Quasi M-convex Functions
and Minimization Algorithms

Kazuo MUROTAs and Akiyoshi SHIOURA2

1Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
murota@kurims.kyoto-u.ac.jp

2Department of Mechanical Engineering
Sophia University
7-1 Kioi-cho, Chiyoda-ku
Tokyo 102-8554, Japan
shioura@me.sophia.ac.jp

Abstract: We introduce a class of discrete quasiconvex functions, called quasi M-convex functions, by generalizing the concept of M-convexity due to Murota (1996). We investigate the structure of quasi M-convex functions with respect to level sets, and show that various greedy algorithms work for the minimization of quasi M-convex functions.

Keywords: quasiconvex function, discrete optimization, matroid, base polyhedron.

1 Introduction

The concept of convexity for sets and functions plays a central role in continuous optimization (or nonlinear programming with continuous variable), and has various applications in the areas of mathematical economics, engineering, operations research, etc. [2, 12, 15]. The importance of convexity relies on the fact that a local minimum of a convex function is also a global minimum. Due to this property, we can find a global minimum of a convex function by iteratively moving in descent directions, i.e., so-called descent algorithms work for the convex function minimization. Therefore, convexity for a function is a sufficient condition for the success of descent methods. Most of descent methods, however, work for a fairly larger class of functions called quasiconvex functions.

Let \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be defined over a nonempty convex set, i.e., \( \text{dom} \ f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\} \) is a nonempty convex set. A function \( f \) is said to be quasiconvex if it satisfies

\[
 f(\alpha x + (1 - \alpha)y) \leq \max\{f(x), f(y)\}
\]

for all \( x, y \in \text{dom} \ f \) and \( 0 < \alpha < 1 \), and semistrictly quasiconvex if it satisfies

\[
 f(\alpha x + (1 - \alpha)y) < \max\{f(x), f(y)\}
\]

for all \( x, y \in \text{dom} \ f \) with \( f(x) \neq f(y) \) and \( 0 < \alpha < 1 \). It is easy to see that convexity implies semistrict quasiconvexity, and semistrict quasiconvexity implies quasiconvexity under the assumption of lower semicontinuity. Although (semistrict) quasiconvexity is a weaker property than convexity, it still has nice properties as follows:

- strict local minimality leads to global minimality for quasiconvex functions,
- local minimality leads to global minimality for semistrictly quasiconvex functions,
- level sets of quasiconvex functions are convex sets.

Due to these properties, quasiconvexity also plays an important role in continuous optimization. See [1] for more accounts on quasiconvexity.

In the area of discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered, with a view to identifying the discrete structure that guarantees the success of descent methods, i.e., so-called "greedy algorithms." Examples of discrete convexity are "discretely-convex functions" by Miller [7], "integrally-convex functions" by Favati–Tardella [3], and "M-convex/L-convex functions" by Murota [8, 9, 10].

A function \( f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is called M-convex if \( \text{dom} \ f \neq \emptyset \) and \( f \) satisfies the following property:

(M-EXC) \( \forall x, y \in \text{dom} \ f, \forall u \in \text{supp}^+(x - y), \)

\[
 f(\alpha x + (1 - \alpha)y) \leq \max\{f(x), f(y)\}
\]

for all \( x, y \in \text{dom} \ f \) and \( 0 < \alpha < 1 \), and semistrictly M-convex if it satisfies

\[
 f(\alpha x + (1 - \alpha)y) < \max\{f(x), f(y)\}
\]

for all \( x, y \in \text{dom} \ f \) with \( f(x) \neq f(y) \) and \( 0 < \alpha < 1 \). It is easy to see that convexity implies (semistrict) M-convexity, and (semistrict) M-convexity implies quasiconvexity under the assumption of lower semicontinuity.
We first review some fundamental results on M-convex functions in Section 2. Then, we show some properties for level sets of quasi M-convex functions, and prove that the class of quasi M-convex functions is closed under various fundamental operations in Section 3. Finally, we show that greedy algorithms work for the minimization of (semistrictly) quasi M-convex functions in Section 4. We also show a proximity theorem on (semistrictly) quasi M-convex functions, which guarantee that the so-called "scaling technique" is applicable to the quasi M-convex function minimization.

2 Review of Fundamental Results on M-convex Functions

We denote by $\mathbb{R}$ the set of reals, and by $\mathbb{Z}$ the set of integers. Also, we denote by $\mathbb{R}_{++}$ the set of positive reals. Throughout this paper, we assume that $V$ is a nonempty finite set of cardinality $n (> 0)$. For $w \in V$, we denote by $\chi_w \in \{0,1\}^V$ the characteristic vector of $w$.

Let $x \in \mathbb{R}^V$. For $S \subseteq V$, we define $x(S) = \sum_{w \in S} x(v)$. We also define

$\text{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\}$,
$\text{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\},$
and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w$. M-convex functions have various desirable properties as discrete convexity:

(i) local minimality leads to global minimality for M-convex functions,
(ii) M-convex functions can be extended to ordinary convex functions,
(iii) various duality theorems hold,
(iv) M-convex functions are conjugate to L-convex functions.

In particular, the property (i) shows that greedy algorithms work for the M-convex function minimization. However, we see from results in continuous optimization that strong properties such as M-convexity are not required for the success of greedy algorithms, and that some property like "quasi M-convexity" will suffice.

The main aim of this paper is to introduce the concept of quasi M-convex functions by generalizing the concept of M-convexity. To define quasi M-convexity, we use the following weaker properties than (M-EXC):

(QM) $\forall x, y \in \text{dom} f$, $\forall u \in \text{supp}^+(x-y)$, $\exists v \in \text{supp}^-(x-y)$:
$f(x - xu + xu) \leq f(x)$ or $f(y + xu - xu) \leq f(y)$.

(SSQM) $\forall x, y \in \text{dom} f$, $\forall u \in \text{supp}^+(x-y)$, $\exists v \in \text{supp}^-(x-y)$:
(i) $f(x - xu + xu) \geq f(x)$
$\Rightarrow f(y + xu - xu) \leq f(y)$, and
(ii) $f(y + xu - xu) \geq f(y)$
$\Rightarrow f(x - xu + xu) \leq f(x)$.

