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A method of computations of fundamental groups of 3-dimensional manifolds

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1 Introduction.

By the theorem of Hilden-Montesinos (Hilden [7], Montesinos [9]), for every 3-dimensional compact oriented manifold Y, there exists a topological branched covering

$$h : Y \longrightarrow S^3$$

of the 3-sphere S^3 of degree 3 branching at a knot B_h, whose monodromy around the knot is given only by transpositions.

We regard the knot B_h as a braid, for every knot (and link) is isotopic in S^3 to a braid. We may identify S^3 with $\partial(\overline{\Delta(0,\, a')} \times \overline{\Delta(0,\, b')})$, where $\Delta(0,\, a')$ is the disc in the complex plane $\mathbb{C}$ with the center 0 and the radius a'. We may assume that B_h is contained in $\partial\overline{\Delta(0,\, a')} \times \Delta(0,\, b')$ as in Figure 1.

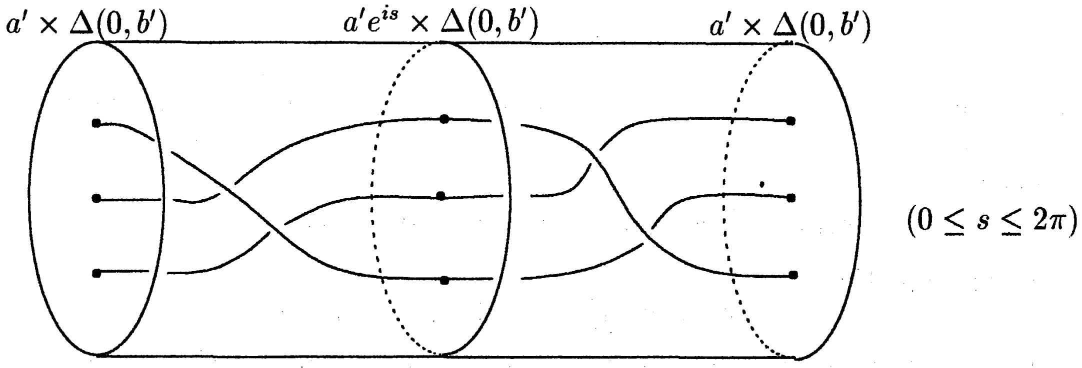

Figure 1:

Let B be the cone over B_h connecting every point of B_h with the origin of $\mathbb{C}^2$. Let

$$f : X \longrightarrow \Delta(0,\, a) \times \Delta(0,\, b)$$

be the topological finite branched covering branching at B with the same monodromy as h . (Such a branched covering exists by Fox completion (Fox [5]). In fact X is a cone over Y.) Since X is a topological cone over Y,

$$\pi_1(X - \{x\}, p_0) \simeq \pi_1(Y, p_0), \quad (x = f^{-1}((0, 0))).$$

Put

$$\begin{aligned} X_t &= f^{-1}(t \times \Delta(0, b)), \\ f_t &= f|_{X_t} : X_t \longrightarrow t \times \Delta(0, b). \end{aligned}$$

Then every $f_t\,(t \neq 0)$ is a finite branched covering of the disc $t \times \Delta(0, b)$, and f can be regarded as a topological degenerating family of finite branched coverings of discs: $f = \{f_t\}$. Its topological type is determined by the pair

$$(\Phi_t,\, \theta(\delta)), \quad (\delta : s \longmapsto a'e^{is},\, (0 \le s \le 2\pi))$$

of the monodromy Φ_t of f_t (for a fixed $t \neq 0$) and the braid monodromy $\theta(\delta)$ of f. But they must satisfy the following equality (Namba [10]):

$$\Phi_t \circ \theta(\delta) = \Phi_t,$$

where $\theta(\delta)$ is regarded as an automorphism of $\pi_1(t \times \Delta(0, b) - B_f,\, q_0)$ (see Section 2).

Conversely, let

$$\Phi : \pi_1(\Delta(0, b) - \{n \text{ points}\}, q_0) \longrightarrow S_d$$

be a representation whose image is a transitive subgroup of the d-th symmetric group S_d. Let σ be a braid which satisfies

$$\Phi \circ \sigma = \Phi.$$

We denote the n-points by $\{q_1, \dots , q_n\}$ and let $\gamma_1, \dots , \gamma_n$ be the lassos as in Figure 2.

