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Fenchel's problem and some examples of algebraic varieties

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This talk is concerned with a topic in the theory of brached coverings, called "Fencel's problem" in Namba [9] : the problem on the existence of finite Galois coverings of complex manifolds with given branch divisors. This problem originates in Fenchel's conjecture on compact Riemann surfaces, proved by Bundgaard-Nielsen [2], Fox [3] early in 1950's. I shall give some higher dimensional analogues of it, where compact Riemann surfaces are replaced by the complex projective plane, products of compact Riemann surfaces, or complex tori. I shall then mention some examples of projective manifolds with ample canonical bundles obtained as an application of our results.

1. First we recall some basic definitions and facts about finite Galois coverings in the category of complex-analytic spaces. Let M be a connected complex manifold. A *finite Galois covering* of M is a finite, surjective, proper holomorphic mapping $f: X\to M$ from an irreducible, normal, complex space X onto M, such that the *covering transformation group* of f acts transitively on each fibre of f. Here, the covering transformation group means the group of biholomorphic mappings $\mu: X\to X$ such that $f\circ\mu=f$. When $f: X\to M$ is a finite Galois covering, we call it frequently the *Galois group* of f. A finite Galois covering is called an *abelian* (resp. *cyclic*, *solvable*) covering if its Galois group is abelian (resp. cyclic, solvable).

Let D be an irreducible hypersurface of a connected complex manifold M and $f: X\to M$ a finite Galois covering. The *ramification index* of f along D is defined as follows. Take a nonsingular point p of D. Then every point $q\in f^{-1}(p)$ is nonsingular point of both X and $f^{-1}(D)$. Moreover, if W is a sufficiently small connected open neighbourhood of p with a coordinate system $(w_1,\dots,w_n)$ such that $p=(0,\dots,0)$ and $D\cap W=\{(w_1,\dots,w_n)\in W \mid w_n=0\}$, then there is a coordinate system $(z_1,\dots,z_n)$ in the connected component $U\subset f^{-1}(W)$ with $q\in U$ and a positive integer e such that $q=(0,\dots,0)$ and

$$f: (z_1,\dots,z_n)\in U\mapsto (w_1,\dots,w_n)=(z_1,\dots,z_{n-1},z_n^e)\in W.$$

We note that e is determined only by D, independently of the choice of p and U. We call e the ramification index of f along D. If $e=1$, f is said to be unramified along D.

Now let $D_1,\dots,D_n$ be k distinct irreducible hypersurfaces on M. For a positive divisor $D=e_1D_1+\cdots+e_nD_n$ with $e_j\geqq 2$ $(1\leqq j\leqq k)$, a finite Galois covering $f: X\to M$ is said to *branch at* D, if for each j $(1\leqq j\leqq k)$, the ramification index of f along D_j is e_j and f unramified along any irreducible hypersurface other than $D_1,\dots,D_n$.

Take a reference point $*$ in $M\setminus(D_1\cup\cdots\cup D_n)$. We denote by γ_j $(1\leqq j\leqq k)$ a lasso round D_j: the homotopy class of a closed path in $M\setminus(D_1\cup\cdots\cup D_n)$ which starts from $*$, moves to a point near the nonsingular locus of D_j, turns once round D_j in the positive direction, and return to $*$ (Figure 1).

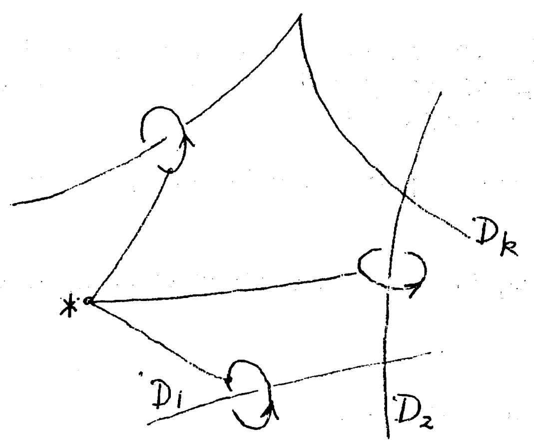

Figure 1

The following is fundamental.