We define a quasi M-convex (resp. semistrictly quasi M-convex) function as a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ with dom $f \neq \emptyset$ satisfying (QM) (resp. (SSQM)). We show that various nice properties hold for (semistrictly) quasi M-convex functions, which justifies the definitions of quasi M-convexity above.
For a set $S \subseteq \mathbb{Z}^V$, the function $\delta_S : \mathbb{Z}^V \to \{0, +\infty\}$ given by
\[
\delta_S(x) = \begin{cases} 
0 & (x \in S), \\
+\infty & (x \notin S)
\end{cases}
\]
is called the indicator function of $S$.

Let $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$. A function $\varphi$ is called quasiconvex if it satisfies
\[
\varphi(\beta) \leq \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \\
\quad (\forall \alpha_1, \alpha_2, \beta \in \mathbb{Z} \text{ with } \alpha_1 < \beta < \alpha_2).
\]
Similarly, $\varphi$ is called semistrictly quasiconvex if it is a quasiconvex function and satisfies
\[
\varphi(\beta) < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \\
\quad (\forall \alpha_1, \alpha_2, \beta \in \mathbb{Z} \text{ with } \alpha_1 < \beta < \alpha_2, \varphi(\alpha_1) \neq \varphi(\alpha_2)).
\]

(2.1)

Remark 2.1. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, semistrict quasiconvexity implies quasi convexity under the assumption of lower semicontinuity [1, 2]. For a function $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$, on the other hand, the property (2.1) alone does not imply the quasiconvexity in general. For convenience, we assume quasiconvexity in the definition of semistrict quasiconvexity for $\varphi$.

Theorem 2.2. Let $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$.

(1) $\varphi$ is quasiconvex if and only if for all $\alpha_1, \alpha_2 \in \text{dom } \varphi$ with $\alpha_1 < \alpha_2$ we have $\varphi(\alpha_1 + 1) \leq \varphi(\alpha_1)$ or $\varphi(\alpha_2 - 1) \leq \varphi(\alpha_2)$.

(ii) $\varphi$ is semistrictly quasiconvex if and only if for all $\alpha_1, \alpha_2 \in \text{dom } \varphi$ with $\alpha_1 < \alpha_2$ we have both
\[
\varphi(\alpha_1 + 1) \geq \varphi(\alpha_1) \Rightarrow \varphi(\alpha_2 - 1) \leq \varphi(\alpha_2), \text{ and}
\varphi(\alpha_2 - 1) \geq \varphi(\alpha_2) \Rightarrow \varphi(\alpha_1 + 1) \leq \varphi(\alpha_1).
\]

A function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is said to be nondecreasing if $\varphi(\alpha) \leq \varphi(\beta)$ holds for all $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and strictly increasing if for all $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we have either $\varphi(\alpha) < \varphi(\beta)$ or $\varphi(\alpha) = \varphi(\beta) = +\infty$.

A function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is called M-convex if $\text{dom } f \neq \emptyset$ and $f$ satisfies the following property:

\begin{enumerate}
\item[(M-EXC)] \quad $\forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):$
\end{enumerate}
\[
f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).
\]

Note that the inequality (2.2) can be rewritten as follows in terms of directional differences:
\[
\Delta f(x; v, u) + \Delta f(y; u, v) \leq 0. \tag{2.3}
\]

M-convex functions can be characterized by the following (seemingly) weaker property:

\begin{enumerate}
\item[(M-EXCw)] \quad $\forall x, y \in \text{dom } f$ with $x \neq y$, $\exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y)$ satisfying (2.2).
\end{enumerate}

Theorem 2.3 ([9, Th. 3.1]). For $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$, we have (M-EXC) $\iff$ (M-EXCw).

We also define the set version of M-convexity as follows. A set $B \subseteq \mathbb{Z}^V$ is called M-convex if $B \neq \emptyset$ and it satisfies
\begin{enumerate}
\item[(B-EXC)] \quad $\forall x, y \in B, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):$
\end{enumerate}
\[
x - \chi_u + \chi_v \in B \quad \text{and} \quad y + \chi_u - \chi_v \in B.
\]

Note that an M-convex set is nothing but the (set of integral vectors in) an integral base polyhedron [4]. For $x \in B$ and $u, v \in V$, the exchange capacity associated with $x, v$ and $u$ is defined as
\[
\delta_B(x, u, v) = \max\{\alpha \in \mathbb{R} \mid x + \alpha(\chi_v - \chi_u) \in B\}.
\]

M-convex sets can be characterized also by the following (seemingly) weaker property:

\begin{enumerate}
\item[(B-EXCw)] \quad $\forall x, y \in B$ with $x \neq y$, $\exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):$
\end{enumerate}
\[
x - \chi_u + \chi_v \in B \quad \text{and} \quad y + \chi_u - \chi_v \in B.
\]

Theorem 2.4 ([16]). For $B \subseteq \mathbb{Z}^V$, we have (B-EXC) $\iff$ (B-EXCw).

3 Quasi M-convex Functions

3.1 Definitions

To extend the concept of M-convexity to quasi M-convexity, we relax the condition (2.3) while keeping the possible sign patterns of values $\Delta f(x; v, u)$ and $\Delta f(y; u, v)$ in mind. Table 1 shows the possible sign patterns of those values.

Let $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be a function. Then, we call $f$ a quasi M-convex function if $\text{dom } f \neq \emptyset$ and it satisfies the following property:

\begin{enumerate}
\item[(QM)] \quad $\forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):$
\end{enumerate}
\[
\Delta f(x; v, u) \leq 0 \quad \text{or} \quad \Delta f(y; u, v) \leq 0.
\]
Similarly, we call $f$ a semistrictly quasi $M$-convex function if $\text{dom} f \neq \emptyset$ and it satisfies the following property:

$(\text{SSQM}) \forall x, y \in \text{dom} f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\begin{align*}
&\text{(i) } \Delta f(x; v, u) \geq 0 \implies \Delta f(y; u, v) \leq 0, \text{ and} \\
&\text{(ii) } \Delta f(y; u, v) \geq 0 \implies \Delta f(x; v, u) \leq 0.
\end{align*}

Note that (SSQM) can be rewritten as follows:

$\forall x, y \in \text{dom} f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y)$ satisfying at least one of the following:

(i) $\Delta f(x; v, u) < 0$, (ii) $\Delta f(y; u, v) < 0$, (iii) $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$.