Then

$$\pi_1(\Delta(0, b) - \{q_1, \dots , q_n\}, q_0) = <\gamma_1, \dots , \gamma_n>$$

is a free group. Put

$$A_j = \Phi(\gamma_j) \quad (j = 1, 2, \dots , n).$$

We regard the braid σ as a link which is contained in $\partial\overline{\Delta(0, a')} \times \Delta(0, b')$ as in Figure 1. By the condition $\Phi \circ \sigma = \Phi$, we can construct a topological branched covering

$$h : Y \longrightarrow \partial(\overline{\Delta(0, a')} \times \overline{\Delta(0, b')})$$

branching at the link σ whose monodromy is Φ. More precisely, we can construct a topological branched covering Y' of $\partial\overline{\Delta(0, a')} \times \Delta(0, b')$ branching at the link σ whose monodromy is Φ. We then attach solid tori to Y' at the part corresponding to the mutually prime cyclic decomposition of the permutation

$$A_\infty = \Phi(\gamma_n \cdots \gamma_1)^{-1} = (A_n \cdots A_1)^{-1}$$

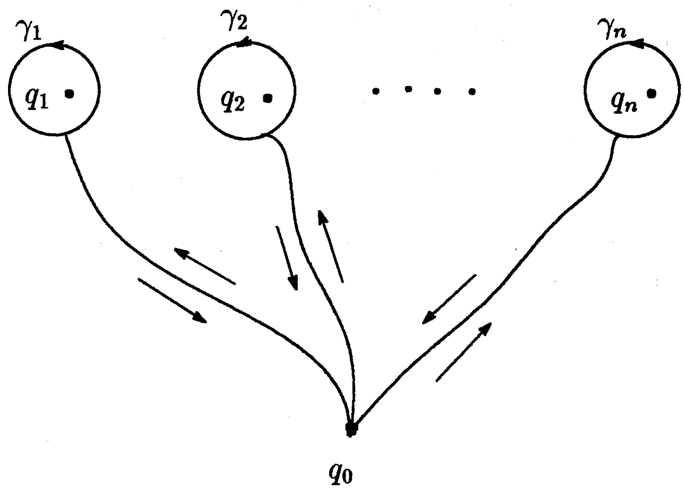

Figure 2:

over $\partial\overline{\Delta(0, a')} \times \partial\overline{\Delta(0, b')}$. Then we get a 3-dimensional compact oriented manifold Y and a topological finite branched covering

$$h : Y \longrightarrow \partial(\overline{\Delta(0, a')} \times \overline{\Delta(0, b')})$$

of the 3-sphere branching at the link σ whose monodromy is Φ .

We then construct the topological cone X of Y as above and construct a topological finite branched covering

$$f : X \longrightarrow \Delta(0, a) \times \Delta(0, b)$$

such that

$$\Phi_f = \Phi, \quad \theta(\delta) = \sigma.$$

This is regarded as a topological degenerating family of finite branched coverings of discs.

Thus to construct topological degenerating families of finite branched coverings of discs (hence to construct 3-dimensional compact oriented manifolds) is reduced to find out the pair (Φ, σ) as above such that $\Phi \circ \sigma = \Phi$.

2 Monodromy of a branched covering of degree 3 of the disc and its canonical forms.

Let X and Y be Riemann surfaces and let $f : X \longrightarrow Y$ be a finite branched covering, that is, a surjective proper finite holomorphic mapping. A point p of X is called a ramification point of f if f is not biholomorphic around p. Its image $q = f(p)$ is called a branch point of f. The set of all ramification points (resp. branch points) is denoted by R_f (resp. B_f) and is called the ramification locus (resp. branch locus). Then

$$f : X - f^{-1}(B_f) \longrightarrow Y - B_f$$

is an unbranched covering, whose mapping degree is called the degree of f and is denoted by $deg f$. (X, f) (or simply f) is called a finite branched covering of Y.

Definition 1. *Two finite branched coverings*

$$f : X \longrightarrow Y, \qquad f' : X' \longrightarrow Y$$

are said to be isomorphic if there is a biholomorphic mapping ψ which makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{id} & Y \end{array}$$

Definition 2. *Two finite branched coverings*

$$f : X \longrightarrow Y, \qquad f' : X' \longrightarrow Y$$

are said to be equivalent (resp. topologically equivalent) if there are biholomorphic mappings (resp. orientation preserving homeomorphisms) ψ and φ which make the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\varphi} & Y \end{array}$$

Let B_n be the Artin braid group of n strings. Then B_n is expressed as follows:

$$B_n =< \sigma_1, \dots, \sigma_{n-1} \;\Big|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for } |i-j| \geq 2 > . \end{array}$$

Let $\{q_1, \dots, q_n\}$ be a set of n distinct points in $\mathbb{C}$. The fundamental group $\pi_1(\mathbb{C} - \{q_1, \dots, q_n\}, q_0)$ is the free group

$$\pi_1(\mathbb{C} - \{q_1, \dots, q_n\}, q_0) =< \gamma_1, \dots, \gamma_n >$$

generated by the lassos $\gamma_1, \dots, \gamma_n$ as in Figure 2.

