is put in the form

$$X_H = \frac{\partial H^{(\mathrm{geo})}}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H^{(\mathrm{geo})}}{\partial x^i}\frac{\partial}{\partial p_i} = g^{ij}p_jD_i,$$

which turns out to be equal to $\widetilde{\boldsymbol{\xi}}$ with the condition $p_i = g_{ij}\xi^j$. Substituting the Hamiltonian (27) into the equation of deviation (26), we obtain the following

$$\begin{aligned}
\frac{\mathrm{d}X^i}{\mathrm{d}s} &= -\Gamma^i_{jk}\xi^kX^j + g^{ij}X^{\bar{j}},\\
\frac{\mathrm{d}X^{\bar{i}}}{\mathrm{d}s} &= -R_{jk\ell i}\xi^k\xi^\ell X^j + \Gamma^j_{ik}\xi^kX^{\bar{j}}.
\end{aligned} \tag{28}$$

If $\widetilde{\boldsymbol{X}}$ is a lifted vector field, and if $\boldsymbol{P} = g^{ij} X^{\bar{j}} \dfrac{\partial}{\partial x^i} \in \mathfrak{X}(M)$ is imposed, the above equation reduces to the Jacobi equation. Thus we have fount that equation (28) is the first-order differential equation associated with the Jacobi equation, which we call the lifted Jacobi equation.

It is to be noted that equation (28) is equivalently written as

$$\widetilde{\nabla}_{\widetilde{\boldsymbol{\xi}}} \widetilde{\boldsymbol{X}} = \widetilde{\nabla}_{\widetilde{\boldsymbol{X}}} X_H. \tag{29}$$

4.2 Lyapunov vectors

In this subsection, we are to verify that Lyapunov vectors to be constructed on the geometric method set up above satisfies the requirement mentioned in section 1. Let $T_s T^*M$ be the tangent space to T^*M at $(x(s), p(s)) \in T^*M$. We denote that gradient vector field by $(\mathrm{d}H^{(\mathrm{geo})})^* = p_i D_{\bar{i}}$, which is dual to the one-form $\mathrm{d}H^{(\mathrm{geo})} = \xi^i \theta^{\bar{i}}$. With this notation, we define a subspace of $T_s T^*M$ to be a space spanned by $\widetilde{\boldsymbol{\xi}}$ and $(\mathrm{d}H^{(\mathrm{geo})})^*$, and its orthogonal complement as follows:

$$H_s = \{\widetilde{\boldsymbol{X}} \in T_s T^*M \mid \widetilde{\boldsymbol{X}} = (\alpha \widetilde{\boldsymbol{\xi}} + \beta (\mathrm{d}H^{(\mathrm{geo})})^*)|_s,\ \alpha, \beta \in \boldsymbol{R}\},$$
$$V_s = \{\widetilde{\boldsymbol{X}} \in T_s T^*M \mid \widetilde{\boldsymbol{g}}(\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{\xi}})|_s = 0,\ \mathrm{d}H^{(\mathrm{geo})}(\widetilde{\boldsymbol{X}})|_s = 0\}.$$

We show

1. that a lifted Jacobi field which is in H_0 at an initial time remains to be in H_s at any instance s,
2. and that a lifted Jacobi field which is in V_0 at an initial time remains in V_s at any instance s.

On the basis of these facts, we can construct a set of Lyapunov vectors $\{\boldsymbol{V}_1, \boldsymbol{V}_2, \cdots, \boldsymbol{V}_{2m}\}$ satisfying the requirement mentioned in section 1. In this subsection, we adopt the Riemannian manifold $(M_J, \boldsymbol{g}_J)$ introduced in subsection 2.3, and hence $m = N$.