Theorem (cf. Namba [9]). *There is an one-to-one correspondence between the set of finte Galois coverings $f: X \to M$ branching at $D = e_1 D_1 + \cdots + e_k D_k$ and the set of normal subgroups N of the fundamental group $\pi_1(M \setminus (D_1 \cup \cdots \cup D_k), *)$ with finite indices such that the order of the image of γ_j in the quotient group $\pi\ (M \setminus (D_1 \cup \cdots \cup D_k), *)/N$ is e_j $(1 \leqq j \leqq k)$.*

The quotient group $\pi_1(M \setminus (D_1 \cup \cdots \cup D_k), *)/N$ is nothing but the Galois group of f.

2. We now start a discussion on Fenchel's problem. Let M be a compact Riemann surface of genus g and $p_1, \ldots, p_k$ distinct points on M. For integers $e_j \geqq 2$ $(1 \leqq j \leqq k)$, we consider the positive divisor $D = e_1 p_1 + \cdots + e_k p_k$. Fenchel's conjecture, proved by Bundgaard-Nielsen [2], Fox [3], asserts that *there is a finite Galois covering $f: X \to M$ branching at D if and only if one of the following conditions is satisfied:*

(i) $g \geqq 1$,

(ii) $g = 0$ *and* $k \geqq 3$,

(iii) $g = 0$, $k = 2$ *and* $e_1 = e_2$.

(The case where $g \geqq 1$ is due to Bundgaar-Nielsen; The case where $g = 0$ is due to Fox.)

As a generalization to higher dimension, we consider the following problem raised in the book of Namba [9], called Fenchel's problem : *Given a connected complex manifold M and distinct irreducible hypersurfaces $D_1, \ldots, D_k$ on M, we consider the positive divisor $D = e_1 D_1 + \cdots + e_k D_k$ $(e_j \geqq 2,\ 1 \leqq j \leqq k)$. Then, give a condition on $e_1, \ldots, e_k$ for the existence of a finite Galois covering $f: X \to M$ branching at D.*

In group-theoretic terms, as we explained in section 1, the existence of such a covering is equivalent to that of a finite quotient group of $\pi_1(M \setminus (D_1 \cup \cdots \cup D_k), *)$ such that each order of the image of γ_j is $e_j \geqq 2$ $(1 \leqq j \leqq k)$. ($*$: a reference point, γ_j : lassos round D_j .)

When M is a compact Riemann surface of genus g, $\pi_1(M \setminus \{p_1, \ldots, p_k\}, *)$ is isomorphic to

(i) $\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_k \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = 1 \rangle$, if $g \geqq 1$,

(ii) $\langle \gamma_1, \ldots, \gamma_k \mid \gamma_1 \cdots \gamma_k = 1 \rangle$, if $g = 0$,

where $[\ ,\ ]$ being the commutator.

Bunggaard-Nielsen, or Fox proved the assertion by finding out adroitly finite quotients of $\pi_1(M \setminus \{p_1, \ldots, p_k\}, *)$ satisfying the requirements. However, when M is of higher dimension, $\pi_1(M \setminus (D_1 \cup \cdots \cup D_k), *)$ is in general too complicated to handle.

(Remark 1) Bundgaard-Nilelsen's proof implies that, in the case g$\geqq$1, f can be so chosen as to be solvable.

(Remark 2) Their proof does not make it clear which finite groups occur as the quotients, or the Galois groups. Recently Matsuno [7], improving their proof, gave a method to compute the Galois groups effectively.

(Remark 3) In some special cases, even if M is of higer dimension, the fundamental groups can be dealt with. See Matsuno's report in this volume.

3. By a combination of Fox's result [3] and a technique on linear pencils,Kato [6], Namba [9], evading the difficulty of the fundamental groups in higher dimension, give some solutions for Fenchel's problem, for example, in the following cases:

- M: the complex projective plane $\boldsymbol{P}^2$,
  $D_1, \ldots, D_k$: lines such that there is at least one point of multiplicity $\geqq 3$ on each line (Figure 2),

- M: $\boldsymbol{P}^2$,
  $D_1, \ldots, D_k$: conics such that for each D_j there is another D_k touching at 2 distinct points (Figure 3),

- M : $\boldsymbol{P}^2$,
  $D_1, \ldots, D_k$: 3 lines circumscribing a conic (Figure 4).

