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Mini-Maximizers for Reaction-Diffusion Systems with Skew-Gradient Structure

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1 Introduction

In this paper, we deal with reaction-diffusion systems with *skew-gradient* structure, which was introduced in [15] as a generalized activator-inhibitor system. Let us consider $(m+n)$-component reaction-diffusion systems of the form

$$\begin{cases} Su_t = C\Delta u + f(u,v) & \text{in } \Omega, \\ Tv_t = D\Delta v + g(u,v) & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}u = 0 = \dfrac{\partial}{\partial\nu}v & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1, \cdots, u_m)^t$ and $v(x,t) = (v_1, \cdots, v_n)^t$, Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ stands for the outward normal derivative on $\partial\Omega$, S and C are mth order positive definite symmetric matrices, T and D are nth order positive definite symmetric matrices. We assume that for some C^3-function $H(u,v) : \mathbf{R}^{m+n} \to \mathbf{R}$, the nonlinear terms $f = (f_1, \cdots, f_m)^t : \mathbf{R}^{m+n} \to \mathbf{R}^m$ and $g = (g_1, \cdots, g_n)^t : \mathbf{R}^{m+n} \to \mathbf{R}^n$ are expressed as

$$f(u,v) = +\nabla_u H(u,v), \qquad g(u,v) = -\nabla_v H(u,v), \tag{1.2}$$

where ∇_u and ∇_v are gradient operators with respect to u and v, respectively, i.e.,

$$\nabla_u := (\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_m})^t, \qquad \nabla_v := (\frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n})^t.$$

In this case, we say that the system (1.1) has skew-gradient structure.

Any steady state $(u,v)=(\varphi(x),\psi(x))$ of (1.1) satisfies the system of elliptic equations

$$\begin{cases} C\Delta\varphi + f(\varphi,\psi)=0 & \text{in } \Omega, \\ D\Delta\psi + g(\varphi,\psi)=0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}\varphi = 0 = \dfrac{\partial}{\partial\nu}\psi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

We note that the solution of this problem corresponds to a critical point of the functional

$$E[u,v] := \int_\Omega \left\{ \frac{1}{2}\langle C\nabla u, \nabla u\rangle - \frac{1}{2}\langle D\nabla v, \nabla v\rangle - H(u,v) \right\} dx, \tag{1.4}$$

where ∇ is a gradient operator with respect to x and

$$\langle C\nabla u, \nabla u\rangle := \sum_{i,j=1}^{m} c_{ij}\nabla u_i \cdot \nabla u_j, \quad \langle D\nabla v, \nabla v\rangle := \sum_{i,j=1}^{n} d_{ij}\nabla v_i \cdot \nabla v_j,$$

with $C=(c_{ij})$ and $D=(d_{ij})$. In fact, (1.3) is the Euler-Lagrange equation for $E[u,v]$. We say that $(u,v)=(\varphi,\psi)$ is a *mini-maximizer* of $E[u,v]$ if $u=\varphi$ is a minimizer of $E[u,\psi]$ and $v=\psi$ is a maximizer of $E[\varphi,v]$. (More precise definitions will be given in the next section.) The purpose of this paper is to study the relation between a stability property of $(u,v)=(\varphi,\psi)$ as a steady state of (1.1) and a mini-maximizing property as a critical point of $E[u,v]$.

When v is fixed to $\psi(x)$ in the first equation of (1.1), then we have a system for u

$$\begin{cases} Su_t = C\Delta u + f(u,\psi) & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

For any solution $u(x,t)$ of this equation, we have

$$\begin{aligned} \frac{d}{dt}E[u(x,t),\psi(x)] &= \int_\Omega \left\{ \langle C\nabla u, \nabla u_t\rangle - f(u,\psi)\cdot u_t \right\} dx \\ &= \int_\Omega \left\{ -C\Delta u \cdot u_t - f(u,\psi)\cdot u_t \right\} dx \\ &= -\int_\Omega Su_t \cdot u_t dx \le 0. \end{aligned}$$

Hence (1.5) describes a gradient flow of $E[u,\psi]$. Therefore, $u=\varphi$ is a steady state of (1.5) if and only if $u=\varphi$ is a critical point of $E[u,\psi]$, and is stable if and only if it is a local minimizer of $E[u,\psi(x)]$.