The braid group B_n acts on this group as follows:

$$\begin{aligned} \sigma_i(\gamma_i) &= \gamma_i^{-1}\gamma_{i+1}\gamma_i \\ \sigma_i(\gamma_{i+1}) &= \gamma_i \\ \sigma_i(\gamma_j) &= \gamma_j \qquad (j \neq i, i+1). \end{aligned}$$

Note that this action is faithful (Birman [1]). A similar assertion holds if we replace $\mathbb{C}$ by a disc $\Delta(0, b)$.

The following theorem is well known:

Theorem 1. *Put* $B = \{q_1, \ldots, q_n\} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. *For any homomorphism* $\Phi : \pi_1(\mathbb{P}^1 - B, q_0) \longrightarrow S_d$ *whose image* $Im\Phi$ *is transitive, there exists a unique (up to isomorphisms) finite branched covering* $f : X \longrightarrow \mathbb{P}^1$ *such that*

$$B_f \subset B, \qquad \Phi_f = \Phi.$$

For the proof of Theorem 1, see Forster [4]. There is a higher dimensional analogy of the theorem (Grauert-Remmert [6]).

Theorem 2. *For two finite branched coverings* $f : X \longrightarrow \mathbb{P}^1$, $f' : X' \longrightarrow \mathbb{P}^1$ *such that* $B_f = B_{f'} = \{q_1, \ldots, q_n\} \subset \mathbb{C}$ *, they are topologically equivalent if and only if there is a braid* σ *in* B_n *such that* $\sigma^*(\Phi_f) = \Phi_f \circ \sigma = \Phi_{f'}$. *Here the equality is that as representation classes.*

For the proof of Theorem 2, see Namba [10].

Remark . Theorem 2 holds even if $\mathbb{P}^1$ is replaced by the compelx plane $\mathbb{C}$ or a disc in $\mathbb{C}$.

Every branched covering

$$f : X \longrightarrow \Delta(0, b)$$

of degree d can be extended to a branched covering

$$\hat{f} : \hat{X} \longrightarrow \mathbb{P}^1$$

of degree d in the following canonical manner: Put

$$B_f = \{q_1, \ldots, q_n\}, \quad A_j = \Phi_f(\gamma_j) \quad (j = 1, \ldots, n),$$

where γ_j is a lasso as in Figure 2. Let γ_∞ be the lasso around the point ∞ as in Figure 3.

Then

$$\pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n, \infty\}, q_0) = < \gamma_1, \ldots, \gamma_n, \gamma_\infty \,|\, \gamma_\infty \gamma_n \cdots \gamma_1 = 1 > .$$

Put

$$A_\infty = (A_n \cdots A_1)^{-1}.$$

We define a homomorphism

$$\Phi : \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n, \infty\}, q_0) \longrightarrow S_d$$

by

$$\Phi(\gamma_j) = A_j \quad (j = 1, \cdots n), \qquad \Phi(\gamma_\infty) = A_\infty.$$

Then the branched covering

$$\hat{f} : \hat{X} \longrightarrow \mathbb{P}^1$$

corresponding to Φ (see Theorem 1) is an extension of f.

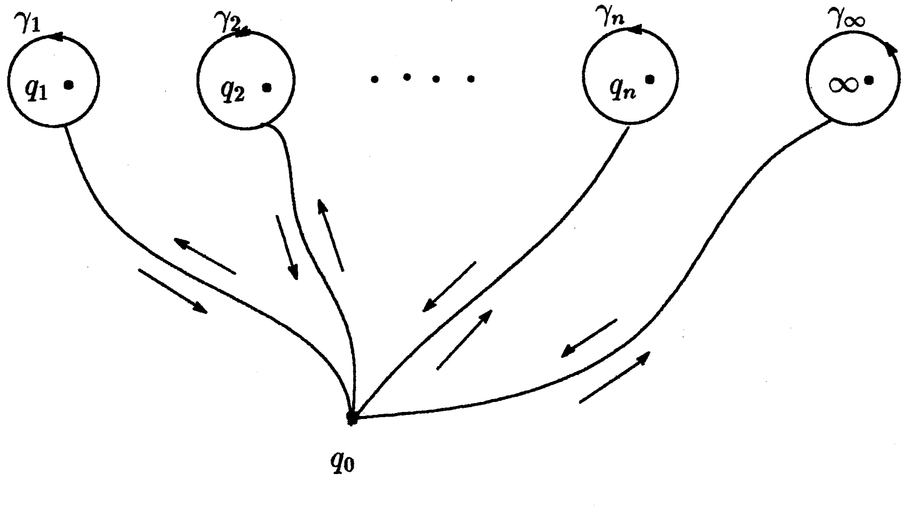

Figure 3:

Note that if $A_\infty = 1$, then $\hat{f}$ does not branch at the point ∞.