We also consider weaker properties than (QM) and (SSQM):

$(\text{QM}_w) \forall x, y \in \text{dom} f \text{ with } x \neq y, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\begin{align*}
&\Delta f(x; v, u) \leq 0 \text{ or } \Delta f(y; u, v) \leq 0. \\
&(\text{SSQM}_w) \forall x, y \in \text{dom} f \text{ with } x \neq y, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\begin{align*}
&\Delta f(x; v, u) \geq 0 \implies \Delta f(y; u, v) \leq 0, \text{ and} \\
&\Delta f(y; u, v) \geq 0 \implies \Delta f(x; v, u) \leq 0.
\end{align*}

The set version of quasi $M$-convexity can be obtained by translating the properties (QM) and (QM$_w$) for the indicator function $\delta_B : \mathbb{Z}^V \to \{0, +\infty\}$ of a set $B \subseteq \mathbb{Z}^V$ in terms of $B$.

$(\text{Q-EXC}) \forall x, y \in B, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\begin{align*}
&x - \chi_u + \chi_v \in B \text{ or } y + \chi_u - \chi_v \in B.
\end{align*}

$(\text{Q-EXC}_w) \forall x, y \in B \text{ with } x \neq y, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\begin{align*}
&x - \chi_u + \chi_v \in B \text{ or } y + \chi_u - \chi_v \in B.
\end{align*}

Note that the properties (Q-EXC) and (Q-EXC$_w$) are the same as (EXC) and (EXC$_w$) discussed in [14], respectively.

### Theorem 3.1

Let $B \subseteq \mathbb{Z}^V$.

(i) (Q-EXC) for $B \iff (\text{QM})$ for $\delta_B$.

(ii) (Q-EXC$_w$) for $B \iff (\text{QM}_w)$ for $\delta_B$.

(iii) (B-EXC) for $B \iff (\text{SSQM})$ for $\delta_B \iff (\text{SSQM}_w)$ for $\delta_B$.

We show some examples of quasi $M$-convex functions below.

### Example 3.2

Let $\psi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$. We define $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{+\infty\}$ by

\[
\begin{align*}
&f(x_1, x_2) = \left\{ \begin{array}{ll}
\psi(x_1), & (x_1 + x_2 = 0), \\
+\infty, & (x_1 + x_2) \neq 0.
\end{array} \right. \\
&\text{(3.1)}
\end{align*}
\]

By Theorem 2.2, $f$ satisfies (QM) (or (QM$_w$)) if and only if $\psi$ is quasiconvex, and $f$ satisfies (SSQM) (or (SSQM$_w$)) if and only if $\psi$ is semistrictly quasiconvex.

### Example 3.3

Let $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be an M-convex function, and $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a nondecreasing function. We define a function $\tilde{f} : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ by

\[
\tilde{f}(x) = \left\{ \begin{array}{ll}
\varphi(f(x)), & (x \in \text{dom } f), \\
+\infty, & (x \notin \text{dom } f).
\end{array} \right. \\
\text{(3.2)}
\]

Then, $\tilde{f}$ satisfies (QM). Furthermore, if $\varphi$ is strictly increasing, then $\tilde{f}$ satisfies (SSQM).

### Example 3.4

Let $B \subseteq \mathbb{Z}^V$ be an M-convex set, $v \in \mathbb{R}^V$, and $\alpha \in \mathbb{R}$. Then, the set $S = \{z \in B \mid \langle p, z \rangle \leq \alpha\}$ satisfies (Q-EXC). Moreover, the function $f : S \to \mathbb{R}$ defined by $f(x) = \langle p, z \rangle (x \in S)$ satisfies (SSQM).

### Remark 3.5

The concept of (semistrict) quasi $M$-convexity can be naturally extended to functions $f : S \to T$ with $S \subseteq \mathbb{Z}^V$ and a totally ordered set $T$ with total order $\prec$. For example, the property (SSQM) is rewritten for such functions as follows:

$\forall x, y \in S, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\begin{align*}
&\text{(i) if either } z - \chi_u + \chi_v \notin S, \text{ or } z - \chi_u + \chi_v \in S \text{ and } f(x - \chi_u + \chi_v) \geq f(x), \text{ then } y + \chi_u - \chi_v \in S \text{ and } f(y + \chi_u - \chi_v) \leq f(y), \text{ and} \\
&\text{(ii) if either } y + \chi_u - \chi_v \notin S, \text{ or } y + \chi_u - \chi_v \in S \text{ and } f(y + \chi_u - \chi_v) \geq f(y), \text{ then } z - \chi_u + \chi_v \in S \text{ and } f(z - \chi_u + \chi_v) \leq f(z),
\end{align*}

where for $p, q \in T$ the notation $p \preceq q$ means $p - q$ or $p = q$. It is easy to see that the properties...
of (semistrictly) quasi M-convex functions shown in this paper still holds true. For simplicity and convenience, we assume, in this paper, that the codomain of a function is \( \mathbb{R} \cup \{+\infty\} \).

\[ \square \]

**Example 3.6.** Suppose that \( V = \{1, 2, \ldots, n\} \) \((n \geq 1)\). Let \( a : V \to \mathbb{Z} \cup \{-\infty\}, b : V \to \mathbb{Z} \cup \{+\infty\}, \) and \( \alpha \in \mathbb{Z} \) satisfy \( a(v) \leq b(v) \) \((v \in V)\) and \( \sum_{i \in V} a(i) \leq \alpha \leq \sum_{i \in V} b(i) \). For \( i \in V \), let \( f_i : [a(i), b(i)] \to \mathbb{R} \) be a semistrictly quasiconvex function. We define \( B \subseteq \mathbb{Z}^V \) and \( f : B \to \mathbb{R}^V \) by

\[
B = \{ x \in \mathbb{Z}^V \mid x(V) = \alpha, a \leq x \leq b \},
\]

\[
f(x) = (f_i(x(i)) \mid i \in V) \quad (x \in B),
\]

where the total order \( \prec \) on the codomain \( \mathbb{R}^V \) of \( f \) is given by the lexicographic order, i.e., for each \( p, q \in \mathbb{R}^V \), \( p \prec q \) holds if there exists some \( k \) \((1 \leq k \leq n)\) such that \( p_i = q_i \) for \( i = 1, \ldots, k-1 \) and \( p_k < q_k \). Then, \( f \) satisfies (SSQM) in the extended sense (see Remark 3.5).