The proof of the above facts is carried out as follows: First it is easy to observe that $\widetilde{\boldsymbol{\xi}}$ and $(\mathrm{d}H^{(\mathrm{geo})})^* + s\,\widetilde{\boldsymbol{\xi}}$ are solutions to the lifted Jacobi equation (28), which is in H_s. Accordingly, a lifted Jacobi field $\widetilde{\boldsymbol{X}} = (\alpha - \beta\, s)\widetilde{\boldsymbol{\xi}} + \beta((\mathrm{d}H^{(\mathrm{geo})})^* + s\,\widetilde{\boldsymbol{\xi}})$ is in H_s at any instance s. To prove the second fact, we consider the temporal evolution of $\widetilde{\boldsymbol{g}}(\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{\xi}})$ for a lifted Jacobi field $\widetilde{\boldsymbol{X}}$. The first- and second-order derivations of $\widetilde{\boldsymbol{g}}(\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{\xi}})$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\boldsymbol{g}}(\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{\xi}}) = \mathrm{d}H^{(\mathrm{geo})}(\widetilde{\boldsymbol{X}}),$$
$$\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{d}H^{(\mathrm{geo})}(\widetilde{\boldsymbol{X}}) = 0,$$

respectively. Here we used the equations (24), (28) and (13). The above equations result in

$$\widetilde{g}(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{\xi}})|_s = \widetilde{g}(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{\xi}})|_{s=0} + s\ \mathrm{d}H^{(\mathrm{geo})}(\widetilde{\boldsymbol{X}})|_{s=0},$$
$$\mathrm{d}H^{(\mathrm{geo})}(\widetilde{\boldsymbol{X}})|_s = \text{constant},$$

which implies that $\widetilde{\boldsymbol{X}}(s) \in V_s$ if $\widetilde{\boldsymbol{X}}(0) \in V_0$.

We conclude this section with a construction of Lyapunov vectors satisfying the three requirements. The first $N-1$ linearly independent solutions to the lifted Jacobi equation are chosen in V_s, which are orthogonalized to give first $N-1$ Lyapunov vectors $\{\boldsymbol{V}_1, \cdots, \boldsymbol{V}_{N-1}\}$. The N-th and $(N+1)$-th Lyapunov vectors are chosen as $\widetilde{\boldsymbol{\xi}}$ and $(\mathrm{d}H^{(\mathrm{geo})})^*$, respectively. This is because they are mutual orthogonal and because $\widetilde{\boldsymbol{\xi}}$ and $(\mathrm{d}H^{(\mathrm{geo})})^* + s\ \widetilde{\boldsymbol{\xi}}$ are solutions to the lifted Jacobi equation and further orthogonal to the first $N-1$ Lyapunov vectors. The remaining $N-1$ Lyapunov vectors are chosen in V_s which are orthogonal to $\widetilde{\boldsymbol{\xi}}$ and $(\mathrm{d}H^{(\mathrm{geo})})^*$ by the very definition. Consequently, we can obtain a set of Lyapunov vectors satisfying the three requirements by choosing $\widetilde{\boldsymbol{X}}_i(0) \in V_0$ $(i = 1, \cdots, N-1, N+2, \cdots, 2N)$, $\widetilde{\boldsymbol{X}}_N(0) = \boldsymbol{\xi}(0)$ and $\widetilde{\boldsymbol{X}}_{N+1}(0) = (\mathrm{d}H^{(\mathrm{geo})})^*(0)$ at the initial time $s = 0$.

5 Numerical Calculations

We numerically compare Lyapunov exponents and Lyapunov vectors obtained by the geometric method and ones by usual method in a model system with 3 degrees of freedom. We use the configuration space and the Jacobi metric introduced in subsection 2.3.

5.1 Model

We introduce a model with 3 degrees of freedom which has interactions of Hénon-Heiles type,

$$\begin{aligned} H(q,p) &= \sum_{i=1}^{3} \left[\frac{1}{2} p_i^2 + V_{HH}(q^i, q^{i+1}) \right], \\ V_{HH}(x,y) &= x^2 y - \frac{1}{3} y^3, \end{aligned} \tag{30}$$

where $q^4 = q^1$.

Temporal evolutions of $(q(t), p(t))$ and tangent vectors in usual method are performed by 4th order symplectic integrator [Yos93], which gives explicit discretization keeping symplectic properties. On the other hand, discretization of the lifted Jacobi equation (28) must be implicit if we keep the symplectic properties, and hence we use 6th order symplectic implicit Runge-Kutta method (Kuntzmann & Butcher method, see Appendix A) [HNW93]. We set the time slice as $h = 2.5 \times 10^{-6}$. Initial conditions on T^*M are $q^j(0) = 0$ and $p_j(0) = \alpha\gamma_j$ $(j = 1, 2, 3)$, where each random variable γ_j follows the uniform distribution function on the interval $[0, 1]$ and the constant α is determined as satisfying the energy condition $\sum_{j=1}^{3}(p_j(0))^2/2 = E$.