Let
$$f : X \longrightarrow \Delta(0,\, b)$$
be a branched covering of the disc $\Delta(0,\, b)$ of degree 3. Let γ_j $(j = 1, \ldots, n)$ be the lassos as in Figure 2. Put $A_j = \Phi_f(\gamma_j)$ $(j = 1, \ldots, n)$. Suppose that every A_j is a transposition in the 3rd symmetric group S_3. As above, we extend the covering to that of $\mathbb{P}^1$ which is denoted by the same notation f for simplicity. Let γ_∞ be the lasso around the point ∞ and put

$$A_\infty = (A_n \cdots A_1)^{-1} = \Phi_f(\gamma_\infty)$$

as above. There are three cases:

Case 1. $A_\infty = 1$. In this case, the extended covering does not branch at ∞.

Case 2. A_∞ is a transposition. In this case, the point ∞ is a branch point, that is there is a point over ∞ with the ramification index is 2. Since we may change the monodromy with an equivalent representation , we may assume that $A_\infty = (1\ 2)$.

Case 3. A_∞ is a cyclic permutation. In this case, the point ∞ is a branch point. We may assume that $A_\infty = (1\ 3\ 2)$.

Under these assumptions, we have the following theorem:

Theorem 3. *Under the above assumptions, the covering f is topologically equivalent to one of the following canonical forms: Arranging $A_1, A_2, \ldots, A_n$ in this order:*

Case 1: $(1\ 2), (1\ 2), (2\ 3), (2\ 3), \underbrace{(2\ 3), (2\ 3), \ldots, (2\ 3)}_{2g}$

Case 2: $(1\ 2), (2\ 3), (2\ 3), \underbrace{(2\ 3), (2\ 3), \dots, (2\ 3)}_{2g}$

Case 3: $(1\ 2), (2\ 3), \underbrace{(2\ 3), (2\ 3), \dots, (2\ 3)}_{2g}$

where g is the genus of the Riemann surface X.

3 Isotropy subgroups of the braid groups.

Let

$$\Phi : < \gamma_1, \dots, \gamma_n > \longrightarrow S_d$$

be a representation of the free group $< \gamma_1, \dots, \gamma_n >$ of n generators into the d-th symmetric group S_d whose image $Im\Phi$ is transitive.

By the discussion in Section 1, it is important to consider the braid $\sigma \in B_n$ such that $\Phi \circ \sigma = \Phi$, where the equality is not as representaion classes but is just as representaions. (The action of the braid σ on the free group $< \gamma_1, \dots, \gamma_n >$ is defined in Section 2.) Put

$$I(\Phi) = \{\sigma \in B_n \,|\, \Phi \circ \sigma = \Phi\},$$

the isotropy subgroup of B_n for Φ.

Since the number of representaions Φ is finite (in fact is less than $(d!)^n$), $I(\Phi)$ is a subgroup of B_n of finite index.

Note that the following equality holds:

$$I(\Phi \circ \tau) = \tau^{-1} I(\Phi) \tau.$$

Put

$$\Phi(\gamma_j) = A_j \quad (j = 1, 2, \cdots, n).$$

Now, let Φ be the representation of the canonical forms as in Theorem 3.

For Case 2 and 3 (i.e, $A_1 = (1\ 2)$, $A_2 = \cdots = A_n = (2\ 3)$), by the theorem of Birman-Wajnryb (Birman-Wajnryb [2]) $I(\Phi)$ is generated by the following elements:

$$\sigma_1^3,\ \sigma_2,\ \dots,\ \sigma_{n-1},$$
$$\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_1 \ (n \geq 5).$$

The following theorem for Case 1 (i.e, $A_1 = A_2 = (1\ 2)$, $A_3 = \cdots = A_n = (2\ 3)$) is the main result.

Theorem 4. *For Case 1, $I(\Phi)$ is generated by the following elements:*

$$\sigma_1,\ \sigma_2^3,\ \sigma_3,\ \dots,\ \sigma_{n-1},\ \sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2,$$
$$\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-2}\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_3\sigma_2^2\sigma_3\sigma_4^2\sigma_3\sigma_2 \ (n \geq 6).$$

Remark . For Case 1, the generators of the isotropy subgroup $I(\Phi)$ of $B_n(S^2)$ are described in Birman-Wajnryb ([2]) but not of B_n.