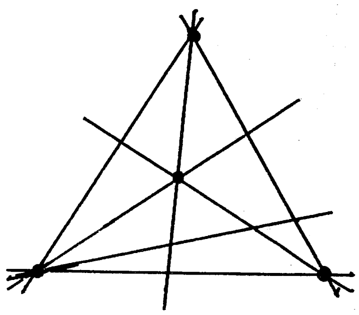

Figure 2

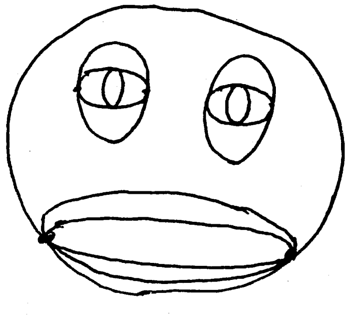

Figure 3

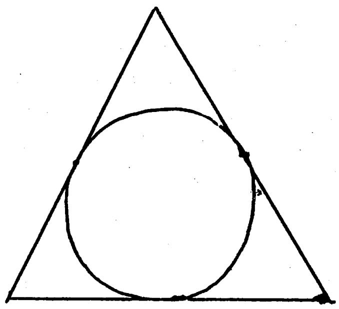

Figure 4

In fact, the following theorems are proved.

Theorem ([6]). *Let $D_1, \dots, D_k$ be distinct lines on $\boldsymbol{P}^2$. We put $\Delta = \{p \in B = D_1 \cup \cdots \cup D_k \mid m_p(B) \geqq 3\}$, where $m_p(B)$ is the multiplicity of the curve B at p. Suppose that $D_j \cap \Delta \neq \phi$, for $1 \leqq j \leqq k$. Then, for any integers $e_j \geqq 2$ $(1 \leqq j \leqq k)$, there is a finite Galois covering $f: X \to \boldsymbol{P}^2$ branching at $D = e_1 D_1 + \cdots + e_k D_k$.*

Theorem ([9], Theorem 1.5.8). *Let $D_1, \dots, D_k$ be distinct irreducible conics on $\boldsymbol{P}^2$. Suppose that, for each D_j, there is another D_k such that D_j and D_k are tangent at two dintinct poits. Then, for any integers $e_j \geqq 2$ $(1 \leqq j \leqq k)$, there is a finite Galois covering $f: X \to \boldsymbol{P}^2$ branching at $D = e_1 D_1 + \cdots + e_k D_k$.*

Theorem ([9], Proposition 1.5.9). *Let D_1, D_2, D_3 be 3 distinct lines on $\boldsymbol{P}^2$ circumscribing an irreducible conic C. Then, for any integers $a, b \geqq 2$, there is a finite Galois covering $f: X \to \boldsymbol{P}^2$ branching at $a(D_1 + D_2 + D_3) + bC$.*

(For further discussions in this line, refer Namba [9].)

4. Let us now give some other results on Fenchel's problem. We treat the following cases:

- M : $\boldsymbol{P}^2$,
  $D_1, \dots, D_k$: lines in a near-pencil arrangement, that is, lines passing through one point and another line not passing the point (Figure 5),

- M : $\boldsymbol{P}^2$,
  $D_1, \dots, D_k$: n ($\geqq 3$) lines circumscribing a conic (Figure 6),

- M : a product of compact Riemann surfaces of genus $\geqq 1$,
  $D_1, \dots, D_k$: any hypersurfaces,

- M : a complex torus,
  $D_1, \dots, D_k$: any hypersurfaces.