5.2 Results

Convergence of Lyapunov exponents is confirmed in figure 1 for $E = 0.04$, where $\Lambda_i(t)$ is defined as

$$\Lambda_i(t) = \frac{1}{t}\ln\frac{||V_i(t)||}{||V_i(0)||}$$

and $\lim_{t\to\infty}\Lambda_i(t) = \lambda_i$. We got similar convergence also for $E = 0, 01,\ 0.02$ and 0.03, and energy dependence of Lyapunov exponents are shown in figure 2. Lyapunov exponents by the geometric method and usual method are good agreement with each other.

The requirements introduced in section 1 are checked in figures 3 and 4. The former shows temporal evolutions of inner products between normalized Lyapunov vectors and $\widetilde{\boldsymbol{\xi}}$, and the latter shows temporal evolutions of inner products between normalized Lyapunov vectors and the normalized gradient vector of Hamiltonian. Initial conditions of tangent vectors are $\boldsymbol{V}_3(0) = \widetilde{\boldsymbol{\xi}}$, $\boldsymbol{V}_4(0) = (\mathrm{d}H^{(\mathrm{geo})})^*$ and $\boldsymbol{V}_i(0) \in V_0$ $(i = 1, 2, 5, 6)$. In the geometric method, Lyapunov vectors except for the 3rd one are always orthogonal to tangential direction of a trajectory, and the 3rd one points to the tangential direction at any time. Moreover, Lyapunov vectors except for the 4th one are always orthogonal to the gradient direction of the Hamiltonian function, and the 4th one points to the gradient direction at any time. That is, the three requirements are satisfied in the geometric method with a little numerical error. On the other hand, the requirements are not satisfied in usual method.

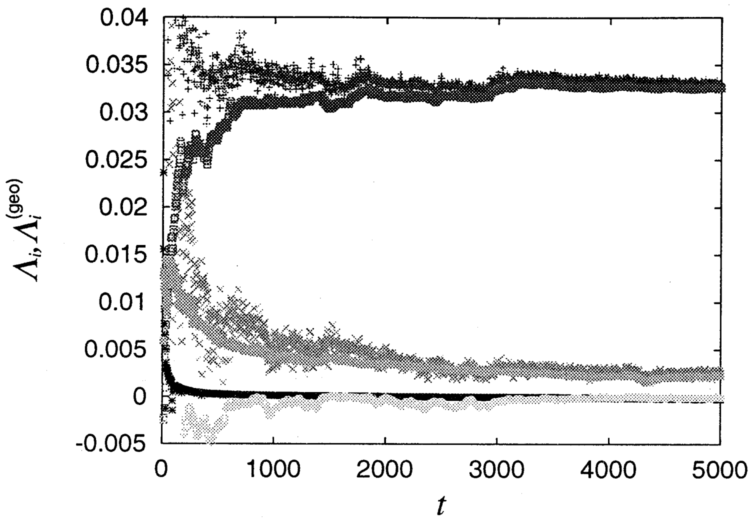


図 1: Convergence of Lyapunov exponents. $E = 0.04$. Curves represent $\Lambda_1^{(geo)}$,Λ_1, $\Lambda_2^{(geo)}$,Λ_2, $\Lambda_3^{(geo)}$,Λ_3 from top to bottom, where $\Lambda_i^{(geo)}$ are obtained by the geometric method and Λ_i are usual method.