Similarly, when u is fixed to $\varphi(x)$ in the second equation of (1.1), then we have a system for v

$$\begin{cases} Tv_t = D\Delta v + g(\varphi,v) & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

For any solution $v(x,t)$ of this equation, we have

$$\begin{aligned}\frac{d}{dt}E[\varphi(x),v(x,t)] &= \int_\Omega \{-\langle D\nabla v, \nabla v_t\rangle + g(\varphi,v)\cdot v_t\}\,dx \\ &= \int_\Omega \{D\Delta v\cdot v_t + g(\varphi,v)\cdot v_t\}\,dx \\ &= \int_\Omega Tv_t\cdot v_t dx \geq 0.\end{aligned}$$

Hence (1.6) describes a gradient flow of $-E[\varphi,v]$. Therefore, $v=\psi$ is a steady state of (1.6) if and only if $v=\psi$ is a critical point of $E[\varphi,v]$, and is stable as a steady state of (1.6) if and only if it is a maximizer of (1.6).

On the other hand, it follows from (1.2) that

$$f_v := \nabla_v f = \left(\frac{\partial f_i}{\partial v_j}\right) = \left(+\frac{\partial^2 H}{\partial u_i \partial v_j}\right)$$

and

$$g_u := \nabla_u g = \left(\frac{\partial g_i}{\partial u_j}\right) = \left(-\frac{\partial^2 H}{\partial u_j \partial v_i}\right).$$

Hence

$$f_v = -g_u^t.$$

Thus, roughly speaking, the reaction-diffusion system with skew-gradient structure is a sort of activator-inhibitor system which consists of two gradient systems coupled in a skew-symmetric way.

Even if $u=\varphi$ is a minimizer of $E[u,\psi]$ and $v=\psi$ is a maximizer of $E[\varphi,v]$, due to the interaction between u and v, it does not automatically mean that $(u,v)=(\varphi,\psi)$ is stable as a steady state of (1.1). In fact, if (u,v) is a solution of (1.1), then

$$\begin{aligned}&\frac{d}{dt}E[u(x,t),v(x,t)] \\ &= \int_\Omega \left\{\frac{1}{2}\langle C\nabla u, \nabla u_t\rangle - \frac{1}{2}\langle D\nabla v, \nabla v_t\rangle - f(u,v)\cdot u_t + g(u,v)v_t\right\}dx \\ &= \int_\Omega \{-Su_t\cdot u_t + Tv_t\cdot v_t\}\,dx.\end{aligned}$$

Hence $E[u,v]$ is not necessarily nonincreasing or nondecreasing in t, and cannot be used as a Liapunov functional. Nonetheless, we can show that $(u,v)=(\varphi,\psi)$ is stable as a steady state of (1.1) for any S and T if it is a mini-maximizer of $E[u,v]$.

Let (φ,ψ) be a solution of (1.3). As is well-known [7], stability of $(u,v)=(\varphi,\psi)$ as a steady state of (1.1) can be determined by analyzing the eigenvalue problem

$$\begin{cases}\lambda SU = C\Delta U + f_u U + f_v V, \\ \lambda TV = D\Delta V + g_u U + g_v V,\end{cases} \tag{1.7}$$

on Ω under the Neumann boundary conditions, where f_u, f_v, g_u and g_v are evaluated at (φ, ψ). Since this is not a self-adjoint eigenvalue problem, there may exist complex eigenvalues. Usually, in such a situation, it is extremely difficult to locate the eigenvalues. However, if $(u, v) = (\varphi, \psi)$ is a mini-maximizer of $E[u, v]$, we can show by using the skew-symmetric structure that any eigenvalue has a negative real part regardless of the choice of S and T. Conversely, if $(u, v) = (\varphi, \psi)$ is not a mini-maximizer of $E[u, v]$, then there exists a positive eigenvalue for some S and T.

A remarkable property is that any mini-maximizer must be spatially homogeneous if the domain Ω is convex. This kind of result was proved by Casten and Holland [1] and Matano [10] for scalar reaction-diffusion equation, and by Jimbo and Morita [5] and Lopes [9] for gradient systems. From this property together with the spectral characterization of mini-maximizers, we can derive quite a general instability criterion for some activator-inhibitor systems.

