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KNESER FAMILIES IN SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Dedicated to Professor Norio Kikuchi on his sixtieth birthday

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1. Introduction. In the theory of ordinary differential equations, it is well known that a family F of all solution curves for an initial value problem

$$x' = f(t,x), \qquad x(\sigma) = x_0 \qquad (x_0 \in \mathbf{R}^n) \tag{1}$$

has the Kneser's property, namely, a cross section $\{x(\tau); x \in F\}$ of F with the hyperplane $t = \tau$ is compact and connected if $|\sigma - \tau| > 0$ is sufficiently small. In 1967, Hukuhara [1] extended this local property to a global one under suitable assumptions. Separately from differential equations, he constructed a family of continuous mappings having some topological properties which are required for solution curves of (1) and called it Kneser family. He further proved the Nagumo's existence theorem to boundary value problems for second order ordinary differential equations from the viewpoint of Kneser family. By applying the theory of Kneser family directly, Kikuchi, Hayashi and the author obtained a variation of Nagumo's existence theorem and succeeded in solving a boundary layer problem in [4].

Solution curves of (1) are lying in finite dimensional spaces and are continuable to both right and left, however, those of a partial differential equation are lying in infinite dimensional spaces in some sense and are not always continuable to the left. Recently, Kikuchi and the author [3] proved that a family of solution curves for a semilinear parabolic partial differential equation has Kneser's property. Considering these facts, we shall extend Hukuhara's result to infinite dimensional spaces in Sections 2 and 3, and it will be shown that our extension is applicable to solution curves of a semilinear parabolic partial differential equation in Section 4.

2. Family of characteristics. Let X be a Banach space with norm $\|\cdot\|$, and let d denote a metric in $\mathbf{R} \times X$ defined by $d((t,x),(s,y)) = |t-s| + \|x-y\|$. For two nonvoid closed subsets A and B of $\mathbf{R} \times X$, we denote the Hausdorff distance between A and B by $d_H(A,B)$, namely,

$$d_H(A,B) := \inf\{\varepsilon > 0; N_\varepsilon(A) \supset B, N_\varepsilon(B) \supset A\},$$

where

$$N_\varepsilon(A) = \{(t,x) \in \mathbf{R} \times X; d((t,x),A) < \varepsilon\},$$
$$d((t,x),A) = \inf\{d((t,x),(s,y)); (s,y) \in A\}.$$

Let $\boldsymbol{E}$ be a family of all X-valued continuous mappings defined on compact intervals which are allowed to be one point. We denote the domain of $f \in \boldsymbol{E}$ by I_f. When $I_f = [\alpha, \beta]$, the points $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$ are called, respectively, left end point and right end point of f. The graph of f is denoted by Γ_f, namely, $\Gamma_f = \{(t, f(t)) \in \mathbf{R} \times X; t \in I_f\}$. Here, we define a metric ρ in $\boldsymbol{E}$ by

$$\rho(f,g) := d_H(\Gamma_f, \Gamma_g) \qquad \text{for } f, g \in \boldsymbol{E}.$$

For two elements f and g in $\boldsymbol{E}$, f is called a *part* of g or g is called an *extension* of f when $\Gamma_f \subset \Gamma_g$ holds. Let $\boldsymbol{F}$ be a subset of $\boldsymbol{E}$. An element f of $\boldsymbol{F}$ is called *right maximal* in $\boldsymbol{F}$ provided that the right end point of every extension of f in $\boldsymbol{F}$ coincides with that of f. Similarly, we can define a *left maximal* element of $\boldsymbol{F}$. A subset $\mathcal{D}(\boldsymbol{F})$ of $\mathbf{R} \times X$ defined by $\mathcal{D}(\boldsymbol{F}) := \bigcup\{\Gamma_f; f \in \boldsymbol{F}\}$ is called the *fundamental domain* of $\boldsymbol{F}$, and the boundary of $\mathcal{D}(\boldsymbol{F})$ is denoted by $\mathcal{B}(\boldsymbol{F})$. For a subset $\mathcal{E}$ of $\mathcal{D}(\boldsymbol{F})$, we denote by $\boldsymbol{F}^+(\mathcal{E})$ a family of all elements $g \in \boldsymbol{F}$ whose left end points belong to $\mathcal{E}$ and of all parts of such the elements g, that is, $\boldsymbol{F}^+(\mathcal{E})$ is expressed by

$$\boldsymbol{F}^+(\mathcal{E}) = \{f \in \boldsymbol{E}; \exists g \in \boldsymbol{F}, \Gamma_f \subset \Gamma_g, \text{ left end point of } g \text{ belongs to } \mathcal{E}\}.$$

The fundamental domain $\mathcal{D}(\boldsymbol{F}^+(\mathcal{E}))$ of $\boldsymbol{F}^+(\mathcal{E})$ is denoted by $\mathcal{Z}^+(\mathcal{E})$. Furthermore, the sets $\boldsymbol{F}^+(\{p\})$ and $\mathcal{Z}^+(\{p\})$ are denoted, respectively, by $\boldsymbol{F}^+(p)$ and $\mathcal{Z}^+(p)$, where $p \in \mathcal{D}(\boldsymbol{F})$.