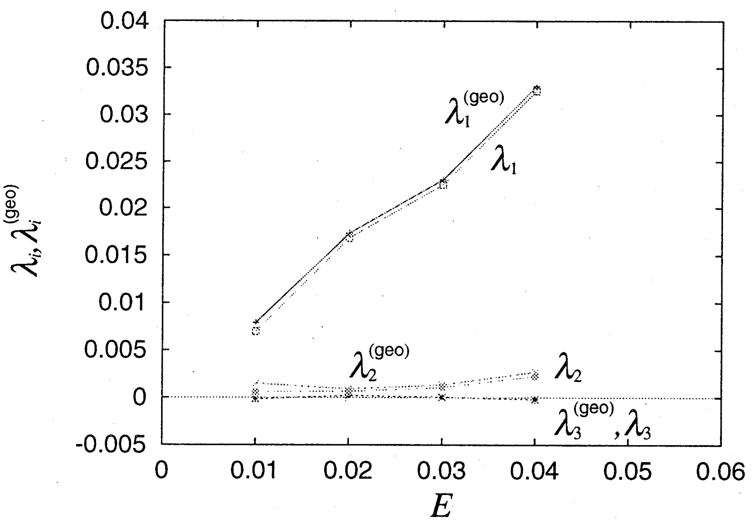
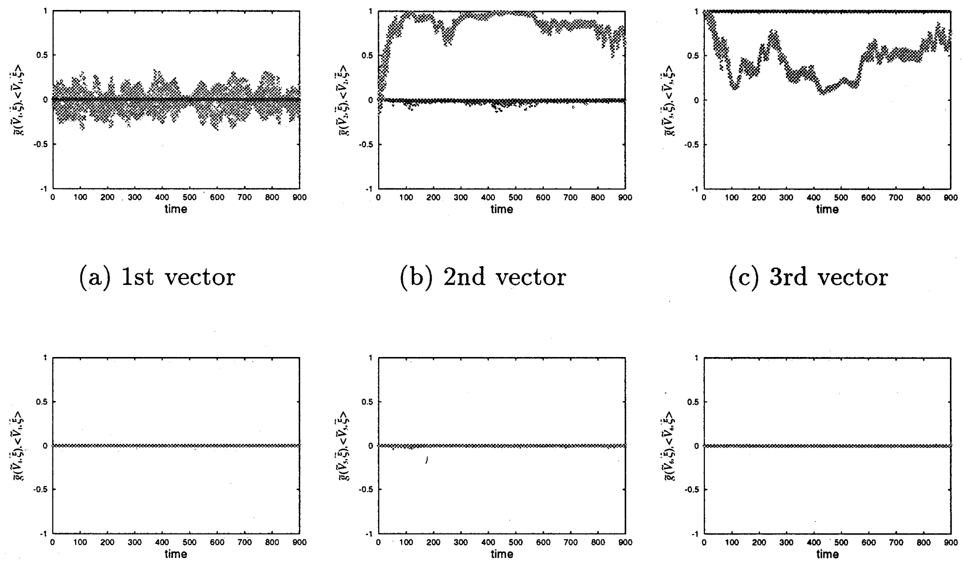


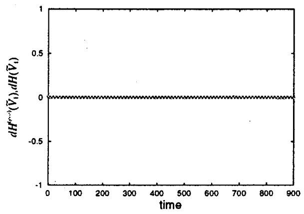
図 2: Comparison of Lyapunov exponents obtained by the geometric method and by usual method. $\lambda_i^{(geo)}$ are obtained by the geometric method and λ_i are usual method, and both the methods give good agreement.



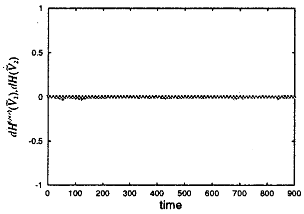
(a) 1st vector (b) 2nd vector (c) 3rd vector

(d) 4th vector (e) 5th vector (f) 6th vector

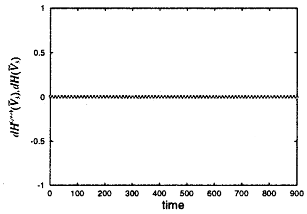
図 3: Temporal evolutions of inner products between $\widetilde{\boldsymbol{\xi}}$ and normalized Lyapunov vectors. Dark curves are from the geometric method $\widetilde{g}(\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{\xi}})$, and gray curves are usual method $< \widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{\xi}} >$. The 1st and 2nd Lyapunov vectors are always orthogonal to tangential direction of a trajectory, $\widetilde{\boldsymbol{\xi}}$, in the geometric method, but not always orthogonal in usual method. Moreover, the 3rd Lyapunov vector always points to the direction of $\widetilde{\boldsymbol{\xi}}$ in the geometric method, but does not in usual method. In (d),(e),(f), dark lines are hidden by gray lines.