This paper is organized as follows. In Section 2, we give some definitions and preliminary results. In Section 3, the stability of steady states of skew-gradient systems is precisely investigated. In Section 4, we prove that if the domain is convex, any mini-maximizer must be spatially homogeneous. Then we derive a general criterion for the instability of spatially inhomogeneous steady states. Finally, in Section 5, we apply our results to the diffusive FitzHugh-Nagumo system and the Gierer-Meinhardt system.

2 Definitions and preliminaries

In this section, we give precise definitions concerning critical points of $E[u, v]$, and then describe their fundamental properties.

We say that $(u, v) = (\varphi, \psi)$ is a *mini-maximizer* of $E[u, v]$ if $u = \varphi$ is a local minimizer of $E[u, \psi]$ and $v = \psi$ is a local maximizer of $E[\varphi, v]$. More precisely, $(u, v) = (\varphi, \psi)$ is a mini-maximizer of $E[u, v]$ if

$$E[U, \psi] \geq E[\varphi, \psi]$$

for any U in a neighborhood of φ in $H^1(\Omega)$, and

$$E[\varphi, V] \leq E[\varphi, \psi]$$

for any V in a neighborhood of ψ in $H^1(\Omega)$. A critical point $u = \varphi$ of $E[u, \psi]$ is said to be *nondegenerate* if the linearized operator

$$\mathcal{A} := C\Delta + f_u \tag{2.1}$$

is invertible, where $f_u = f_u(\varphi, \psi)$ is an $m \times m$ symmetric matrix given by

$$f_u := \nabla_u f = \left(\frac{\partial f_i}{\partial u_j}\right) = \left(+\frac{\partial^2 H}{\partial u_i \partial u_j}\right).$$

Similarly, a critical point $v = \psi$ of $E[\varphi, v]$ is said to be *nondegenerate* if the linearized operator

$$\mathcal{B} := D\Delta + g_v \tag{2.2}$$

is invertible, where $g_v = g_v(\varphi, \psi)$ is an $n \times n$ symmetric matrix given by

$$g_v := \nabla_v g = \left(\frac{\partial g_i}{\partial v_j}\right) = \left(-\frac{\partial^2 H}{\partial v_i \partial v_j}\right).$$

Finally, we say that $(u, v) = (\varphi, \psi)$ is a *nondegenerate* critical point of $E[u, v]$ if $u = \varphi$ and $v = \psi$ are nondegenerate critical points of $E[u, \psi]$ and $E[\varphi, v]$, respectively.

Next, we describe some properties of the eigenvalue problem

$$\begin{cases} \lambda S U = \mathcal{A} U & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} U = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $\mathcal{A}$ is given by (2.1).

Lemma 2.1 *All eigenvalues of* (2.3) *are real. Moreover, there exists a maximal eigenvalue λ^u with finite multiplicity that is characterized by*

$$\lambda^u = \sup_{U \in H^1(\Omega)} \frac{\displaystyle\int_\Omega \{-\langle C\nabla U, \nabla U \rangle + f_u U \cdot U\}\, dx}{\displaystyle\int_\Omega SU \cdot U dx},$$

and the supremum is attained by an eigenfunction of (2.3) *associated with λ^u.*

Proof. Since f_u is symmetric, it is shown by a standard argument that $\mathcal{A}$ is self-adjoint and all eigenvalues of (2.3) are real. Moreover, from the variational principle for self-adjoint eigenvalue problems, there exists a maximal eigenvalue with finite multiplicity that is characterized as above. □

We see from the above lemma that the maximal eigenvalue λ^u depends on S but its sign does not depend on S. We say that $u = \varphi$ is *linearly stable* if $\lambda^u < 0$ and *linearly unstable* if $\lambda^u > 0$ as a steady state of (1.5).

