数理解析研究所講究録
1185 巻 2001 年 63-71

An algebraic approach to matching problems

James F. Geelen

Department of Combinatorics & Optimization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
jfgeelen@math.uwaterloo.ca

Abstract: In Tutte's seminal paper on matching, he associates a skew–symmetric matrix with a graph; this matrix is now known as the *Tutte matrix.* The rank of the Tutte matrix is exactly twice the size of a maximum matching in the graph. This formulation easily leads to an efficient randomized algorithm for matching. The Tutte matrix is also useful in obtaining a min–max theorem and an efficient deterministic algorithm. We review these results and look at similar formulations of other problems; namely, linear matroid intersection, linear matroid parity, path matching, and matching forests.

Keywords: matching, matroid intersection, Tutte matrix, mixed matrix

1 Introduction

Two problems that play a fundamental role in combinatorial optimization are the maximum cardinality matching problem and the matroid intersection problem. This paper surveys matrix–rank formulations of these and related problems. We begin by considering the size of a maximum cardinality matching in a bipartite graph, which is a special case of both of the aforementioned problems.

Let $G=(V,E)$ be a bipartite graph with bipartition (V_r, V_c), and let $(z_e : e\in E)$ be algebraically independent commuting indeterminates. We let $\nu(G)$ denote the size of a maximum cardinality matching of G. Now, define a V_r by V_c matrix X, such that $X_{i,j}=z_e$ if $ij=e\in E$ and $X_{i,j}=0$ otherwise. We call X the *bipartite–matching matrix* of G. If $|V_r|=|V_c|$, then, by considering the determinate expansion for X, it is straightforward to see that X is nonsingular if and only if G has a perfect matching. More generally, $\nu(G)=\operatorname{rank} X$. Thus, we have reformulated the bipartite matching problem as a matrix–rank problem. This formulation is not a panacea, as the matrix in question has indeterminate entries, so it is non–trivial to compute its rank. However, we shall see that such formulations easily provide efficient randomized algorithms, and may be used to obtain min–max theorems and efficient deterministic algorithms.

We now progress to matching in general graphs; for this we introduce "Pfaffians". Let K be a V by V skew–symmetric matrix, where $V=\{1,\ldots,n\}$. The determinant of K is the square of its *Pfaffian.* The Pfaffian has an expansion somewhat like the permutation expansion of the determinant. Let $G(K)$ denote the graph (V,E') where $E'=\{ij : K_{ij}\neq 0\}$, and let $\mathcal{M}_K$ denote the set of perfect matchings of $G(K)$. The *Pfaffian* of K, denoted *Pf*(K), is defined as follows:

$$\mathrm{Pf}(K) := \sum_{M\in\mathcal{M}_K} \sigma_M \prod_{\substack{uv\in M\\ u<v}} K_{uv}, \qquad (1)$$

where σ_M takes the value 1 or -1 as appropriate; see Godsil [7]. For $X\subseteq V$, we will denote $K[X,X]$ by $K[X]$.

Now, let $G=(V,E)$ be a simple graph, and let $(z_e : e\in E)$ be algebraically independent commuting indeterminates. We define a V by V skew–symmetric matrix T, called the *Tutte matrix* of G, such that $T_{ij}=\pm z_e$ if $ij=e\in E$, and $T_{ij}=0$ otherwise. By (1), it is immediate that T is nonsingular if and only if G admits a perfect matching. In fact, $\operatorname{rank} T=2\nu(G)$. The Tutte matrix was introduced by Tutte, in 1947, in his seminal paper on matching [20].

Finally we consider matroid intersection. (Those readers not familiar with matroids are directed to Cook, Cunningham, Pulleyblank, and

Schrijver [2] for an excellent introduction.) Matroids are a natural abstraction of linear independence. Basically, a matroid $M = (V, \mathcal{I})$ is determined by a finite *ground set* V and a collection $\mathcal{I}$ of *independent sets*, that satisfy certain axioms. If Q is a V_r by V_c matrix, then $M(Q)$ denotes the matroid on ground set V_c where $A \subseteq V_c$ is independent if the columns of Q indexed by A are linearly independent. We call $M(Q)$ a *linear matroid.* Similarly, we can define a matroid on the rows of Q; this matroid is denoted by $M^r(Q)$. (We often use the following elementary fact: *if X_r is a basis of $M^r(Q)$ and X_c is a basis of $M(Q)$ then $Q[X_r, X_c]$ is nonsingular.*)