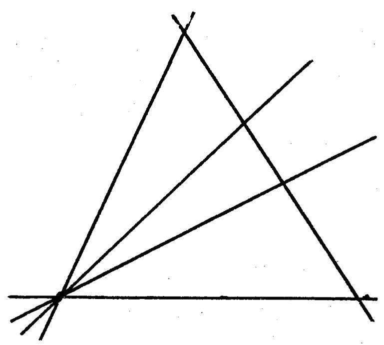

Figure 5

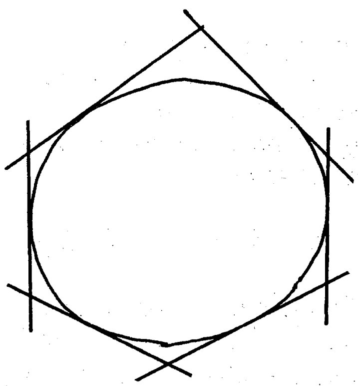

Figure 6

In fact, by means of a topological method, we obtain the following:

Theorem 1 (cf. [10], Prositiotion 7.2, 7.3). *Take a point p on $\boldsymbol{P}^2$. Suppose that n distinct lines $D_1,\ldots, D_n$ pass through p and that another line D_∞ does not. Let $e_1,\ldots, e_n, d$ be integers $\geqq 2$. Then, there is a finite Galois covering $f: X\to \boldsymbol{P}^2$ branching at $D=e_1D_1+\cdots+e_nD_n+dD_\infty$ if and only if one of the following conditions is satiesfied:*

(i) $n \geqq 4$,

(ii) $n=3$ and $e_1^{-1}+e_2^{-1}+e_3^{-1}\leqq 1$,

(iii) $n=3$, $e_1^{-1}+e_2^{-1}+e_3^{-1}>1$ and (e_1, e_2, e_3, d) is one of the following:

$$(e_1, e_2, e_3, d)=\begin{cases}(e\ ,2\ ,2\ ,d)\ ; e\geqq 2 \text{ and } d \text{ divides } 2e,\\ (3\ ,2\ ,2\ ,d)\ ; d \text{ divides } 12,\\ (4\ ,3\ ,2\ ,d)\ ; d \text{ divides } 24,\\ (5\ ,3\ ,2\ ,d)\ ; d \text{ divides } 60.\end{cases}$$

(Remark 4) When I gave my talk, on the necessary condition, I could prove only the first case in the quadruplet in (iii) above. After that, I have carried out the proof of all cases, stimulated by Professor H. Tsuchihashi's comment on maximal coverings of $\boldsymbol{P}^2$.

Theorem 2 ([11]). *Let $D_1,\ldots, D_n$ be n ($\geqq 3$) distinct lines on $\boldsymbol{P}^2$ circumscribing an irreducible conic C.* Let *$e_1,\ldots, e_n$ be integers $\geqq 2$ and d an integer $\geqq 1$. Then, there is a finite Galois covering $f: X\to \boldsymbol{P}$ branching at $D=e_1D_1+\cdots+e_nD_n+2dC$ if $(e_1,\ldots, e_n, d)$ is one of the following:*

(i) $n\geqq 4$,

(ii) $n=3$ and $e_1^{-1}+e_2^{-1}+e_3^{-1}\leqq 1$,

(iii) $n=3$, $d=1$ and $e_1^{-1}+e_2^{-1}+e_3^{-1}>1$.

(Remark 5) I do not know about the necessary conditions.

Theorem 3 ([11]). *Let M_i be compact Riemann surfaces of genus $g_i\geqq 1$ $(1\leqq i\leqq n)$ and $D_1,\ldots, D_k$ distinct irreducible hypersurfaces on $M=M_1\times\cdots\times M_n$. Then, for any integers $e_j\geqq 2$ $(1\leqq j\leqq k)$, there is a solvable covering $f: X\to M$ branching at $D=e_1D_1+\cdots+e_kD_k$.*

Theorem 4. *Let M be an n-dimensional complex torus and $D_1,\ldots, D_k$ distinct irreducible hypersurfaces on M. Then, for any integers $e_j\geqq 2$ $(1\leqq j\leqq k)$, there is a* solvable *covering $f: X\to M$ branching at $D=e_1D_1+\cdots+e_kD_k$.*