Lemma 2.2 *Let (φ, ψ) be a solution of* (1.3). *Then the following holds.*

(i) *$u = \varphi$ is linearly stable as a steady state of (1.5) if and only if it is a nondegenerate local minimizer of $E[u, \psi]$.*

(ii) *If $u = \varphi$ is linearly unstable, then it is not a local minimizer of $E[u, \psi]$.*

Proof. Let $U \in H^1(\Omega)$ be fixed and let $\varepsilon > 0$ be a small parameter. Since φ is a critical point of $E[u,\psi]$, we have

$$\begin{aligned}
&E[\varphi+\varepsilon U,\psi] - E[\varphi,\psi] \\
&= \int_\Omega \Big\{ \frac{1}{2} \langle C\nabla(\varphi+\varepsilon U), \nabla(\varphi+\varepsilon U) \rangle - \frac{1}{2} \langle C\nabla\varphi, \nabla\varphi \rangle \\
&\qquad\qquad - H(\varphi+\varepsilon U,\psi) + H(\varphi,\psi) \Big\} dx \\
&= \varepsilon^2 \int_\Omega \Big\{ \langle C\nabla U, \nabla U \rangle - f_u U \cdot U \Big\} dx + O(\varepsilon^3).
\end{aligned}$$

Suppose that φ is a local minimizer. Then we have

$$\int_\Omega \Big\{ \langle C\nabla U, \nabla U \rangle - f_u U \cdot U \Big\} dx \geq 0$$

for any $U \in H^1(\Omega)$. By Lemma 2.1, this implies $\lambda^u \leq 0$. Moreover, if φ is nondegenerate, then $\lambda^u \neq 0$. Thus, if $u = \varphi$ is a nondegenerate local minimizer of $E[u,\psi]$, then $\lambda^u < 0$. Conversely, if $\lambda^u < 0$, then

$$\int_\Omega \{ \langle C\nabla U, \nabla U \rangle - f_u U \cdot U \}\, dx > 0$$

for any $U \in H^1(\Omega)$ with $U \not\equiv 0$. Hence $u = \varphi$ is a nondegenerate local minimizer. Thus the proof of (i) is complete.

Next, suppose $\lambda^u > 0$. In this case, by Lemma 2.1, we have

$$\int_\Omega \Big\{ \langle C\nabla U, \nabla U \rangle - f_u U \cdot U \Big\} dx < 0$$

for some $U \in H^1(\Omega)$ with $U \not\equiv 0$. Then

$$E[\varphi+\varepsilon U,\psi] - E[\varphi,\psi] < 0$$

if $\varepsilon > 0$ is sufficiently small. Hence $u = \varphi$ is not a local minimizer. This proves (ii). □

Next, we consider the eigenvalue problem

$$\begin{cases} \lambda T V = \mathcal{B} V & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} V = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

The following lemmas can be obtained in the same manner as the above lemmas for (2.3).

Lemma 2.3 *All eigenvalues of* (2.4) *are real. Moreover, there exists a maximal eigenvalue* λ^v *with finite multiplicity that is characterized by*

$$\lambda^v = \sup_{V \in H^1(\Omega)} \frac{\int_\Omega \Big\{ - \langle D\nabla V, \nabla V \rangle + g_v V \cdot V \Big\} dx}{\int_\Omega TV \cdot V dx},$$

and the supremum is attained by an eigenfunction of (2.3) *associated with* λ^v.

We note that the maximal eigenvalue λ^v depends on T but its sign does not depend on T. We say that $u = \psi$ is linearly stable if $\lambda^v < 0$ and is linearly unstable if $\lambda^v > 0$ as a steady state of (1.6).

Lemma 2.4 *Let* (φ, ψ) *be a solution of* (1.3). *Then the following holds.*

(i) $v = \psi$ *is linearly stable as a steady state of* (1.6) *if and only if it is a nondegenerate local maximizer of* $E[\varphi, v]$.

(ii) *If* $v = \psi$ *is linearly unstable, then* $v = \psi$ *is not a local maximizer of* $E[\varphi, v]$.

We see from the above lemmas that (φ, ψ) is a nondegenerate mini-maximizer of $E[u, v]$ if and only if both $u = \varphi$ and $v = \psi$ are linearly stable.

3 Stability of steady states

Let (φ, ψ) be a solution of (1.3). In order to study the stability of $(u, v) = (\varphi, \psi)$ as a steady state of (1.1), we rewrite the eigenvalue problem (1.7) as

$$\begin{cases} \lambda SU = \mathcal{A}U + f_v V, \\ \lambda TV = \mathcal{B}V + g_u U, \end{cases} \tag{3.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the operators defined by (2.1) and (2.2), respectively, $f_v = f_v(\varphi, \psi)$ and $g_u = g_u(\varphi, \psi)$. We note that the eigenvalue λ and the eigenfunction (U, V) of (3.1) may be complex-valued. We say that $(u, v) = (\varphi, \psi)$ is *linearly stable* as a steady state of (1.1) if for some $\delta > 0$, all eigenvalues of (3.1) satisfy $\Re\{\lambda\} < -\delta$. Conversely, the steady state is said to be *linearly unstable* if there exists an eigenvalue of (3.1) with a positive real part. It is well-known [7] that the linearly stable (resp. unstable) steady state is stable (resp. unstable) in the sense of Lyapunov.