Let M_1 and M_2 be matroids on a common ground set V. The *intersection problem* for M_1 and M_2 is the problem of finding a maximum cardinality common independent set of M_1 and M_2. Now consider the intersection problem for linear matroids $M_1 := M(Q_1)$ and $M_2 := M(Q_2)$ on a common ground set $V := \{1, \ldots, n\}$. Let $(z_1, \ldots, z_n)$ be algebraically independent commuting indeterminates, and consider the following matrix:

$$Z := \left(\begin{array}{c|ccc} 0 & & Q_1 & \\ \hline & z_1 & & \\ Q_2^t & & \ddots & \\ & & & z_n \end{array} \right).$$

Murota [18] proved that the size of a maximum common independent set of M_1 and M_2 is rank $Z - n$. Thus we have a matrix–rank formulation of the linear matroid intersection problem.

If X is a V_r by V_c bipartite–matching matrix and Q is a V_r by V_c matrix over the rationals, then we call $Q + X$ a *mixed matrix.* The matrix Z, above, is a mixed matrix. Thus, the problem of determining the rank of a mixed matrix contains the linear matroid intersection problem. Murota [18] studies mixed matrices extensively, and shows that computing their rank is in fact equivalent to linear matroid intersection. In particular, we can compute the rank of $Q + X$ using Edmonds' [5] matroid intersection algorithm.

Randomized algorithms

The matrix–rank formulations do not immediately provide efficient algorithms, as we cannot efficiently perform basic operations on a matrix with indeterminate entries. For example, the determinant of a mixed matrix is a polynomial that may have exponentially many terms. Lovász [15] overcomes this problem by replacing the indeterminates with rational values; of course the rank may decrease, but, fortunately, this is unlikely. If K is a matrix with indeterminate entries, then an *evaluation* of K is any matrix obtained from K by replacing the indeterminates with rational values.

Theorem 1.1 *Let $Q + X$ be an n by m mixed matrix and let $\tilde{X}$ be an evaluation of X with entries chosen independently and at random from $\{1, \ldots, m+n\}$, then rank $(Q+X) =$ rank $(Q+\tilde{X})$ with probability at least $\frac{1}{2}$.*

Theorem 1.2 *Let T be an n by n Tutte matrix, and let $\tilde{T}$ be an evaluation of T with entries chosen independently and at random from $\{1, \ldots, n\}$, then rank $T =$ rank $\tilde{T}$ with probability at least $\frac{1}{2}$.*

We prove Theorems 1.1 and 1.2 in Section 2. These theorems provide efficient randomized algorithms for computing the rank of a mixed matrix and determining the size of a maximum matching. The reader may not be comfortable with a one in two chance of failure, but the odds improve significantly with repeated trials. Among n independent evaluations, one has the correct rank with probability at least $1 - 1/2^n$.

Let T be the Tutte matrix of a graph G, and let $\tilde{T}$ be an evaluation of T. We know that $\nu(G) \geq (\text{rank } \tilde{T})/2$, however, knowing the evaluation $\tilde{T}$ does not provide a more efficient method of finding a matching of size $(\text{rank } \tilde{T})/2$. Nevertheless, Cheriyan [1] shows that random evaluations of the Tutte matrix contain a lot of information about matching structure.

Min–max theorems

Consider a V_r by V_c mixed matrix $Q + X$. If $Y_r \subseteq V_r$, $Y_c \subseteq V_c$ and $X[Y_r, Y_c] = 0$ then

$$\text{rank } Q+X \leq \text{rank } Q[Y_r, Y_c] + |V_r - Y_r| + |V_c - Y_c|.$$

Since $Q[Y_r, Y_c]$ is a rational matrix, we can easily evaluate the right side of the inequality above. Murota [17] proved that, for an appropriate choice of Y_r and Y_c, this inequality is in fact attained with equality; this generalized the result of

Hartfiel and Loewy [12], who characterized singular mixed matrices.