4 Riemann pictures and symplectic basis for canonical forms.

In this section, we introduce a picture, (we call it a Riemann picture), which represents a finite branched covering of a disc topologically (see Namba-Takai [11]). We explain it by an example:

Let us consider Case 1 of genus 1.

Let X be a Riemann surface of genus 1. Let $f : X \longrightarrow \mathbb{C}$ be a branched covering of degree 3 with the monodromy Φ of canonical form of Case 1. Put $B_f = \{q_1, q_2, \ldots, q_6\}$. Let q_0 be a reference point. We take the lassos γ_j around q_j as in Figure 2. We extend the covering to the branched covering of $\mathbb{P}^1$ in a canonical way as in Section 2. In this case, we have

$$\pi_1(\mathbb{P}^1 - B_f, q_0) = < \gamma_1, \gamma_2, \ldots, \gamma_6, \gamma_\infty \mid \gamma_\infty \gamma_6 \cdots \gamma_2 \gamma_1 = 1 >,$$
$$A_1 = A_2 = (1\ 2) \quad A_2 = \cdots A_6 = (2\ 3), \quad A_\infty = id \quad (A_j = \Phi(\gamma_j)).$$

Consider the picture (Figure 4) in which the circle part of every lasso γ_j in Figure 2 is degenerated to the point q_j:

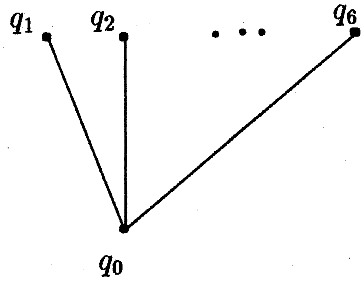

Figure 4:

We then pull the picture in Figure 4 back over the covering f and get the following picture in Figure 5 which we call the Riemann picture of f:

In Figure 5, the points ①, ②, ③ are the inverse images of the reference point q_0 while the points $1, \ldots, 6$ and ∞ are the inverse images of $q_1, \ldots, q_6$ and ∞ respectively. Note that around every point ①, ②, ③, the pathes connecting to the points $1, \ldots, 6$ and ∞ in this order are arranged clockwisely. On the other hand, around every point $1, \ldots, 6$ and ∞, the pathes connecting to the points ①, ②, ③ are arranged counterclockwisely in order to be compatible with the monodromy. (We omit unramified points in the picture.)

The covering (X, f) can be topologically expressed by this picture.

Put

$$\begin{aligned}
\xi_3 &= [1, 21][\infty, 11][1, 12], \\
\xi_2 &= [\infty, 22], \\
\xi_1 &= [6, 23][\infty, 33][6, 32], \\
\alpha &= [3, 23][4, 32], \\
\beta &= [5, 23][4, 32].
\end{aligned}$$

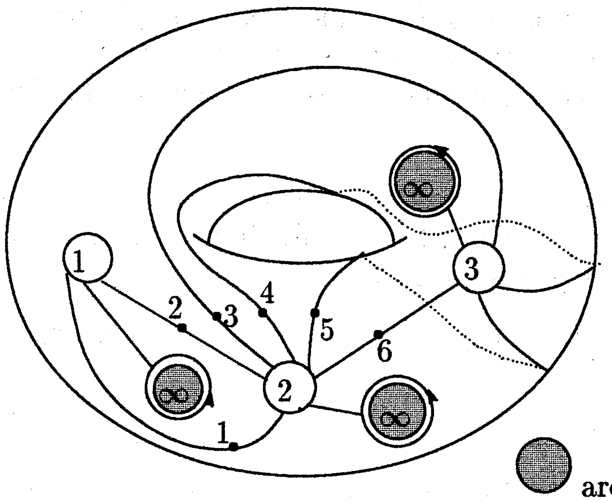

Figure 5:

Here the notation [6, 23] for example means the path in Figure 5 whose initial point is ② and the terminal point is ③ passing through the branch point 6. Then these are loops with the initial point ②. We can observe the following relations:

$$\begin{aligned} \beta\alpha\beta^{-1}\alpha^{-1}\xi_3\xi_2\xi_1 &= 1, \\ <\alpha, \beta> &= 1, \end{aligned}$$

where the notation $<\ ,\ >$ means the intersection number. We pull back the relation

$$\gamma_\infty\gamma_6\cdots\gamma_1 = 1$$

over f and get the following three relations:

$$\begin{aligned} &[\infty, 11][2, 12][1, 21] = 1, \\ &[\infty, 22][6, 23][5, 32][4, 23][3, 32][2, 21][1, 12] = 1, \\ &[\infty, 33][6, 32][5, 23][4, 32][3, 23] = 1. \end{aligned}$$

The above relation

$$\beta\alpha\beta^{-1}\alpha^{-1}\xi_3\xi_2\xi_1 = 1$$

can be induced from these three relations.