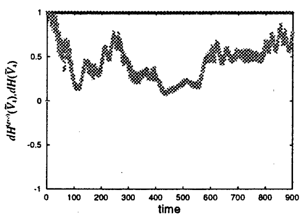
(a) 1st vector



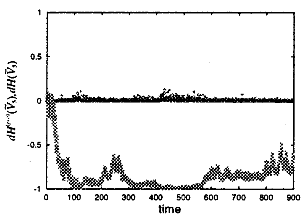
(b) 2nd vector



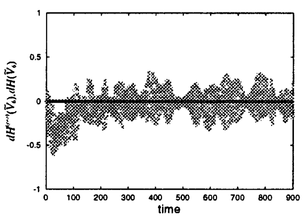
(c) 3rd vector



(d) 4th vector



(e) 5th vector



(f) 6th vector

図 4: Temporal evolutions of inner products gradient of the Hamiltonian function and normalized Lyapunov vectors. Dark curves are from the geometric method $\mathrm{d}H^{(\mathrm{geo})}(\widetilde{\boldsymbol{V}})$, and gray curves are usual method $\mathrm{d}H(\widetilde{\boldsymbol{V}})$. The 4th Lyapunov vector always points to gradient direction of the Hamiltonian function, but not always in usual method. Moreover, the 5th and 6th Lyapunov vectors are always orthogonal to the gradient direction in the geometric method, but not in usual method. In (a),(b),(c), dark lines are hidden by gray lines.

6 Summary

In this paper we developed a geometric method to calculate Lyapunov exponents and Lyapunov vectors for natural Hamiltonian systems with N degrees of freedom. A former geometric method uses Jacobi equations to consider orbital instability, but the Jacobi equations are second order differential equations while Lyapunov exponents and vectors are defined through first order differential equations. We therefore lifted the Jacobi equations from Riemannian manifolds to their cotangent bundles in order to recover first order differential equations.

Lyapunov exponents and vectors may change the values from usual method which uses linearized Newton's equations of motion, when the independent parameter s depends on position as the lifted Jacobi equation, i.e. $\mathrm{d}s = \phi(x)\mathrm{d}t$. We numerically compared the values of Lyapunov exponents between the two equations for a model system with 3 degrees of freedom, and obtain the same Lyapunov exponents. We guess that the Lyapunov exponents are invariant if $\phi(x(t)) < \infty$ where $x(t)$ is an arbitrary trajectory.

In the geometric method, we can choose Lyapunov vectors satisfy the following requirements: (i) Lyapunov vectors except for mth and $(m+1)$th vectors are always orthogonal to both tangent direction of trajectory and gradient direction of Hamiltonian function, (ii) Nth Lyapunov vector points to the tangent direction of trajectory, and (iii) $(N+1)$th Lyapunov vector points to the gradient direction of Hamiltonian function. From such Lyapunov vectors, we can obtain purely (un)stable directions in phase spaces without influences of zero Lyapunov exponents corresponding to the Nth and $(N+1)$th Lyapunov vectors, while we cannot do by usual method in general systems (one of exception is a system consists of harmonic oscillators).

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A Symplectic Implicit Runge-Kutta method

Let us consider a dynamical system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x,t), \tag{31}$$

and discretizing this equation with time slice h. The s-stage Runge-Kutta method is represented as

$$\begin{aligned} x' &= x + h\sum_{i=1}^{s} b_i k_i \\ k_i &= f(x + h\sum_{j=1}^{s} a_{ij}k_j, t + c_i h) \quad (i = 1, \cdots, s) \end{aligned}$$

where (x,t) goes to $(x', t+h)$ after one step, and a_{ij}, b_i and c_i are real constants with $\sum_{i=1}^{s} c_i = 1$. Note that the second equation is implicit. The 6th order Kuntzmann & Butcher method is defined as table 1.

$\frac{1}{2} - \frac{\sqrt{15}}{10}$	$\frac{5}{36}$	$\frac{2}{9} - \frac{\sqrt{15}}{15}$	$\frac{5}{36} - \frac{\sqrt{15}}{30}$
$\frac{1}{2}$	$\frac{5}{36} + \frac{\sqrt{15}}{24}$	$\frac{2}{9}$	$\frac{5}{36} - \frac{\sqrt{15}}{24}$
$\frac{1}{2} + \frac{\sqrt{15}}{10}$	$\frac{5}{36} + \frac{\sqrt{15}}{30}$	$\frac{2}{9} + \frac{\sqrt{15}}{15}$	$\frac{5}{36}$
	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$

表 1: Kuntzmann & Butcher method, order 6. The upper right block is the matrix (a_{ij}), the left column is the vector (b_i) and the lower raw is the vector (c_i).