**Proof.** Let \( x, y \in B \) be distinct vectors. Also, let \( u \in \text{supp}^+(x - y), v \in \text{supp}^-(x - y) \) be any elements, and w.l.o.g. assume that \( u < v \). Then, we have \( x - x_u + x_v \in B \) and \( y + x_u - x_v \in B \). If \( f_u(x(u) - 1) < f_u(x(u)) \) or \( f_u(y(u) + 1) < f_u(y(u)) \), holds, then we have \( f(x - x_u + x_v) < f(x) \) or \( f(y + x_u - x_v) < f(y) \). Otherwise, we have \( f_u(x(u) - 1) = f_u(x(u)) \) and \( f_u(y(u) + 1) = f_u(y(u)) \) by Theorem 2.2. If \( f_v(x(v) + 1) < f_v(x(v))) \) or \( f_v(y(v) - 1) < f_v(y(v)) \), holds, then we have \( f(x - x_u + x_v) < f(x) \) or \( f(y + x_u - x_v) < f(y) \). Otherwise, we have \( f_v(x(v) + 1) = f_v(x(v)) \) and \( f_v(y(v) - 1) = f_v(y(v)) \), from which follows \( f(x - x_u + x_v) = f(x) \) and \( f(y + x_u - x_v) = f(y) \). \[ \square \]

The relationship among various properties for sets and functions is summarized as follows. Note that the claim (i) of Theorem 3.7 is already shown in [14, Remark 11].

**Theorem 3.7.** (i) For \( S \subseteq \mathbb{Z}^V \), we have

\[
(B-\text{EXC}) \quad \Rightarrow \quad (\text{Q-EXC})
\]

\[
(B-\text{EXC}_w) \quad \Rightarrow \quad (\text{Q-EXC}_w).
\]

(ii) For \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \), we have

\[
(M-\text{EXC}) \quad \Rightarrow \quad (\text{SSQM}) \quad \Rightarrow \quad (\text{QM})
\]

\[
(M-\text{EXC}_w) \quad \Rightarrow \quad (\text{SSQM}_w) \quad \Rightarrow \quad (\text{QM}_w).
\]

### 3.2 Level Sets

We show various properties for level sets of quasi M-convex functions.

The following two theorems claim that level sets of quasi M-convex functions have quasi M-convexity. Furthermore, the weaker version of quasi M-convexity (QM\(_w\)) for functions can be characterized by quasi M-convexity (Q-EXC\(_w\)) of level sets.

**Lemma 3.8 ([14]).** Let \( B \subseteq \mathbb{Z}^V \).

(i) If \( B \) satisfies (Q-EXC\(_w\)), then \( x(V) = y(V) \) for all \( x, y \in \text{dom} f \).

(ii) (Q-EXC\(_w\)) \iff (Q-EXC\(_w^+\)):

\[
(\text{Q-EXC}_{w^+}) \forall x, y \in B, x \neq y, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y) \text{ : } x - x_u + x_v \in B.
\]

**Theorem 3.9.** A function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) satisfies (QM\(_w\)) if and only if the level set \( L(f, \alpha) \) satisfies (Q-EXC\(_w\)) for all \( \alpha \in \mathbb{R} \cup \{+\infty\} \). In particular, if \( f \) satisfies (QM\(_w\)), then \( \text{dom} f \) and \( \arg \min f \) satisfy (Q-EXC\(_w\)).

**Proof.** \([\Rightarrow]\) Let \( \alpha \in \mathbb{R} \cup \{+\infty\} \), and \( x, y \in L(f, \alpha) \) be vectors with \( x \neq y \). Applying (QM\(_w\)) to \( z \) and \( y \), we have \( \Delta f(z; u, v) \leq 0 \) or \( \Delta f(y; u, v) \leq 0 \) for some \( u \in \text{supp}^+(x - y) \) and \( v \in \text{supp}^-(x - y) \). Therefore, we have \( x - x_u + x_v \in L(f, \alpha) \) or \( y + x_u - x_v \in L(f, \alpha) \).

\([\Leftarrow]\) Let \( x, y \in \text{dom} f \), and we may assume that \( f(x) \geq f(y) \). By Lemma 3.8 (ii), the level set \( L(f, f(x)) \) satisfies (Q-EXC\(_w^+\)), from which follows \( x - x_u + x_v \in L(f, f(x)) \) for some \( u \in \text{supp}^+(x - y) \) and \( v \in \text{supp}^-(x - y) \). This implies that \( f(x - x_u + x_v) \leq f(x) \) which yields (QM\(_w\)) for \( L(f, f(x)) \).

**Theorem 3.10.** Let \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) be a function satisfying (QM). Then, the level set \( L(f, \alpha) \) satisfies (Q-EXC) for all \( \alpha \in \mathbb{R} \cup \{+\infty\} \). In particular, \( \text{dom} f \) and \( \arg \min f \) satisfy (Q-EXC).

**Proof.** The proof is similar to that for the "only if" part of Theorem 3.9.

**Theorem 3.11.** Suppose \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) satisfies (SSQM\(_w\)). Then \( \arg \min f \) satisfies (B-EXC), i.e., \( \arg \min f \) is an M-convex set if it is nonempty.
An M-convex function can be characterized also by quasi M-convexity for level sets of a function perturbed by linear functions. For any function \( f : Z^V \to R \cup \{+\infty\} \) and any vector \( p \in R^V \), the function \( f[p] : Z^V \to R \cup \{+\infty\} \) is given by

\[
\begin{align*}
f[p](x) &= f(x) + \sum_{v \in V} p(v)x(v) \quad (x \in Z^V).
\end{align*}
\]

**Theorem 3.12** ([14, Th. 1]). A function \( f : Z^V \to R \cup \{+\infty\} \) satisfies (M-EXC) if and only if \( L(f[p], \alpha) \) satisfies (Q-EXC) for all \( p \in R^V \) and \( \alpha \in R \cup \{+\infty\} \).