Definition 1. A subfamily $\boldsymbol{F}$ of $\boldsymbol{E}$ is called a *family of characteristics* if the following conditions (C_1) through (C_5) are fulfilled, and each element of $\boldsymbol{F}$ is called a *characteristic.*

(C_1) Every part of a characteristic is also a characteristic.

(C_2) If two characteristics f and g take the same value at $t = \tau$, then a mapping which coincides with f for $t \le \tau$ and with g for $t \ge \tau$ is also a characteristic.

(C_3) $\mathcal{D}(\boldsymbol{F})$ is a closed subset of $\mathbf{R} \times X$.

(C_4) All right end points of right maximal characteristics in $\boldsymbol{F}$ belong to $\mathcal{B}(\boldsymbol{F})$.

(C_5) If $\mathcal{E}$ is a compact subset of $\mathcal{D}(\boldsymbol{F})$, then $\boldsymbol{F}^+(\mathcal{E})$ is a compact subset of $\boldsymbol{E}$.

If $\boldsymbol{F}$ is a family of characteristics and if $\mathcal{D}'$ is a closed subset of $\mathcal{D}(\boldsymbol{F})$, then a family $\boldsymbol{F}(\mathcal{D}')$ defined by $\boldsymbol{F}(\mathcal{D}') := \{f \in \boldsymbol{F}; \Gamma_f \subset \mathcal{D}'\}$ forms a family of characteristics.

3. Kneser family. Throughout this section, we always assume that $\boldsymbol{F}$ denotes a family of characteristics. We shall classify all points of $\mathcal{B} = \mathcal{B}(\boldsymbol{F})$. Right endpoint of a right maximal characteristic is called a *right extreme point* of $\boldsymbol{F}$. Similarly, we define a *left extreme point* of $\boldsymbol{F}$. The set of all rigth extreme points of $\boldsymbol{F}$ is called the *right boundary* and is denoted by $\mathcal{B}^r = \mathcal{B}^r(\boldsymbol{F})$. By (C_4), we have that $\mathcal{B}^r \subset \mathcal{B}$. The set of all left extreme points which belong to $\mathcal{B}(\boldsymbol{F})$ is denoted by $\mathcal{B}^l = \mathcal{B}^l(\boldsymbol{F})$. We denote by $\mathcal{B}^+ = \mathcal{B}^+(\boldsymbol{F})$ the set of all points $p \in \mathcal{B} \setminus \mathcal{B}^r$ with the property that every point q of $\mathcal{Z}^+(p) \setminus \{p\}$ belongs to $\mathrm{Int}\mathcal{D}$ when q is sufficiently near to p. In other words, $p \in \mathcal{B}^+$ if and only if p is an isolated point of $\mathcal{Z}^+(p) \cap \mathcal{B}$. Finally, we put $\mathcal{B}_+ = \mathcal{B}_+(\boldsymbol{F}) := \mathcal{B} \setminus (\mathcal{B}^r \cup \mathcal{B}^+)$. It is clear that $p \in \mathcal{B}_+$ if and only if p is an accumulation point of $\mathcal{Z}^+(p) \cap \mathcal{B}$. Thus, $\mathcal{B}$ is expressed by $\mathcal{B} = \mathcal{B}^r \cup \mathcal{B}^+ \cup \mathcal{B}_+$ as a disjoint union.

For a subset $\mathcal{S}$ of $\mathbf{R} \times X$ and a $\tau \in \mathbf{R}$, we define two sets $\mathcal{S}_\tau$ and $\mathcal{S}|_\tau$, respectively, by

$$\mathcal{S}_\tau := \{(t,x) \in \mathcal{S}; t \leq \tau\} \qquad \text{and} \qquad \mathcal{S}|_\tau := \{(t,x) \in \mathcal{S}; t = \tau\}.$$

For any $\tau \in \mathbf{R}$, we denote $\boldsymbol{F}(\mathcal{D}_\tau)$ by $\boldsymbol{F}_\tau$, where $\mathcal{D} = \mathcal{D}(\boldsymbol{F})$. Furthermore, for any compact subset $\mathcal{E}$ of $\mathcal{D}$, we put

$$\mathcal{Z}_\tau^+(\mathcal{E}) := \mathcal{Z}^+(\mathcal{E})_\tau \qquad \text{and} \qquad \boldsymbol{F}_\tau^+(\mathcal{E}) := \boldsymbol{F}(\mathcal{Z}_\tau^+(\mathcal{E})).$$

Here notice that $\boldsymbol{F}_\tau$ and $\boldsymbol{F}_\tau^+(\mathcal{E})$ are family of characteristics.

Definition 2. Let $p = (\alpha, \xi)$ be a point of $\mathcal{D} = \mathcal{D}(\boldsymbol{F})$. We call p a *Kneser point* if one of the following conditions (K_1) through (K_3) holds.