Theorem 1.3 *Let $Q + X$ be a V_r by V_c mixed matrix. If $Y_r \subseteq V_r$, $Y_c \subseteq V_c$ and $X[Y_r, Y_c] = 0$ then*

$$\text{rank } Q+X \leq \text{rank } Q[Y_r, Y_c]+|V_r-Y_r|+|V_c-Y_c|.$$

Moreover, there exists $Y_r^* \subseteq V_r$ and $Y_c^* \subseteq V_c$ such that $X[Y_r^*, Y_c^*] = 0$ and

$$\text{rank } Q+X \leq \text{rank } Q[Y_r^*, Y_c^*]+|V_r-Y_r^*|+|V_c-Y_c^*|.$$

For a bipartite–matching matrix X, the above theorem is Kőnig's Theorem: *The rank of X is equal to the minimum number of lines required to cover all of the nonzero entries in X.*

We now consider a canonical version of Theorem 1.3 that appears in [9]; we require the following definitions. Let Q be a V_r by V_c matrix. An element $i \in V_r$ is an *avoidable row* if rank $Q[V_r - \{i\}, V_c]$ = rank Q. Thus, i is an avoidable row if and only if i is not a coloop of $M^r(Q)$. We define *avoidable columns* similarly. Now let $D^r(Q)$ and $D^c(Q)$ denote the set of avoidable rows and columns respectively. For $i \in V_c - D^c(Q)$, it is straightforward to see that $D^r(Q) \subseteq D^r(Q[V_r, V_c - \{i\}])$. We let $A^c(Q)$ denote the set of elements $i \in V_c - D^c(Q)$ such that $D^r(Q) = D^r(Q[V_r, V_c - \{i\}])$, and let $A^r(Q) = A^c(Q^t)$. These definitions were motivated by the Dulmage–Mendelsohn decomposition of a bipartite graph [4].

Theorem 1.4 *Let $Q + X$ be a V_r by V_c mixed matrix. If $Y_r^* = D^r(Q + T)$ and $Y_c^* = V_c - A_c$ then $X[Y_r^*, Y_c^*] = 0$ and*

$$\text{rank } Q+X \leq \text{rank } Q[Y_r^*, Y_c^*]+|V_r-Y_r^*|+|V_c-Y_c^*|.$$

When applied to a bipartite–matching matrix, the above theorem implies Dulmage and Mendelsohn's decomposition theorem [4].

Since Tutte [20] introduced the Tutte matrix to prove his matching theorem, it should be little surprise that the Tutte matrix helps in proving the following min–max theorem.

Theorem 1.5 (Tutte–Berge Theorem) *For any graph $G = (V, E)$,*

$$2\nu(G) = \min_{A \subseteq V} |V| - (\text{odd}(G - A) - |A|).$$

(Here, odd(H) denotes the number of components of H that have an odd number of vertices.)

Let T be the Tutte matrix of a graph G. The matroid $M(T)$ is called the *matching matroid* of G. A subset X of V is a *matchable set* of G if $G[X]$, the subgraph of G induced by X, has a perfect matching. Note that, if X is a matchable set, then X is independent in $M(T)$. The converse need not hold, since, unless G is trivial, $M(T)$ has independent sets of odd cardinality which cannot be matchable. However, consider a basis X of $M(T)$. By skew–symmetry, X is also a basis of $M^r(T)$. Therefore, $T[X]$ is nonsingular, and, hence, X is a matchable set. Therefore, the bases of $M(T)$ are the maximum cardinality matchable sets of G.

Let $D(G)$ denote the set of vertices $v \in V$ such that $\nu(G - v) = \nu(G)$, and let $A(G)$ denote the set of vertices in $V - D(G)$ that have a neighbour in $D(G)$. It is easy to see that $D^r(T) = D(G)$, however, we shall see that more surprising fact that $A^r(T) = A(G)$. The following canonical version of the Tutte–Berge Theorem is tantamount to the Edmonds–Gallai Decomposition Theorem; see [16].

Theorem 1.6 *For any graph $G = (V, E)$,*

$$2\nu(G) = |V| - (\text{odd}(G - A(G)) - |A(G)|).$$

Moreover, $D(G) = D(G - A(G))$.