Combining Theorems 3.9 and 3.12, we have the following property.

**Corollary 3.13.** A function \( f : Z^V \to R \cup \{+\infty\} \) satisfies (M-EXC) if and only if \( f[p] \) satisfies (QM) for all \( p \in R^V \).

### 3.3 Operations

The classes of (semistrictly) quasi M-convex functions are closed under several fundamental operations.

Let \( f : Z^V \to R \cup \{+\infty\} \). For any subset \( U \subseteq V \), define \( f_U : Z^U \to R \cup \{+\infty\} \) by

\[
f_U(y) = f(y, 0_{V \setminus U}) \quad (y \in Z^U),
\]

where \( 0_{V \setminus U} \in R^{V \setminus U} \) denotes the vector with each component equal to zero. For any functions \( a : V \to Z \cup \{-\infty\} \) and \( b : V \to Z \cup \{+\infty\} \), define \( f^b_a : Z^V \to R \cup \{+\infty\} \) by

\[
f^b_a(x) = \begin{cases} f(x) & (a \leq x \leq b), \\ +\infty & \text{(otherwise).} \end{cases}
\]

**Theorem 3.14.** Let \((\ast \text{QM})_*\) denote one of (QM), (QM\(_w\)), (SSQM), or (SSQM\(_w\)), and let \( f : Z^V \to R \cup \{+\infty\} \) be a function with \((\ast \text{QM})_*\) as functions in \( x \).

(i) For any \( a \in Z^V \) and \( \nu > 0 \), the functions \( \nu \cdot f(a - x) \) and \( \nu \cdot f(a + x) \) satisfy \((\ast \text{QM})_*\) as functions in \( x \).

(ii) For any \( U \subseteq V \), the function \( f_U : Z^U \to R \cup \{+\infty\} \) satisfies \((\ast \text{QM})_*\).

(iii) For any \( a : V \to Z \cup \{-\infty\} \) and \( b : V \to Z \cup \{+\infty\} \) with \( a \leq b \), the function \( f^b_a : Z^V \to R \cup \{+\infty\} \) satisfies \((\ast \text{QM})_*\).

(iv) Let \( f_i : Z^{V_i} \to R_{++} \cup \{+\infty\} \) (\( i = 1, 2 \)) be functions with \((\ast \text{QM})_*\). Then, the function \( f : Z^{V_1} \times Z^{V_2} \to R_{++} \cup \{+\infty\} \) defined by

\[
f(x_1, x_2) = f_1(x_1)f_2(x_2) \quad ((x_1, x_2) \in Z^{V_1} \times Z^{V_2})
\]
satisfies \((\ast \text{QM})_*\).

**Proof.** We prove (iv) only. We consider the case when \((\ast \text{QM})_*\) is (SSQM). Let \( x = (x_1, x_2), y = (y_1, y_2) \in dom f_1 \times dom f_2 \), and let \( u \in \text{supp}^+(x - y) \), where \( u \in \text{supp}^+(x_1 - y_1) \) w.l.o.g. Then, there exists \( v \in \text{supp}^-(x_1 - y_1) \) such that

\[
\Delta f_1(x_1; v, u) \geq 0 \Longrightarrow \Delta f_1(y_1; u, v) \leq 0, \quad \Delta f_1(y_1; u, v) \geq 0 \Longrightarrow \Delta f_1(x_1; v, u) \leq 0.
\]

This implies that

\[
\Delta f_1(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \quad \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.
\]

Hence, (SSQM) holds for \( f \).

**Remark 3.15.** The class of (semistrictly) quasi M-convex functions is not closed under addition; in particular, it is not closed under addition of a linear function.

**Theorem 3.16.** For \( f : Z^V \to R \cup \{+\infty\} \) and \( \varphi : R \to R \cup \{+\infty\} \), define \( \tilde{f} : Z^V \to R \cup \{+\infty\} \) by (3.2).

(i) If \( f \) satisfies (QM) (resp. (QM\(_w\))) and \( \varphi \) is nondecreasing, then \( \tilde{f} \) satisfies (QM) (resp. (QM\(_w\))).

(ii) If \( f \) satisfies (SSQM) (resp. (SSQM\(_w\))) and \( \varphi \) is strictly increasing, then \( \tilde{f} \) satisfies (SSQM) (resp. (SSQM\(_w\))).

**Remark 3.17.** A quasi M-convex function \( \tilde{f} : Z^V \to R \cup \{+\infty\} \) is not necessarily given as the form (3.2). As an example, let \( \tilde{f} : Z^3 \to R \cup \{+\infty\} \) be a function given by

\[
\text{dom } \tilde{f} = \{(0, 0, 0), (1, 0, -1), (2, 0, -2), (2, 1, -3), (2, 2, -4)\}.
\]

\[
\tilde{f}(x_1, x_2, x_3) = -x_1 + x_2 \quad (x \in \text{dom } \tilde{f}).
\]

Although \( \tilde{f} \) satisfies (SSQM), it cannot be represented in the form (3.2) with an M-convex function \( f : Z^3 \to R \cup \{+\infty\} \) and a nondecreasing function \( \varphi : R \to R \cup \{+\infty\} \).

**Theorem 3.18.** Let \( f : Z^V \to R \cup \{+\infty\} \) and \( g : Z^V \to R \cup \{-\infty\} \) be functions such that \( g(x) > 0 \) for all \( x \in \text{dom } f \). Suppose that the function \( f(-a \cdot g) \) satisfies (QM\(_w\)) for all \( \alpha \in R \cup \{+\infty\} \).

Then, the function \( r : Z^V \to R \cup \{+\infty\} \) given by

\[
r(x) = \begin{cases} f(x)/g(x) & (x \in \text{dom } f), \\ +\infty & (x \notin \text{dom } f), \end{cases}
\]

satisfies (QM\(_w\)).
Proof. Clear from Theorem 3.9.

Remark 3.19. The statement of Theorem 3.18 cannot be strengthened by replacing (QMw) with (QM), even if $f$ and $g$ are linear functions.