5. Theorem 3 or 4 is deduced from the following propositions respevtively.

Proposition 5.1 ([11]). *Suppose that M and $D_1,\ldots, D_k$ are the same as is stated in Theorem 3. Then, for any integers $e\geqq 2$, there is a finite Galois covering $f: X\to M$ branching at $D=e(D_1+\cdots+D_k)$ whose Galois group is isomorphic to a meta-abelian group G equipped with the following exact sequence:*

$$1\to \boldsymbol{Z}/e\boldsymbol{Z}\to G\to (\boldsymbol{Z}/e\boldsymbol{Z})^{2(g_1+\cdots+g_n)}\to 1.$$

(Here, $\boldsymbol{Z}$ denotes the ring of integers)

Proposition 5.2. *Suppose that M and $D_1,\ldots, D_k$ are the same as is stated in Theorem 4. Then, for any integers $e\geqq 2$, there is a finite Galois covering $f: X\to M$ branching at $D=e(D_1+\cdots+D_k)$ whose Galois group is isomorphic to a meta-abelian group G equipped with the following exact sequence:*

$$1\to \boldsymbol{Z}/e\boldsymbol{Z}\to G\to (\boldsymbol{Z}/e\boldsymbol{Z})^{2n}\to 1.$$

We give a brief outline of the proof of Proposition 5.1. (For details, see [11].)
Let $\phi_i: M_i'\to M_i$ $(1\leqq i\leqq n)$ be the unramified coverings corresponding to the kernels of $\boldsymbol{Z}/e\boldsymbol{Z}$-Hurewicz homomorphism : $\pi_1(M_i)\twoheadrightarrow H_1(M_i, \boldsymbol{Z}/e\boldsymbol{Z})$. Put

$$\phi=\phi_1\times\cdots\times\phi_n: M'=M_1'\times\cdots\times M_n'\to M.$$

We denote by $[D]$ the line bundle on M associated to $D=D_1\cup\cdots\cup D_k$. Let $\phi^*[D]$ be the pull-back of $[D]$ by ϕ, $c_1(\phi^*[D])$ its first Chern class and $\overline{c_1(\phi^*[D])}$ its image under natural homomorphism $H^2(M', \boldsymbol{Z}) \to H^2(M', \boldsymbol{Z}/e\boldsymbol{Z})$. By Kunneth formula and projection formula, we see that $\overline{c_1(\phi^*[D])} \cap \sigma = 0$ for any 2-dimensional cycle $\sigma \in H_2(M', Z/eZ)$, where $\cap$ means the cap product. From this it follows that $\overline{c_1(\phi^*[D])}=0$, which implies that there is a line bundle L on M' such that $L^{\otimes e} = \phi^*[D]$ as C^∞-line bundles. As a consequence, we can construct $\boldsymbol{Z}/e\boldsymbol{Z}$- covering $\mu: X \to M'$ branching at $e(\phi^* D)$. Moreover, we see that each covering transformation of ϕ lifts to a biholomorphism of X up to covering transformations of μ. Thus, the composite $f=\phi\circ\mu: X \to M$ is a finite Galois covering having the properties in the statement of Proposition 5.1.

Proposition 5.2 is proved in a similar way.

6. We now give some examples of projective manifolds with ample canonical bundles and calculate their Chern numbers. We denote by K_M the canonical divisor of a compact complex manifold M and by $c_1(M)$ (resp. $c_2(M)$) the first (resp. second) Chern class of the tangent bundle of M.

For n-dimensional projective manifolds M with K_M ample, the inequality

$$(-1)^n c_1^n(M) \leq (-1)^n \frac{2(n+1)}{n} c_1^{n-2} c_2(M)$$

holds, where the equality holds if and only if the universal cover of M is the unit ball (see Yau [12], also Miyaoka [8] for surfaces of general type).