The Riemann picture of a general (X, f) is defined as in the above example, that is, a pull-back over f of the graph on $\mathbb{P}^1$ of Figure 2 degenerated the circle part of every lasso to the branch point.

Remark . 1. The Riemann picture is determined by (X, f) up to orientation preserving homeomorphisms of X.
2. As noted above, we can draw the Riemann picture of (X, f) even when only the monodromy $\Phi = \Phi_f$ is given and (X, f) is not explicitely given.
3. In Namba [10], we have introduced another picture in order to express (X, f)

topologically, which we called a Klein picture. Klein pictures and Riemann pictures are dual in a sense. Klein pictures are useful to observe the degeneration of branched coverings, while Riemann picutres are useful to compute fundamental groups as will be seen in Section 5.

We draw the Riemann pictures of the canonical forms in Theorem 3 (see Figures 6, 7 and 8 for $g = 3$), from which we easily find canonical generators $\{\alpha_i,\ \beta_i,\ \xi_i\}$ of the fundamental group of X such that

Case 1: $\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_{g-1}^{-1}\cdots\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}\xi_3\xi_2\xi_1 = 1,$

Case 2: $\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_{g-1}^{-1}\cdots\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}\xi_2\xi_1 = 1,$

Case 3: $\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_{g-1}^{-1}\cdots\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}\xi = 1.$

Here $\{\alpha_i,\ \beta_i\ (i = 1, \dots, g)\}$ is a symplectic basis in homology level of the extension $\hat{X}$ of X:

$$<\alpha_i,\ \beta_j>= \delta_{ij}, \quad <\alpha_i,\ \alpha_j>= 0, \quad <\beta_i,\ \beta_j>= 0$$

$(i,\ j = 1, \dots, g)$, where $<\ ,\ >$ means the intersection number.