3.4 Characterization by Local Exchange Properties

An M-convex function is characterized by a localized version of the property (M-EXC):

\[(M-\text{EXC}-\text{loc}) \forall x, y \in \text{dom} f \text{ with } ||x - y||_1 = 4, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y) \text{ satisfying (2.2)}.\]

Theorem 3.20 ([9, Th. 3.1], [14, Th. 2]).

Let $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ be a function such that dom $f$ is a nonempty set with (Q-EXCw). Then, (M-EXC) $\iff$ (M-EXC-loc).

We show that (semistrict) quasi M-convexity can be characterized also by the localized version of (SSQM) and (QM).

\[(SSQMw-\text{loc}) \forall x, y \in \text{dom} f \text{ with } ||x - y||_1 = 4, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
   \begin{align*}
   (i) & \Delta f(x; v, u) \geq 0 \Rightarrow \Delta f(y; u, v) \leq 0, \\
   (ii) & \Delta f(y; u, v) \geq 0 \Rightarrow \Delta f(x; v, u) \leq 0.
   \end{align*}\]

\[(SSQMw_2-\text{loc}) \forall x, y \in \text{dom} f \text{ with } ||x - y||_1 = 4, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
   \begin{align*}
   (i) & \Delta f(x; v, u) \geq 0 \Rightarrow \Delta f(y; u, v) \leq 0, \\
   (ii) & \Delta f(y; u, v) \geq 0 \Rightarrow \Delta f(x; v, u) \leq 0.
   \end{align*}\]

Theorem 3.21. Let $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ be a function such that dom $f$ satisfies (Q-EXCw).

Then, 
(i) (SSQM) $\iff$ (SSQM-loc).
(ii) (SSQMw) $\iff$ (SSQMw-loc).


Remark 3.22. The localized version of (QM) does not characterize (QM) in general. Let $f : Z^2 \to Z \cup \{+\infty\}$ be a function such that

\[
\text{dom} f = \{(0, 0), (1, -1), (2, -2), (3, -3)\},
\]

\[
f(0, 0) = f(3, -3) = 0, f(1, -1) = f(2, -2) = 1.
\]

Then, dom $f$ satisfies (Q-EXC), and (QM) holds for any $x, y \in \text{dom} f$ with $||x - y||_1 = 4$. However, (QM) does not hold for $x = (0, 0)$ and $y = (3, -3).

4 Minimization of Quasi M-convex Functions

4.1 Theorems

Global minimality of quasi M-convex functions is characterized by local minimality.

Theorem 4.1. Let $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ and $x \in \text{dom} f$.

(i) Assume (QMw) for $f$. Then, $\Delta f(x; v, u) > 0$ \((\forall u, v \in V, u \neq v) \iff f(x) < f(y) (\forall y \in Z^V \setminus \{x\})
\)

(ii) Assume (SSQMw) for $f$. Then, $\Delta f(x; v, u) \geq 0$ \((\forall u, v \in V) \iff f(x) \leq f(y) (\forall y \in Z^V)
\)

Proof. We show the sufficiency of (ii) only. Assume, to the contrary, that there exists some $y \in \text{dom} f$ such that $f(y) < f(x)$. We further assume that $y$ minimizes the value $||y - x||_1$ among all such vectors. By (SSQMw), there exist some $v' \in \text{supp}^+(x - y)$ and $v' \in \text{supp}^-(x - y)$ such that if $\Delta f(x; v', u') \geq 0$ then $\Delta f(y; v', u') \leq 0$. Since $\Delta f(x; v', u') \geq 0$ holds true, we have $f(y + x_{v'} - x_{v'}) \leq f(y) \leq f(x)$ and $||y + x_{v'} - x_{v'} - x||_1 < ||y - x||_1$, a contradiction to the choice of $y$.

If $f$ satisfies (SSQM), then any vector in dom $f$ can be easily separated from some minimizer of $f$ (cf. [13, Th. 2.2, Cor. 2.3]).

Theorem 4.2. Let $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ be a function with (SSQM). Assume $\arg \min f \neq \emptyset$.

(i) For $x \in \text{dom} f$ and $v \in V$, let $u \in V$ satisfy

\[
f(x - x_u + x_v) = \min_{z \in V} f(x - x_s + x_v).
\]

Then, there exists $x_\ast \in \arg \min f$ with $x_\ast(u) \leq x(u) - 1 + x_\ast(u)$.

(ii) For $x \in \text{dom} f$ and $u \in V$, let $v \in V$ satisfy

\[
f(x - x_u + x_v) = \min_{z \in V} f(x - x_u + x_t).
\]

Then, there exists $x_\ast \in \arg \min f$ with $x_\ast(v) \geq x(v) - x_u(v) + 1$.

Proof. We prove (i) only. Put $z' = x - x_u + x_v$.

Assume, to the contrary, that there is no $z \in \arg \min f$ with $x(u) \leq x'(u)$. Let $x_\ast \in \arg \min f$ with minimum $x_\ast(u)$. Then, we have $x_\ast(u) > x'(u)$. By applying (SSQM) to $x_\ast$, $z'$, and $u$, we have some $w \in \text{supp}^-(x_\ast - z')$ such that if
$$\Delta f(x_{*}; w, u) > 0$$ then $$\Delta f(x'; w, u) < 0$$. Due to the choice of $$x_{*}$$, we have $$\Delta f(x_{*}; w, u) > 0$$. Hence, it holds that

$$f(x') > f(x' + \chi_{u} - \chi_{w}) = f(x - \chi_{w} + \chi_{v}),$$

a contradiction to the definition of $$u \in V$$. □

**Corollary 4.3.** Let $$f: \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$$ be a function with (SSQM). Also, let $$x \in \text{dom} f \setminus \arg \min f$$, and $$u, v \in V$$ satisfy

$$f(x - \chi_{u} + \chi_{v}) = \min_{x, s \in V} f(x - \chi_{s} + \chi_{v}).$$

Then, there exists $$x_{*} \in \arg \min f$$ with $$x_{*}(u) \leq x(u) - 1$$ and $$x_{*}(v) \geq x(v) + 1$$.

**Remark 4.4.** The statements in Theorem 4.2 do not hold even if $$f$$ satisfies the property (SSQMw) (and not (SSQM)). □

The following theorem shows that a global minimizer of a semistrictly quasi M-convex function exists in the neighborhood of a $$\Delta$$-local minimum. This generalizes [6, Th. 4.1].