First we consider the case where (φ, ψ) is a nondegenerate mini-maximizer of $E[u, v]$.

Theorem 3.1 *Let* $(u, v) = (\varphi, \psi)$ *be a nondegenerate mini-maximizer of* $E[u, v]$. *Then, for any* S *and* T, $(u, v) = (\varphi, \psi)$ *is linearly stable as a steady state of* (1.1).

Proof. From

$$\begin{cases} \lambda SU = \mathcal{A}U + f_v V, \\ \overline{\lambda} T\overline{V} = \mathcal{B}\overline{V} + g_u \overline{U}, \end{cases}$$

and $f_v = -g_u^t$, we have

$$\lambda \int_\Omega SU \cdot \overline{U}\, dx + \overline{\lambda} \int_\Omega T\overline{V} \cdot V\, dx = \int_\Omega \mathcal{A}U \cdot \overline{U}\, dx + \int_\Omega \mathcal{B}\overline{V} \cdot V\, dx. \tag{3.2}$$

Here, since S and T are positive definite symmetric matrices, the integrals

$$\int_\Omega SU \cdot \overline{U}\, dx, \quad \int_\Omega T\overline{V} \cdot V\, dx,$$

must be positive. On the other hand, by partial integration, we have

$$\int_\Omega \mathcal{A}U \cdot \overline{U}\, dx = \int_{\partial\Omega} C\frac{\partial}{\partial\nu}U \cdot \overline{U}\, dx + \int_\Omega \left\{ -\langle C\nabla U, \nabla\overline{U} \rangle + f_u U \cdot \overline{U} \right\} dx. \tag{3.3}$$

The first term in the right-hand side vanishes due to the boundary condition, and the second term satisfies

$$\int_\Omega \left\{ -\langle C\nabla U, \nabla\overline{U} \rangle + f_u U \cdot \overline{U} \right\} dx \le \lambda^u \int_\Omega SU \cdot \overline{U}\, dx$$

by Lemma 2.1. Hence we obtain

$$\int_\Omega \mathcal{A}U \cdot \overline{U}\, dx \le \lambda^u \int_\Omega SU \cdot \overline{U}\, dx.$$

Similarly, we have

$$\int_\Omega \mathcal{B}\overline{V} \cdot V\, dx \le \lambda^v \int_\Omega T\overline{V} \cdot V\, dx.$$

Since $\lambda^u < 0$ and $\lambda^v < 0$ by Lemmas 2.2 and 2.4, there exists $\delta' > 0$ such that

$$\int_\Omega \mathcal{A}U \cdot \overline{U}\, dx + \int_\Omega \mathcal{B}\overline{V} \cdot V\, dx < -\delta' \left\{ \int_\Omega SU \cdot \overline{U}\, dx + \int_\Omega T\overline{V} \cdot V\, dx \right\}.$$

Then it follows from (3.2) that for some $\delta > 0$, all eigenvalues satisfy $\Re\{\lambda\} < -\delta < 0$. This implies the linear stability of (φ, ψ). $\square$

Next, we consider the case where $u = \varphi$ is linearly unstable so that φ is not a local minimizer of $E[u, \psi]$. (The case where $v = \psi$ is linearly unstable can be treated in the same manner.)

Theorem 3.2 *Let (φ, ψ) be a solution of* (1.3). *Suppose that $u = \varphi$ is linearly unstable as a steady state of (1.5). Then for each S fixed, if $\|T^{-1}\|$ is sufficiently small, $(u, v) = (\varphi, \psi)$ is linearly unstable as a steady state of* (1.1).