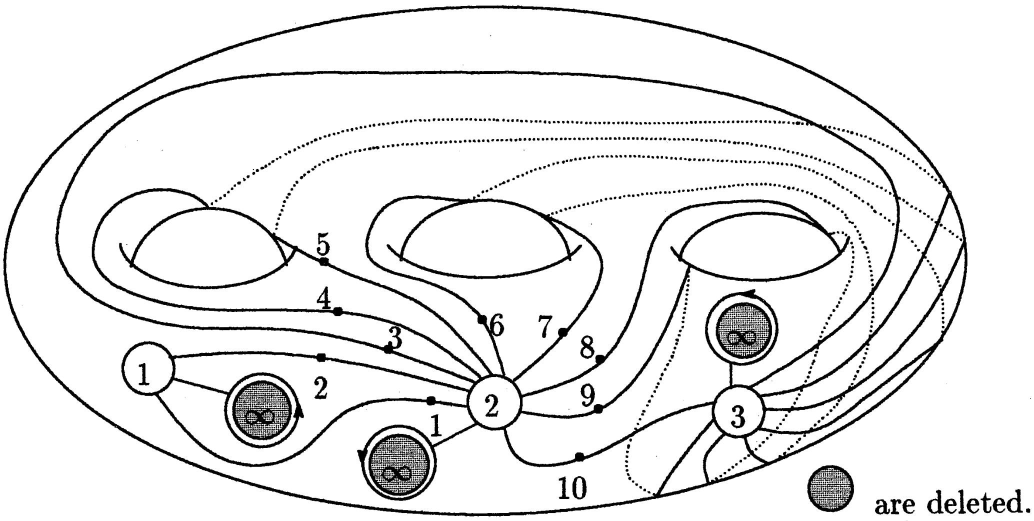

Figure 6: Case 1

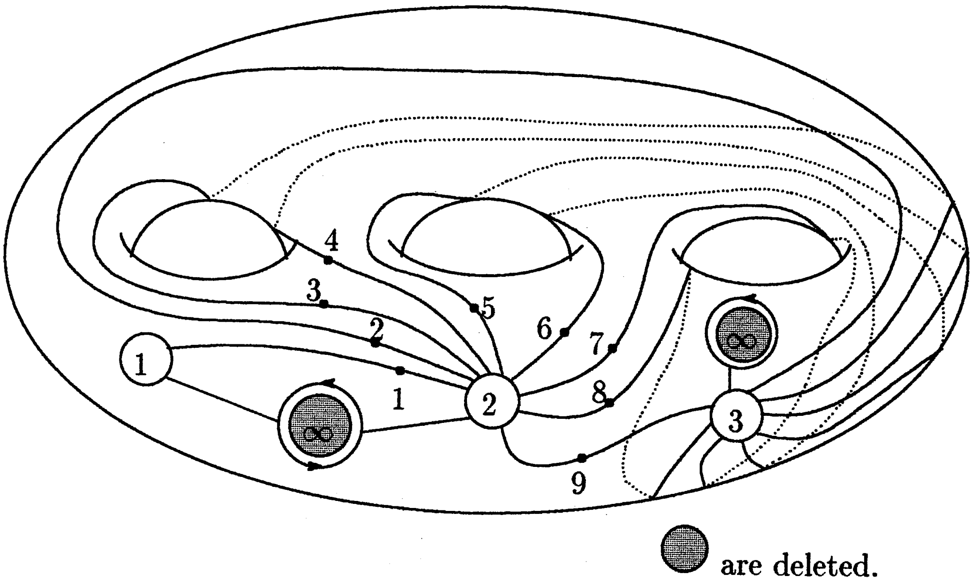

Figure 7: Case 2

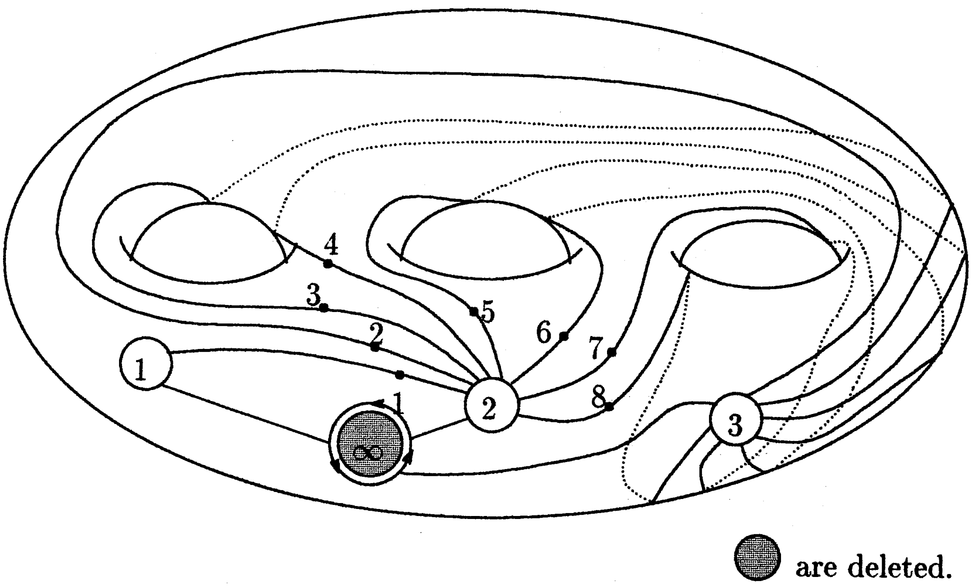

Figure 8: Case 3

In fact we may take as follows:

Case 1:

$$\begin{aligned}
\xi_3 &= [1,\ 21][\infty,\ 11][1,\ 12] \\
\xi_2 &= [\infty,\ 22] \\
\xi_1 &= [2g+4,\ 23][\infty,\ 33][2g+4,\ 32] \\
\alpha_1 &= [3,\ 23][4,\ 32] \\
\beta_1 &= [5,\ 23][4,\ 32] \\
&\quad \cdots\cdots \\
\alpha_j &= [2j+1,\ 23][2j,\ 32]\cdots[3,\ 23][2j+2,\ 32] \\
\beta_j &= [2j+3,\ 23][2j+2,\ 32] \\
&\quad \cdots\cdots \\
\alpha_g &= [2g+1,\ 23][2g,\ 32]\cdots[3,\ 23][2g+2,\ 32] \\
\beta_g &= [2g+3,\ 23][2g+2,\ 32].
\end{aligned}$$

Case 2:

$$\begin{aligned}
\xi_2 &= [\infty,\ 21][\infty,\ 12] \\
\xi_1 &= [2g+3,\ 23][\infty,\ 33][2g+3,\ 32] \\
\alpha_1 &= [2,\ 23][3,\ 32] \\
\beta_1 &= [4,\ 23][3,\ 32] \\
&\quad \cdots\cdots \\
\alpha_j &= [2j,\ 23][2j-1,\ 32]\cdots[2,\ 23][2j+1,\ 32] \\
\beta_j &= [2j+2,\ 23][2j+1,\ 32] \\
&\quad \cdots\cdots \\
\alpha_g &= [2g,\ 23][2g-1,\ 32]\cdots[2,\ 23][2g+1,\ 32] \\
\beta_g &= [2g+2,\ 23][2g+1,\ 32].
\end{aligned}$$