Example 1. Let C be a compact Riemann surface of genus $g \geqq 2$ and $\triangle$ the diagonal of $C\times C$. By Proposition 5.1, we have a meta-abelian covering $f: X \to C\times C$ of degree e^{4g+1} branching at $e\triangle$. Since $\triangle$ is a nonsingular curve on $C\times C$, X is nonsingular surface. By Hurwitz formula (e.g., [1]), we have $K_X = f^*\overline{K}$, where

$$\begin{aligned}\overline{K} &= K_{C\times C} + \tfrac{e-1}{e}\triangle \\ &= (2g-2 \text{ points})\times C + C\times(2g-2 \text{ points}) + \tfrac{e-1}{e}\triangle.\end{aligned}$$

Then,

$$\begin{aligned} c_1^2 = K_X^2 &= (\deg f)\,\overline{K}^2 \\ &= e^{4g+1}\Big(K_{C\times C}^2 + 2\tfrac{e-1}{e} K_{C\times C}\cdot\triangle + \tfrac{(e-1)^2}{e^2}\triangle^2\Big) \\ &= e^{4g-1}(g-1)(8e^2 g - 2(e+1)^2).\end{aligned}$$

Note that $K_X^2 > 0$.

For every irreducible curve D on X, we have $K_X \cdot D = \overline{K}\cdot f_* D > 0$. Nakai's criterion (e.g., [1]) implies that K_X is ample.

The second Chern class c_2, which is equal to the Euler number $e(X)$ in two-dimensional cases, can be calculated by the "cardinality principle" (see, Hirzebruch [4], [5]). Indeed,

$$\begin{aligned} c_2 &= e(X) \\ &= (\deg f)\Big(e(C\times C) - \tfrac{e-1}{e} e(\triangle)\Big) \\ &= e^{4g+1}\Big((2-2g)^2 - \tfrac{e-1}{e}(2-2g)\Big) \\ &= e^{4g-1}(g-1)(4e^2 g - 2e(e+1)).\end{aligned}$$

Thus we get

$$\frac{c_1^2}{c_2} = \frac{4g - (1+1/e)^2}{2g - (1+1/e)}$$

(Remark 6) The composite $X \xrightarrow{f} C\times C \xrightarrow{\text{proj.}} C$ gives a *Kodaira fibration* in the sense of [1].

Example 2. Let C be the compact Riemann surface defined by the equation $y^5 = x(x-1)$, with genus two. The meromorphic function $(x, y) \in C \mapsto x \in P^1$ gives 5-fold cyclic covering ramified completely at the three point $p_0, p_1, p_\infty \in C$ over $0, 1,$ and ∞ (Figure 7).

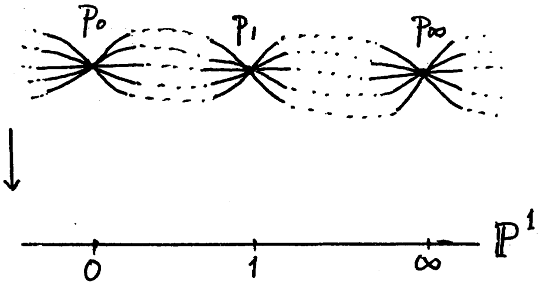

Figure 7

Let σ be a generator of the Galois group ($\cong \mathbf{Z}/5\mathbf{Z}$). For $0 \leqq i \leqq 4$, we put $\Delta_i = \{(p, \sigma^i(p)) \in C \times C \mid p \in C\}$. ($\Delta_0$ is nothing but diagonal Δ, see Figure 8).
By Proposition 5.1, we have a mata-abelian covering $f: X \to C \times C$ of degree 5^9 branching at $5(\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)$.

We have singularities on the fibres of f over three points (p_0, p_0), (p_1, p_1) and (p_∞, p_∞), which can be resolved as follows. Let $M \to C \times C$ be the blow up of $C \times C$ at the 3 points. We denote by E_i ($i = 0, 1, \infty$) the exceptional curves associated to the 3 points, and by $\tilde{\Delta}_i$ the proper transforms of Δ_i ($0 \leqq i \leqq 4$) (Figure 9).