Proof. Let $\delta > 0$ be a small constant, and Λ_δ be a set of complex numbers defined by

$$\Lambda_\delta := \{\lambda \in \mathbf{C}\ ;\ |\lambda - \lambda^u| < \delta\}.$$

Since $\lambda^u > 0$ by assumption, we can take $\delta > 0$ so small that $\Re\{\lambda\} > 0$ for all $\lambda \in \Lambda_\delta$. If $\|T^{-1}\|$ is sufficiently small, the operator $\lambda T - \mathcal{B}$ is invertible for any $\lambda \in \Lambda_\delta$. In this case, the second equation in (3.1) is equivalent to

$$V = (\lambda T - \mathcal{B})^{-1} g_u U.$$

Substituting this in the first equation of (3.1), we obtain

$$\lambda S U = \{\mathcal{A} + \mathcal{A}_1(\lambda, T)\}\, U, \tag{3.4}$$

where

$$\mathcal{A}_1 := f_v(\lambda T - \mathcal{B})^{-1} g_u = T^{-1} f_v\, (\lambda I - T^{-1}\mathcal{B})^{-1} g_u.$$

Since the multiplicity of λ^u is finite and $\mathcal{A}_1$ depends on $\lambda \in \Lambda_\delta$ smoothly, it follows from the perturbation theory of linear operators (see Chapter IV, Section 3.5 of [6]) that the eigenvalue problem

$$\mu S U = \{\mathcal{A} + \mathcal{A}_1(\lambda, T)\}\, U$$

has an eigenvalue $\mu = \mu(\lambda, T)$ that is continuous in $\lambda \in \Lambda_\delta$ and T. Moreover, since $\|\mathcal{A}_1\| \to 0$ as $\|T^{-1}\| \to 0$, we have $\mu(\lambda, T) \to \lambda^u$ uniformly in Λ_δ as $\|T^{-1}\| \to 0$.

Thus, if $\|T^{-1}\|$ is small, we can define a mapping from Λ_δ to itself by $\lambda \mapsto \mu(\lambda, T)$. Since this mapping is continuous in λ, by Brouwer's fixed point theorem, there exists a fixed point in Λ_δ. Namely, $\mu(\lambda, T) = \lambda$ for some $\lambda = \hat{\lambda}(T) \in \Lambda_\delta$. Clearly, $\lambda = \hat{\lambda}(T)$ is an eigenvalue of (3.4). Since $\Re\{\hat{\lambda}(T)\} > 0$, $(u, v) = (\varphi, \psi)$ is linearly unstable. $\square$

4 Convex domains

For reaction-diffusion systems with gradient structure, it was proved by Jimbo and Morita [5] (see also [9]) that if the domain is convex, then any spatially inhomogeneous steady state is linearly unstable. In other words, any local minimizer for the gradient system must be spatially homogeneous.

We will show that the same property holds for reaction-diffusion systems with skew-gradient structure. The following result implies that any mini-maximizer must be spatially homogeneous if the domain is convex.

Theorem 4.1 *Let Ω be a convex domain with C^3-boundary, and let (φ, ψ) be a solution of (1.3). If (φ, ψ) is spatially inhomogeneous, then $\lambda^u > 0$ or $\lambda^v > 0$.*

Proof. We follow the idea of Jimbo and Morita [5]. For $U, V \in H^1(\Omega)$, define

$$J^u[U] = \int_\Omega \Big\{ -\ \langle C\nabla U, \nabla U \rangle\ + f_u U \cdot U \Big\} dx$$

and

$$J^v[V] = \int_\Omega \Big\{ -\ \langle D\nabla V, \nabla V \rangle\ + g_v V \cdot V \Big\} dx.$$

Then we have

$$\begin{aligned} J^u[\varphi_{x_j}] &= \int_\Omega \Big\{ -\langle C\nabla \varphi_{x_j}, \nabla \varphi_{x_j} \rangle\ + f_u \varphi_{x_j} \cdot \varphi_{x_j} \Big\} dx \\ &= -\int_{\partial\Omega} C\varphi_{x_j} \cdot \frac{\partial}{\partial\nu}\varphi_{x_j} dx + \int_\Omega \Big(C\Delta\varphi_{x_j} + f_u \varphi_{x_j} \Big) \cdot \varphi_{x_j}\, dx \end{aligned}$$

and

$$\begin{aligned} J^v[\psi_{x_j}] &= \int_\Omega \Big\{ -\langle D\nabla \psi_{x_j}, \nabla \psi_{x_j} \rangle\ + g_v |\psi_{x_j}|^2 \Big\} dx \\ &= -\int_{\partial\Omega} D\psi_{x_j} \cdot \frac{\partial}{\partial\nu}\psi_{x_j} dx + \int_\Omega \Big(D\Delta\psi_{x_j} + g_v \psi_{x_j} \Big) \cdot \psi_{x_j}\, dx. \end{aligned}$$