Theorems 1.4 and 1.6 shall be proved in Section 3.

Deterministic algorithms

Edmonds' has efficient augmenting path algorithms for both the matching problem [6] and the matroid intersection problem [5]. Therefore, the ranks of Tutte matrices and mixed matrices can be computed efficiently. We describe a different approach, based on evaluations.

Let $Q + \tilde{X}$ be an evaluation of a mixed matrix $Q+X$. For an indeterminate z in X, we denote by $\tilde{X}(z \to a)$ the evaluation of X obtained from $\tilde{X}$ be replacing the old value of z with a; we call this *perturbation.* A heuristic algorithm for finding a good evaluation is to make perturbations if doing so increases the rank. Unfortunately, this method is not guaranteed to produce an evaluation with the same rank as $Q+X$; see [9]. We overcome this problem by considering a more refined ordering on matrices than simply comparing rank.

Let Q_1 and Q_2 be V_r by V_c matrices. We write $Q_1 \succeq Q_2$ if rank $Q_1 >$ rank Q_2, or rank $Q_1 =$ rank Q_2 and $D^r(Q_2) \subseteq D^r(Q_1)$. Similarly, we write $Q_1 \approx Q_2$ if rank $Q_1 =$ rank Q_2 and $D^r(Q_1) = D^r(Q_2)$. If $Q_1 \succeq Q_2$ but $Q_1 \not\approx Q_2$ then we write $Q_1 \succ Q_2$. This gives a quasi-ordering of matrices; if $Q_1 \succ Q_2$ then we say that Q_1 is *more independent* than Q_2.

Another heuristic algorithm for finding a good evaluation is to make perturbations if doing so increases the independence. This algorithm is guaranteed to produce an evaluation with the same rank as $Q + X$; see [9].

Theorem 1.7 *If $Q + \tilde{X}$ is an evaluation of a V_r by V_c mixed matrix $Q + X$, then either rank $Q + \tilde{X} =$ rank $Q+X$, or there exists an indeterminate z in X and $a \in \{1, \ldots, |V_r| + 1\}$ such that $Q + \tilde{X}(z \to a) \succ Q + \tilde{X}$.*

This theorem clearly provides a polynomial–time deterministic algorithm for computing the rank of a mixed matrix. The algorithm is not particularly efficient if implemented naively. However, the are many ways to improve the running time. In fact, it is possible to perform an iteration in the same order of time that it takes to perform a matrix inversion. (This is not straightforward.) The charitable reader will also note that, if the initial evaluation is chosen at random, then it is likely to have near–optimal independence.

We can also compute the rank of a Tutte matrix with a similar algorithm; see [8].

Theorem 1.8 *Let $\tilde{T}$ be an evaluation of a V by V Tutte matrix T, then either rank $\tilde{T} =$ rank T, or there exists an indeterminate z in T and $a \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z \to a) \succ \tilde{T}$.*

We prove Theorems 1.7 and 1.8 in Section 4. The proof of Theorem 1.7 in [8] is quite technical; we present a simpler proof using the techniques of [9].

Path–matching

Above we described matrix–rank formulations for matching and linear matroid intersection, and indicated why these formulations are useful. We conclude the introduction by briefly describing matrix–rank formulations for three other combinatorial problems.

Let T be the Tutte matrix of a graph $G = (V, E)$. Given $A, B \subseteq V$, consider the problem of computing the rank of $T[A, B]$. Cunningham and Geelen [3], show that rank $T[A, B]$ can be computed by a slight variation on the algorithm given by Theorem 1.8. The problem of computing rank $T[A, B]$ has a graphical interpretation; see [3]. We will only consider the special case of deciding whether $T[A, B]$ is nonsingular when $|A| = |B|$. A *perfect path–matching* consists of a set of $|A - B|$ vertex disjoint paths from $A - B$ to $B - A$ and a perfect matching the subgraph induced by the vertices that are not covered by the paths. Cunningham and Geelen [3] show that $T[A, B]$ is nonsingular if and only if $G[A \cup B]$ has a perfect path matching.