**Theorem 4.5.** Let $$f: \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$$ be a function with (SSQM), and $$\Delta$$ be any positive integer. Suppose that $$x_{\Delta} \in \text{dom} f$$ satisfies

$$f(x_{\Delta}) \leq f(x_{\Delta} + \Delta(\chi_{v} - \chi_{w}))$$

for all $$u, v \in V$$. Then, there exists some $$x_{*} \in \arg \min f$$ such that

$$|x_{\Delta}(v) - x_{*}(v)| \leq (n - 1)(\Delta - 1) \quad (v \in V).$$

(4.1)

Proof. It suffices to show that for all $$\varepsilon > 0$$ there exists some $$x_{*} \in \text{dom} f$$ satisfying $$f(x_{*}) \leq \inf f + \varepsilon$$ and (4.1).

Let $$x_{*} \in \text{dom} f$$ satisfy $$f(x_{*}) \leq \inf f + \varepsilon$$, and suppose that $$x_{*}$$ minimizes the value $$||x_{*} - x_{\Delta}||_{1}$$ among all such vectors. In the following, we fix $$v \in V$$ and prove $$x_{\Delta}(v) - x_{*}(v) \leq (n - 1)(\Delta - 1)$$. The inequality $$x_{*}(v) - x_{\Delta}(v) \leq (n - 1)(\Delta - 1)$$ can be shown similarly.

We may assume $$x_{\Delta}(v) > x_{*}(v)$$. We first prove the following two claims.

**Claim 1** There exist $$w_{1}, w_{2}, \cdots, w_{k} \in V \setminus \{v\}$$ and $$y_{0}(= x_{\Delta}), y_{1}, \cdots, y_{k} \in \text{dom} f$$ with $$k = x_{\Delta}(v) - x_{*}(v)$$ such that

$$y_{i} = y_{i-1} - \chi_{v} + \chi_{w_{i}},$$

$$f(y_{i}) < f(y_{i-1}) \quad (i = 1, \cdots, k).$$

[Proof of Claim 1] We show the claim by induction on $$i$$. If $$i - 1 < k$$, then $$v \in \text{supp}^{+}(y_{i-1} - x_{*})$$. By (SSQM) applied to $$y_{i-1}, x_{*},$$ and $$v$$, we have some $$w_{i} \in \text{supp}^{-}(y_{i-1} - x_{*}) \subseteq \text{supp}^{-}(x_{\Delta} - x_{*}) \subseteq V \setminus \{v\}$$ such that if $$\Delta f(x_{*}; v, w_{i}) > 0$$ then $$\Delta f(y_{i-1}; v, w_{i}) < 0$$. By the choice of $$x_{*}$$, we have

$$\Delta f(x_{*}; v, w_{i}) > 0$$ since $$||x_{*} + \chi_{v} - \chi_{w_{i}}||_{1} < ||x_{*} - x_{\Delta}||_{1}$$. Therefore, $$f(y_{i}) = f(y_{i-1} - \chi_{v} + \chi_{w_{i}}) < f(y_{i-1})$$.

[End of Proof for Claim 1]

**Claim 2** For any $$w \in V \setminus \{v\}$$ with $$y_{k}(w) > x_{\Delta}(w)$$ and $$\alpha \in [0, y_{k}(w) - x_{\Delta}(w) - 1]$$, we have

$$f(x_{\Delta} - (\alpha + 1)(\chi_{v} - \chi_{w})) < f(x_{\Delta} - \alpha(\chi_{v} - \chi_{w})).$$

(4.2)

[Proof of Claim 2] We prove (4.2) by induction on $$\alpha$$. Put $$x' = x_{\Delta} - \alpha(\chi_{v} - \chi_{w})$$ for $$\alpha \in [0, y_{k}(w) - x_{\Delta}(w) - 1]$$, and suppose $$x' \in \text{dom} f$$.

Let $$j_{*} '(1 \leq j_{*} ' \leq k)$$ be the largest index such that $$y_{j_{*} '(w)} = x'(w)$$. Then, $$y_{j_{*} '(w)} = y_{k}(w) > x'(w)$$ and $$\text{supp}^{-}(y_{j_{*} '(w)} - x') = \{v\}$$. (SSQM) implies that if $$\Delta f(y_{j_{*} '(w)}; v, w) > 0$$ then $$\Delta f(x'; w, v) < 0$$. By Claim 1, we have $$\Delta f(y_{j_{*} '(w)}; v, w) > 0$$. Hence, (4.2) follows.

[End of Proof for Claim 2]

The $$\Delta$$-local minimality of $$x_{\Delta}$$ implies $$f(x_{\Delta} - \Delta(\chi_{v} - \chi_{w})) \geq f(x_{\Delta})$$, which combined with Claim 2, implies $$y_{k}(w) - x_{\Delta}(w) \leq \Delta - 1$$. Thus,

$$x_{\Delta}(v) - x_{*}(v) = x_{\Delta}(v) - y_{k}(v) \leq \sum_{w \in V \setminus \{v\}} \{y_{k}(w) - x_{\Delta}(w)\} \leq (n - 1)(\Delta - 1),$$

where the second equality is by Lemma 3.8 (i). □

**4.2 Algorithms**

Let $$f: \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$$ be a function such that $$\text{dom} f$$ is a nonempty bounded set, and put

$$L = \max\{||x - y||_{\infty} | x, y \in \text{dom} f\}.$$

Assume (SSQMw) for $$f$$. Then, Theorem 4.1 immediately leads to the following algorithm.

**Algorithm Descent**

Step 0: Let $$z$$ be any vector in dom $$f$$.

Step 1: If $$f(z) = \min_{x \in V} f(x - \chi_{s} + \chi_{t})$$ then stop.

[ $$z$$ is a minimizer of $$f$$ ]
Step 2: Find $u, v \in V$ with $f(x - \chi_u + \chi_v) < f(x)$.
Step 3: Set $z := x - \chi_u + \chi_v$. Go to Step 1. \hfill \Box

Algorithm **Descent** terminates in at most \(|\text{dom } f| \leq (L + 1)^{n-1}\) iterations since it generates distinct $x$ in each iteration.