Differentiating (1.3) by x_j, we obtain

$$\begin{cases} C\Delta\varphi_{x_j} + f_u\varphi_{x_j} + f_v\psi_{x_j} = 0, \\ D\Delta\psi_{x_j} + g_u\varphi_{x_j} + g_v\psi_{x_j} = 0. \end{cases}$$

Hence

$$\Big(C\Delta\varphi_{x_j} + f_u\varphi_{x_j} \Big) \cdot \varphi_{x_j} + \Big(D\Delta\psi_{x_j} + g_v\psi_{x_j} \Big) \cdot \psi_{x_j} = -f_v\psi_{x_j} \cdot \varphi_{x_j} - g_u\varphi_{x_j} \cdot \psi_{x_j} = 0$$

in view of $f_v = -g_u^t$. Thus we obtain

$$J^u[\varphi_{x_j}] + J^v[\psi_{x_j}] = -\int_{\partial\Omega} \Big\{ C\varphi_{x_j} \cdot \frac{\partial}{\partial\nu}\varphi_{x_j i} + D\varphi_{x_j} \cdot \frac{\partial}{\partial\nu}\varphi_{x_j} \Big\} dx.$$

Summing up in j yields

$$\begin{aligned} &\sum_{j=1}^{N} \Big\{ J^u[\varphi_{x_j}] + J^v[\psi_{x_j}] \Big\} \\ &\quad = -\frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial\nu} \Big\{ \langle C\nabla\varphi, \nabla\varphi \rangle\ +\ \langle D\nabla\psi, \nabla\psi \rangle \Big\} dx. \end{aligned} \tag{4.1}$$

Here, from the convexity of Ω and the homogeneous Neumann boundary condition, we have

$$\frac{\partial}{\partial\nu} \langle C\nabla\varphi, \nabla\varphi \rangle\ \le 0, \qquad \frac{\partial}{\partial\nu} \langle D\nabla\psi \cdot \nabla\psi \rangle\ \le 0.$$

(See [10] for more precise argument.)

Suppose here that $\lambda^u \leq 0$ and $\lambda^v \leq 0$. Then $J^u[\varphi_{x_j}] \leq 0$ and $J^v[\psi_{x_j}] \leq 0$ for all j by Lemmas 2.1 and 2.3, respectively. Since the right-hand side of (4.1) is nonnegative, $J^u[\varphi_{x_j}] = 0$ and $J^v[\psi_{x_j}] = 0$ for all j.

Assume that $\varphi_{x_j} \not\equiv 0$ for some j. By Lemma 2.1, $U = \varphi_{x_j}$ must be an eigenfunction of (2.3) associated with $\lambda^u = 0$. Then, φ_{x_j} satisfies

$$\varphi_{x_j} = \frac{\partial}{\partial \nu}\varphi_{x_j} = 0$$

at some point on $\partial\Omega$. By the Calderón unique continuation theorem (see, e.g., [11]), this implies $\varphi_{x_j} \equiv 0$, a contradiction.

Similarly, we can derive a contradiction by assuming that $\psi_{x_j} \not\equiv 0$ for some j. Thus we conclude that $\lambda^u > 0$ or $\lambda^v > 0$. □

The following result is immediately obtained.

Corollary 4.2 *Let Ω be a convex domain with C^3-boundary, and let (φ, ψ) be a solution of* (1.3)*. If (φ, ψ) is spatially inhomogeneous, then $(u, v) = (\varphi, \psi)$ is linearly unstable as a steady state of* (1.1) *for some S and T.*

Proof. By virtue of Theorem 4.1, we see that $\lambda^u > 0$ or $\lambda^v > 0$. Then, by Theorem 3.2, $(u, v) = (\varphi, \psi)$ is linearly unstable as a steady state of (1.1) for some S and T. □

5 Applications

In this section, we give a few applications of the above results. First we consider the diffusive FitzHugh-Nagumo system [2, 12]

$$\begin{cases} u_t = \Delta u + f(u) - v & \text{in } \Omega, \\ \tau v_t = d\Delta v + \varepsilon(u - \gamma v) & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}u = 0 = \dfrac{\partial}{\partial \nu}v & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\tau, d, \varepsilon > 0$ and $\gamma \geq 0$ are positive parameters.