Matching forests

A *mixed graph* is a graph with both directed and undirected edges. We write $G := (V, E, A)$ for a mixed graph with vertex set V, undirected edge set E, and directed edge set A. For $xy \in E$ we call x and y *heads* of xy, and for $xy \in A$ we call y a *head* of xy. A *matching forest* is a subset M of $E \cup A$ such that each vertex is the head of at most one edge in M, and $M \cap A$ does not contain a directed cycle. A vertex v is *covered* by a matching forest M if v is a head of some edge in M. Let $\mu(G)$ denote the maximum number of vertices that can be covered by a a matching forest of G. If G is undirected then $\mu(G) = 2\nu(G)$. If G is directed then $\mu(G)$ is the maximum size of a directed forest. Matching forests were introduced by Giles [11] who found an alternating path algorithm for computing $\mu(G)$.

Let $H = (V, A)$ be a directed graph, and let $(z_e : e \in A)$ be algebraically independent commuting indeterminates. Now define a V by V matrix B where, for $x \neq y$, $B_{xy} = z_e$ if $xy = e \in A$ and $B_{xy} = 0$ otherwise. The diagonal of B is defined so that the row–sums of B are all zero. We call B the *branchings matrix* of H.

Now, let $G = (V, E, A)$ be a mixed graph, let T be the Tutte matrix of (V, E), and let B be the branchings matrix of (V, A). Webb [21] proved that $\mu(G) =$ rank $(T+B)$. Using this formulation Webb also proved a min–max theorem and obtained an efficient deterministic algorithm. The algorithm is more complicated than the algorithm given by Theorem 1.8 for matching. In particular, Webb's algorithm may require perturbing two variables at once (one for a directed edge and another for an undirected edge), and uses a more refined notion of independence for evaluations.

Linear matroid parity

Consider the following problem.

Matroid parity problem *Given a matroid M on the ground set V, and a partition $\Pi = (\pi_1, \ldots, \pi_m)$ of V into pairs, find a maximum size collection $(\pi_{i_1}, \ldots, \pi_{i_k})$ of these pairs such that $\pi_{i_1} \cup \cdots \cup \pi_{i_k}$ is independent in M.*

Lovász showed that the matroid parity problem is intractable (using the usual oracle based approach to matroid algorithms) and NP–hard [16]. More surprisingly, Lovász showed that the matroid parity problem can be solved efficiently if M is linear. Let $\nu_\Pi(M)$ denote the maximum number of pairs in Π whose union is independent in M.

Let Q be a matrix with rows and columns indexed by R and V respectively, and let Π be a partition of V into pairs. Now let T be the Tutte matrix of the graph with vertex set $R \cup V$ and edge set Π, and let

$$K := \begin{array}{c c} & \begin{array}{cc} R & V \end{array} \\ \begin{array}{c} R \\ V \end{array} & \begin{pmatrix} 0 & Q \\ -Q^t & 0 \end{pmatrix} \end{array}.$$

Then, $2\nu_\Pi(M(A)) = \text{rank } (T+K) - |V|$; see [10].

More generally, if T is a V by V Tutte matrix and K is a V by V rational skew–symmetric matrix, then we call $T + K$ a *mixed skew–symmetric matrix*. It seems likely that some analogue of Theorem 1.8 should hold for a mixed skew–symmetric matrix, but this problem remains open. Nevertheless, there is a min–max formula for determining the rank of a mixed skew–symmetric matrix; see [10].

2 Randomized algorithms

In this section we prove Theorems 1.1 and 1.2, for which we require the following lemma which was discovered independently by Zippel [22] and Schwartz [19].

Lemma 2.1 *Let $p(x_1, \ldots, x_k)$ be a nonzero polynomial of degree at most d, and let S be a finite subset of $\mathbf{R}$. If $(\hat{x}_1, \ldots, \hat{x}_k)$ is a random element of S^k, then $p(\hat{x}_1, \ldots, \hat{x}_k) \neq 0$ with probability at least $1 - \frac{d}{|S|}$.*