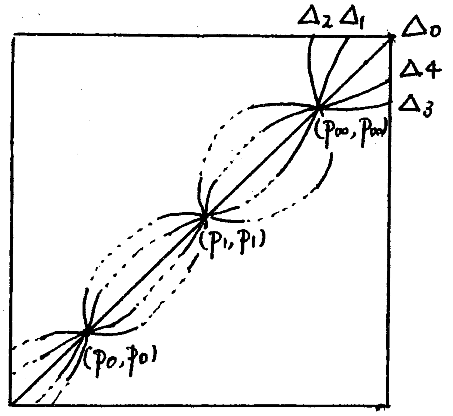

Figure 8

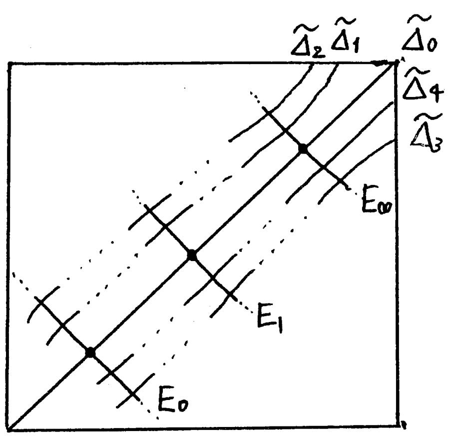

Figure 9

Let $f': X' \to M$ be the pull-back of $f: X \to C \times C$ and Y the normalization of X'. Thus we get a meta-abelian covering $\pi: Y \to M$ of degree 5^9. As we see from the construction in Proposition 5.1, π branches at $5(\tilde{\Delta}_0 + \tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 + \tilde{\Delta}_4)$, unramified along E_i ($i = 0, 1, \infty$). Therefore Y is a nonsingular surface.

We have $K_Y = \pi^* \overline{K}$, where

$$\overline{K} = K_M + 4/5\,(\tilde{\Delta}_0 + \tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 + \tilde{\Delta}_4).$$

It is easy to check that there is no curve isomorphic to the projective line on Y. As a result, K_Y is ample. We obtain by a calculation similar to that in Example 1,

$$c_1 = K_Y^2 = 5^9 \cdot 45, \quad c_2 = e(Y) = 5^9 \cdot 15.$$

We have $c_1^2/c_2 = 3$, which implies the universal cover of Y is the unit ball. (Compare Hirzebruch [4].)

Example 3. Let M be an n-dimensional complex abelian variety and D a nonsingular hypersurface on M. By Proposition 5.2, for any integer $e \geqq 2$, we have a meta-abelian covering $f: X \to M$ of degree e^{2n+1} branching at eD. Since the tangent bundle of M is trivial, we have $K_X = \frac{e-1}{e} f^* D$. Therefore, if D is ample, K_X is also ample divisor.

We have

$$\begin{aligned} c_1^n &= (-K_X)^n \\ &= (-1)^n \left(\frac{e-1}{e}\right)^n (\deg f)\, D^n \\ &= (-1)^n (e-1)^n e^{n+1} D^n. \end{aligned}$$

In order to calculate c_2, we need the following

Lemma 6.1. *Let M be an n-dimensional projective manifold and D a nonsingular hypersurface on M. Suppose that $f: X \to M$ is a finite Galois covering branching at e D. Then we have*

$$c_2(X) = f^*\left(c_2(M) - \frac{e-1}{e} c_1(M) \cdot D + \frac{e-1}{e} D^2\right).$$

This lemma can be proved by induction on n.

By Lemma 6.1, we see that $c_2(X) = \frac{e-1}{e} D^2$. Hence we have

$$c_1^{n-2} c_2 = (-1)^{n-2} (e-1)^{n-1} e^{n+2} D^n.$$

Thus, we get

$$\frac{c_1^n}{c_1^{n-2} c_2} = \frac{e-1}{e}.$$

(Remark 7) Hirzebruch [5] constructed a surface with ample canonical bundle with the universal cover the unit ball as a branched covering of a blow-up of certain abelian surface with complex multiplication. Can we construct a projective manifold of dimension $\geqq 3$ with the same properity as a branched covering of a blow-up of some abelian variety?

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