Theorem 5.1 *Let Ω be a convex domain with C^3-boundary, and let $(u, v) = (\varphi, \psi)$ be a steady state of* (5.1)*. If (φ, ψ) is spatially inhomogeneous, then there exists a constant $\tau_* \geq 0$ such that (φ, ψ) is linearly unstable for all $\tau > \tau_*$.*

Proof. The maximal eigenvalue of

$$\begin{cases} \dfrac{\tau}{\varepsilon}\lambda V = \dfrac{d}{\varepsilon}\Delta V - \gamma V & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} V = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\lambda^v = -\varepsilon\gamma/\tau < 0$. Then, by Theorem 4.1, the maximal eigenvalue of

$$\begin{cases} \lambda U = \Delta U + f_u U & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} U = 0 & \text{on } \partial\Omega, \end{cases}$$

must satisfy $\lambda^u > 0$. Hence, by virtue of Theorem 3.2, the steady state $(u, v) = (\varphi, \psi)$ must be unstable if τ is sufficiently large. □

Next, we consider the Gierer-Meinhardt system [3]

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + \dfrac{u^p}{v^q} + \sigma & \text{in } \Omega, \\ \tau v_t = d\Delta v - v + \dfrac{u^r}{v^s} & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} u = 0 = \dfrac{\partial}{\partial \nu} v & \text{on } \Omega, \end{cases} \tag{5.2}$$

where the exponents are assumed to satisfy $p > 1$, $q, r > 0$, $s, \sigma \geq 0$ and

$$\frac{p-1}{q} < \frac{r}{s+1}.$$

It is easy to verify that there exists a unique positive spatially homogeneous steady state $(u, v) = (\alpha, \beta)$, where α and β are positive numbers satisfying

$$\begin{cases} -\alpha + \dfrac{\alpha^p}{\beta^q} + \sigma = 0, \\ -\beta + \dfrac{\alpha^r}{\beta^s} = 0. \end{cases} \tag{5.3}$$

It was shown first by Takagi [14] and Ni-Takagi [13] that the system on a bounded domain has a spiky stationary solution when $\varepsilon > 0$ is small.

Theorem 5.2 *Let Ω be a convex domain with C^3-boundary, and let $(u, v) = (\varphi, \psi)$ be a positive steady state of* (5.2) *with $p + 1 = r$ and $q + 1 = s$. If (φ, ψ) is spatially inhomogeneous, then there exists a constant $\tau_* \geq 0$ such that the steady state is unstable for all $\tau > \tau_*$.*

Proof. Since $s \geq 0$, the maximal eigenvalue of

$$\begin{cases} q\tau\lambda V = qd\Delta V - q(1 + s\varphi^r/\psi^{s+1})V & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}V = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\lambda^v < 0$. Then, by Theorem 4.1, the maximal eigenvalue of

$$\begin{cases} r\lambda U = r\varepsilon^2\Delta U + r(-1 + p\varphi^{p-1}/\psi^q)U & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}U = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\lambda^u > 0$. Thus, by virtue of Theorem 3.2, the steady state $(u, v) = (\varphi, \psi)$ must be unstable if τ is sufficiently large. $\square$

Notice that both of the above examples are of activator-inhibitor type. In general, any two-component reaction-diffusion system

$$\begin{cases} \tau_1 u_t = d_1\Delta u + f(u, v), \\ \tau_2 v_t = d_2\Delta v + g(u, v), \end{cases} \tag{5.4}$$

has a skew-gradient structure if the nonlinear terms satisfy

$$\frac{\partial f}{\partial v} \equiv -\frac{\partial g}{\partial u} \quad \left(= \frac{\partial H^2}{\partial u \partial v} \right).$$

This implies that if $f_v = -g_u \not\equiv 0$, the skew-gradient system (5.4) is neither a cooperation system nor a competition system so that it is not order-preserving. In the above two examples, we used the fact that $g_v \leq 0$. In this situation, we can obtain the same instability result as Theorems 5.1 and 5.2 for the steady state of (5.4).

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