Proof. The proof is by induction on k. If $k = 1$, then the the result follows from the fact that a nonzero single–variable polynomial of degree d has at most d roots. Suppose that $k > 1$, and that the result holds for polynomials with fewer than k variables. Collecting $p(x)$ in powers of x_k we get

$$p(x_1, \ldots, x_k) = \sum_{i=0}^{d} p_i(x_1, \ldots, x_{k-1}) x_k^i.$$

Since $p(x)$ is not identically zero, there exists $j \in \{0, \ldots, d\}$ such that $p_j(x_1, \ldots, x_{k-1})$ is not identically zero. Choose j maximal such that $p_j(x_1, \ldots, x_{k-1})$ is not identically zero. Thus, by the induction hypothesis, $p_j(\hat{x}_1, \ldots, \hat{x}_{k-1}) \neq 0$ with probability at least $1 - \frac{d-j}{|S|}$. Now, the probability that $p(\hat{x}_1, \ldots, \hat{x}_k) \neq 0$ given that $p_j(\hat{x}_1, \ldots, \hat{x}_{k-1}) \neq 0$ is at least $1 - \frac{j}{|S|}$. Therefore, $p(\hat{x}_1, \ldots, \hat{x}_k) \neq 0$ with probability at least

$$\begin{aligned} (1 - \frac{d-j}{|S|})(1 - \frac{j}{|S|}) &= 1 - \frac{d}{|S|} + \frac{j(d-j)}{|S|^2} \\ &\geq 1 - \frac{d}{|S|}, \end{aligned}$$

as required. ∎

Proof of Theorem 1.1. Let $(Q+X)[A, B]$ be a maximal square nonsingular submatrix of $Q+X$. Then, $\det(Q+X)[A, B]$ is a polynomial of degree at most rank $(Q+X) \leq \frac{m+n}{2}$. Therefore, by Lemma 2.1, $(Q + \tilde{X})[A, B]$ is nonsingular with probability at least 1/2. Thus, rank $(Q + \tilde{X}) =$ rank $(Q + X)$ with probability at least 1/2, as claimed. ∎

Similarly, by considering the Pfaffian of a skew–symmetric matrix we easily obtain Theorem 1.2.

3 Min–max theorems

In this section we prove Theorems 1.4 and 1.6, for which, we require the following lemma.

Lemma 3.1 *Let Q be a V_r by V_c matrix, let $X_r = D^r(Q)$, and let $X_c = V_c - A^c(Q)$. Then*

(1) rank $Q[X_r, V_c] = \text{rank } Q - |V_r - X_r|$, $D^r(Q[X_r, V_c]) = X_r$, and $D^c(Q[X_r, V_c]) = X_c$, and

(2) rank $Q[X_r, X_c] = \text{rank } Q - |V_r - X_r| - |V_c - X_c|$, $D^r(Q[X_r, X_c]) = X_r$, and $D^c(Q[X_r, X_c]) = X_c$.

Proof. Since $V_r - X_r$ is the set of coloops of $M^r(Q)$, we see that, for any basis B of $M^r(Q[X_r, V_c])$ the set $B \cup (V_r - X_r)$ is a basis of $M^r(Q)$. Conversely, for any basis B' of $M^r(Q)$ the set $B' \cap X_r$ is a basis $M^r(Q[X_r, V_c])$. It follows that rank $Q[X_r, V_c] = \text{rank } Q - (|V_r| - |X_r|)$ and $D^r(Q[X_r, V_c]) = X_r$.

Now, it is straightforward to see that $D^c(Q[X_r, V_c])$ contains $D^c(Q)$. Now consider $i \in V^c - D^c(Q)$. Since i is not a coloop of $M(Q)$, rank $Q[V_r, V_c - \{i\}] = \text{rank } Q - 1$. Therefore, rank $Q[X_r, V_c - \{i\}] \geq \text{rank } Q - |V_c - X_c| - 1 = \text{rank } Q[X_r, V_c] - 1$, with equality if and only if the elements of $V_c - X_c$ are coloops of $M^r(Q[V_r, V_c - \{i\}])$. That is, $i \in V_c - D^c(Q)$ is a coloop of $M^r(Q[V_r, V_c - \{i\}])$ if and only if $i \in A^c(Q)$. This completes the proof of (1); (2) follows easily. ∎

Lemma 3.2 *Let Q be a V_r by V_c matrix, $i \in V_r$ and $j \in V_c$. Now let Q' be a matrix obtained by changing the (i,j) entry of Q to $\alpha \neq Q_{i,j}$. If i is an avoidable row of Q and j is an avoidable column of Q then rank $Q' >$ rank Q.*