To the end of this section we assume (SSQM) for $f$. Based on Theorem 4.5, we apply the scaling technique to Algorithm **Descent** to obtain a faster algorithm.

Algorithm **Scaling-Descent**
Step 0: Let $x$ be any vector in dom $f$. Put $\Delta := [L/4n], B := \text{dom } f$.
Step 1: \([\Delta\text{-scaling phase}]\)
  Step 1-1: If
  $$f(x) = \min \{f(x - \Delta(x_s - x_t)) \mid s, t \in V, x - \Delta(x_s - x_t) \in B\}$$
  then go to Step 2.
  Step 1-2: Find $u, v \in V$ with $x - \Delta(x_u - x_v) \in B$ satisfying $f(x - \Delta(x_u - x_v)) < f(x)$.
  Step 1-3: Set $z := x - \Delta(x_u - x_v)$. Go to Step 1-1.
Step 2: If $\Delta = 1$ then stop. \([z \text{ is a minimizer of } f]\)
Step 3: Put
  $$B := B \cap \{y \in ZV \mid |y(v) - x(v)| \leq (n - 1)(\Delta - 1) \ (v \in V)\}$$
  and $\Delta := \lceil \Delta/2 \rceil$. Go to Step 1. \hfill \Box

The number of scaling phases is $[\log L]$, and each scaling phase terminates in $(4n)^{n-1}$ iterations. Therefore, Algorithm **Scaling-Descent** runs in $(4n)^{n-1}[\log L]$ iterations.

We then propose another elaboration of Algorithm **Descent** by exploiting Corollary 4.3

Algorithm **Steepest-Descent**
Step 0: Let $x$ be any vector in dom $f$. Set $B := \text{dom } f$.
Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. \([z \text{ is a minimizer of } f]\)
Step 2: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying
  $$f(x - \chi_u + \chi_v) = \min \{f(x - \chi_s + \chi_t) \mid s, t \in V, x - \chi_s + \chi_t \in B\}. \tag{4.3}$$
Step 3: Set $z := x - \chi_u + \chi_v$ and
  $$B := B \cap \{y \in ZV \mid y(u) \leq z(u) - 1, y(v) \geq z(v) + 1\}. \tag{4.4}$$
Go to Step 1.

By Corollary 4.3, the set $B$ always contains a minimizer of $f$. Hence, Algorithm **Steepest-Descent** finds a minimizer of $f$. To analyze the number of iterations, we consider the value
  $$\sum_{w \in V} \{\max_{y \in B} y(w) - \min_{y \in B} y(w)\}.$$ 
This value is bounded by $nL$ and decreases at least by two in each iteration. Therefore, **Steepest-Descent** terminates in $O(nL)$ iterations. In particular, if $\text{dom } f \subseteq \{0,1\}^V$ then the number of iterations is $O(n^2)$.

It is shown in [13] that the minimization of an M-convex function can be done in polynomial time by the domain reduction method explained below. We show that the domain reduction method also works for the minimization of a function with (SSQM) if its effective domain is a bounded M-convex set.

Given a bounded M-convex set $B$, we define the set $N_B \subseteq B$ as follows. For $w \in V$, define
  $$l_B(w) = \min_{y \in B} y(w), \quad u_B(w) = \max_{y \in B} y(w),$$
  $$l'_B(w) = \left[\frac{1}{n}(1 - \frac{1}{n})l_B(w) + \frac{1}{n}u_B(w)\right],$$
  $$u'_B(w) = \left[\frac{1}{n}l_B(w) + (1 - \frac{1}{n})u_B(w)\right].$$
Then, $N_B$ is defined as
  $$N_B = \{y \in B \mid l'_B \leq y \leq u'_B\}.$$  

**Theorem 4.6 ([13, Th. 2.4]).** $N_B$ is a (nonempty) M-convex set.

The next algorithm maintains a set $B (\subseteq \text{dom } f)$ which is an M-convex set containing a minimizer of $f$. It reduces $B$ iteratively by exploiting Corollary 4.3 and finally finds a minimizer.

Algorithm **Domain-Reduction**
Step 0: Set $B := \text{dom } f$.
Step 1: Find a vector $z \in N_B$.
Step 2: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. \([z \text{ is a minimizer of } f]\)
Step 3: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying (4.3).
Step 4: Set $B$ by (4.4). Go to Step 1. \hfill \Box
We analyze the number of iterations of **Domain Reduction**. Denote by \( B_i \) the set \( B \) in the \( i \)-th iteration, and let \( l_i(w) = l_{B_i}(w) \), \( u_i(w) = u_{B_i}(w) \) (\( w \in V \)). It is clear that \( u_i(w) - l_i(w) \) is nonincreasing w.r.t. \( i \). Furthermore, we have the following property:

**Lemma 4.7 ([13, Lemma 3.1]).**

\[
    u_{i+1}(w) - l_{i+1}(w) < (1 - 1/n) \{ u_i(w) - l_i(w) \}
\]

for \( w \in \{ u, v \} \), where \( u, v \in V \) are the elements found in Step 3.

This lemma implies that Algorithm **Domain Reduction** terminates in \( O(n^2 \log L) \) iterations.

We then consider the time complexity of each step. Steps 2, 3, and 4 can be done in \( O(n^2) \) time. In Step 1, we use the exchange capacity to compute the values \( l_B(w) \) and \( u_B(w) \) and to find a vector in \( N_B \). For any \( w \in V \), the values \( l_B(w) \) and \( u_B(w) \) can be computed by evaluating the exchange capacity at most \( n \) times, provided that a vector in \( B \) is given [4, Th. 3.27]. A vector in \( N_B \) can be found by evaluating the exchange capacity at most \( n^2 \) times, provided that a vector in \( B \) is given [13, Th. 2.5]. The exchange capacity can be computed in \( O(\log L) \) time by binary search. Hence, Step 1 requires \( O(n^2 \log L) \) time.

**Theorem 4.8.** Suppose that \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{ +\infty \} \) satisfies (SSQM) and that \( \text{dom} \ f \) is a bounded \( M \)-convex set. If a vector in \( \text{dom} \ f \) is given, Algorithm **Domain Reduction** finds a minimizer of \( f \) in \( O(n^4 (\log L)^2) \) time.

**References**