Case 3:

$$\begin{aligned}
\xi &= [\infty,\ 21][\infty,\ 13][\infty,\ 32] \\
\alpha_1 &= [2,\ 23][3,\ 32] \\
\beta_1 &= [4,\ 23][3,\ 32] \\
&\quad \cdots\cdots \\
\alpha_j &= [2j,\ 23][2j-1,\ 32]\cdots[2,\ 23][2j+1,\ 32] \\
\beta_j &= [2j+2,\ 23][2j+1,\ 32] \\
&\quad \cdots\cdots \\
\alpha_g &= [2g,\ 23][2g-1,\ 32]\cdots[2,\ 23][2g+1,\ 32] \\
\beta_g &= [2g+2,\ 23][2g+1,\ 32].
\end{aligned}$$

5 Calculations of fundamental groups.

In this section, we compute fundamental groups of 3-dimensional compact oriented manifolds using the local version of the theorem of Zariski-van Kampen (see Dimca [3], Matsuno [8]) and the method of Reidemeister-Schreier (see Rolfsen [12]). One can compute the fundamental group rigorously if one uses the Riemann picture. We explain this using a concrete example:

Let us consider Case 1 of genus 1 for simplicity. If we take the braid σ as

$$\sigma = \sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2^3$$

(σ induces a knot), then we have the equality

$$\Phi \circ \sigma = \Phi$$

where Φ is the monodromy of the canonical form. Hence we may construct a topological degenerating family

$$f : X \longrightarrow \Delta(0,\, a) \times \Delta(0,\, b)$$

of branched coverings of discs constructed from the pair $(\Phi,\, \sigma)$ (see Section 1). Let B_f be the branch locus of f. Let γ_j $(j = 1, \ldots, 6)$ be the lassos as in Figure 2. The local version of the theorem of Zariski-van Kampen asserts that the fundamental group of $\Delta(0,\, a) \times \Delta(0,\, b) - B_f$ is generated by γ_j $(j = 1, \ldots, 6)$ whose generating relations are $\sigma(\gamma_j) = \gamma_j$ $(j = 1, \ldots, 6)$. That is to say

$$\begin{aligned}
&\pi_1(\Delta(0,\, a) \times \Delta(0,\, b) - B_f,\, q_0) =< \gamma_1, \ldots, \gamma_6 \,|\, \sigma(\gamma_j) = \gamma_j \ (j = 1, \ldots, 6) > \\
&=< \gamma_1, \ldots, \gamma_6 \,|\, (\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2^3)\gamma_j = \gamma_j \ (j = 1, \ldots, 6) > \\
&=< \gamma_1, \ldots, \gamma_6 \,|\, \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_1^{-1} = 1,\ \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_5^{-1}\gamma_6^{-1}\gamma_5 \\
&\gamma_1^{-1}\gamma_5^{-1}\gamma_6\gamma_5\gamma_1\gamma_5^{-1}\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_2^{-1} = 1,\ \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1} \\
&\gamma_5^{-1}\gamma_6^{-1}\gamma_5\gamma_1\gamma_5^{-1}\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_3^{-1} = 1,\ \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_3\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1} \\
&\gamma_4^{-1}\gamma_1\gamma_4^{-1} = 1,\ \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_5^{-1} = 1,\ \gamma_5\gamma_6^{-1} = 1 > .
\end{aligned}$$

Now, for fixed $t \neq 0$, the restriction of f is

$$f_t : X_t \longrightarrow t \times \Delta(0,\, b).$$

This is a covering of degree 3 and the genus of X_t is 1. We extend the covering to the branched covering of $\mathbb{P}^1$ in the caconincal way as in Section 2 which is denoted by the same notation for simplicity.

Now the method of Reidemeister-Schreier says that the fundamental group $\pi_1(X - \{x\},\, p_0)$, $(x = f^{-1}((0,\, 0)))$ is generated by these loops ξ_1, ξ_2, ξ_3, α and β (see Section 4) and their generating relations are pull-back over f_t of these of the fundamental group $\pi_1(\Delta(0,\, a) \times \Delta(0,\, b) - B_f,\, q_0)$, expressed by the generators ξ_1, ξ_2, ξ_3, α and β.

We can carry this out observing the Riemann picture in Figure 5. The result is as follows:

$$\pi_1(Y,\, p_0) \simeq \pi_1(X - \{x\},\, p_0) = \{1\},$$

where Y is the 3-dimensional compact oriented manifold on which X is a cone (see Section 1).

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