- (K_1) $p \in \mathcal{B}^r$.
- (K_2) $p \in \mathcal{B}^+ \cup \mathrm{Int}\mathcal{D}$ and $\mathcal{Z}^+(p)|_\tau$ is compact and connected if $\tau - \alpha > 0$ is sufficiently small.
- (K_3) $p \in \mathcal{B}_+$ and the union $\mathcal{Z}^+(p)|_\tau \cup (\mathcal{Z}_\tau^+(p) \cap \mathcal{B})$ is compact and connected if $\tau - \alpha > 0$ is sufficiently small.

Definition 3. $\boldsymbol{F}$ is called a *Kneser family* provided that $\mathcal{B}^+$ is open in $\mathcal{B}$, $\mathcal{B}^+ \subset \mathcal{B}^l$ and that every point of $\mathcal{D}$ is a Kneser point.

We can prove the following theorem by a similar argument as in the proof of Proposition 4.2 in [1] (cf. [2]).

Theorem 1. Let $\boldsymbol{F}$ be a Kneser family. If $\mathcal{E}$ is a compact and connected subset of $\mathcal{D}(\boldsymbol{F})$, then so is $\mathcal{Z}^+(\mathcal{E}) \cap (\mathcal{B}^r \cup \mathcal{B}_+)$.

Remark Hukuhara [1] introduced a useful sufficient condition which guarantees a point of $\mathcal{B}_+$ to be a Kneser point. Though our definition of family of characteristics is different from that in [1], his result is also applicable to ours.

4. Parabolic partial differential equation. Let $T > 0$ be an arbitrary fixed number, and consider the initial boundary value problem for a semilinear parabolic partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{for } t > \sigma,\ x \in \Omega, \\ u(\sigma, x) = u_0(x) & \text{for } x \in \overline{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{for } t > \sigma,\ x \in \partial\Omega, \end{cases} \tag{E}$$

where $0 \leq \sigma \leq T$, Ω is a bounded and open domain with smooth boundary, ν denotes a unit outer normal vector of $\partial\Omega$, $u_0 \in X := C(\overline{\Omega}, \mathbf{R})$ and $f : \mathbf{R} \to \mathbf{R}$ is continuous. We denote the supremum norm of X by $\|\cdot\|$. In this section, we shall apply the result given in Sections 2 and 3 to (E). Here, we further assume the following assumption.

(A) There exist positive constants a and b such that $|f(u)| \leq a + b|u|$ for $u \in \mathbf{R}$.

For any $\sigma \in [0, T]$, let Y_σ be the Banach space $C([\sigma, T] \times \overline{\Omega}, \mathbf{R})$ with supremum norm. By a (*mild*) *solution* u of (E), we shall mean that $u \in Y_\sigma$ is represented by

$$u(t,x) = \int_\Omega U(t-\sigma, x, y) u_0(y)\, dy + \int_\sigma^t ds \int_\Omega U(t-s, x, y) f(u(s,y))\, dy,$$

where U is the *fundamental solution* of $\partial u/\partial t = \Delta u$ with $\partial u/\partial \nu = 0$. In [3], we proved the following theorem for the case where $\sigma = 0$.

Theorem 2. Suppose that (A) holds. Then (E) has at least one solution $u \in Y_\sigma$ and a set $\{u \in Y_\sigma;\ u \text{ is a solution of (E)}\}$ is compact and connected in Y_σ for any $(\sigma, u_0) \in [0, T] \times X$.

For a continuous function $u : [\sigma, \tau] \times \overline{\Omega} \to \mathbf{R}$ with $0 \leq \sigma \leq \tau \leq T$, we denote a function $u(t, \cdot)$ and the interval $[\sigma, \tau]$, respectively, by $\tilde{u}$ and $I_{\tilde{u}}$. Then we obtain a continuous mapping $\tilde{u} : I_{\tilde{u}} \to X$. From Theorem 2, we can easily obtain the following corollary.

Corollary 1. For any $(\sigma, u_0) \in [0,T] \times X$ and $\tau \in [\sigma, T]$, a set

$$\{\tilde{u}(\tau) \in X; u \text{ is a solution of (E)}\}$$

is compact and connected in X.

By virtue of the above corollary, we can prove the following theorem (see [2]).

Theorem 3. If (A) holds, then a family $\boldsymbol{F}$ given by

$$\boldsymbol{F} = \{\tilde{u}; u \text{ is a solution of (E) on } [\sigma,\tau] \times \overline{\Omega}, [\sigma,\tau] \subset [0,T], u_0 \in X\}$$

forms a Kneser family whose fundamental domain is $[0,T] \times X$.

Suppose that $\mathcal{D}$ is a closed subset of $[0,T] \times X$. Then, $\boldsymbol{F}(\mathcal{D})$ is a family of chracteristics. Moreover, if $\mathcal{D}$ is a bounded and closed subset, then the assumption (A) is not essential in Theorem 3, which will be seen in the following corollary.

Corollary 2. Suppose that the function f in (E) is continuous. If $\mathcal{D}$ is a bounded and closed subset of $[0,T] \times X$, then a family $\boldsymbol{F}(\mathcal{D}) = \{\tilde{u} \in \boldsymbol{F}; \Gamma_{\tilde{u}} \subset \mathcal{D}\}$ forms a family of characteristics whose fundamental domain is $\mathcal{D}$.

For the proof, see [2].

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