Proof. Let X be a basis of $M^r(B)$ that does not contain i and let Y be a basis of $M^c(Q)$ that does not contain j. Thus, $Q[X, Y]$ is a maximal nonsingular submatrix of Q. Consequently, $Q[X \cup \{i\}, Y \cup \{j\}]$ is singular. Now,

$$\begin{aligned}\det Q'[X \cup \{i\}, Y \cup \{j\}] &\\ = \det Q[X \cup \{i\}, Y \cup \{j\}] &\\ \pm(\alpha - Q_{ij}) \det Q[X, Y] &\\ = \pm(\alpha - Q_{ij}) \det Q[X, Y] &\\ \neq 0. &\end{aligned}$$

Thus, rank $Q' \geq \text{rank } Q'[X \cup \{i\}, Y \cup \{j\}] > \text{rank } Q[X, Y] = \text{rank } Q$, as required. ∎

Essentially the same proof gives the following result.

Lemma 3.3 *Let $X + Q$ by a mixed matrix. If i is an avoidable row of $X + Q$ and j is an avoidable column of $X + Q$ then $X_{i,j} = 0$.* ∎

Proof of Theorem 1.4. Since deleting a row or a column decreases the rank of a matrix by at most one, we easily see that

$$\text{rank } Q + X \leq \text{rank } Q[Y_r, Y_c] + |V_r - Y_r| + |V_c - Y_c|.$$

Now, by Lemma 3.1,

$$\begin{aligned}\text{rank } Q + X \leq\ & \text{rank } (Q + X)[Y_r^*, Y_c^*] \\ & + |V_r - Y_r^*| + |V_c - Y_c^*|,\end{aligned}$$

moreover, each row and column of $(Q + X)[Y_r^*, Y_c^*]$ is avoidable. Thus, from Lemma 3.2, $X[Y_r^*, Y_c^*] = 0$, as required. ∎

We now consider Theorem 1.6, for which require a little extra matroid theory. Let i and j be distinct elements of a matroid M. If neither i nor j is a coloop of M and $r_M(V - \{i, j\}) < r_M(V)$ then we say that i and j are *in series*. It is well–known, and easy, that series–pairs are transitive. A *series–class* of M is a maximal set of elements that, pairwise, are in series.

Lemma 3.4 (Gallai's Lemma) *If G is a connected graph and $D(G) = V$ then $|V|$ is odd and $2\nu(G) = |V| - 1$.*

Proof. Let T be the Tutte matrix of G. Since $D(G) = V$, $M(T)$ has no coloops. Now, for any edge vw of G, $\nu(G - v - w) < \nu(G)$. Therefore, v and w are in series. Now, by transitivity and since G is connected, $M(T)$ has a single series–class; namely, V. Thus, for any vertex v, $V - \{v\}$ is a matchable set. It follows that $|V|$ is odd and $2\nu(G) = |V| - 1$, as required. ∎

Proof of Theorem 1.6. Let T be the Tutte matrix of G. Recall that $D(G) = D^r(T)$. Now, let $A := A^r(T)$, $X_r := D^r(T)$, and $X_c := V - A$. Note that $X_r \subseteq X_c$. By Lemma 3.1, each line of $T[X_r, X_c]$ is avoidable. By mimicking the proof of Lemma 3.2, we see that no indeterminate occurs exactly once in $T[X_r, X_c]$. Thus, by skew–symmetry, $T[X_r, X_c - X_r] = 0$. That is, $A(G) \subseteq A$.

By Lemma 3.1, we deduce that rank $T = \text{rank } T[V - A] + 2|A|$, and $D^r(T[V - A]) = X_r$. Thus, $2\nu(G) = 2\nu(G - A) + 2|A|$, and $D(G - A) = X_r = D(G)$. Moreover, as $A(G) \subseteq A$, there are no edges in $G - A$ from X_r to $X_c - X_r$. Then, since $D(G-A) = X_r$, $D(G[X_r]) = X_r$ and $G[X_c - X_r]$ has a perfect matching. Using Lemma 3.4 and the fact that $D(G[X_r]) = X_r$, we see that the components of $G[X_r]$ are all odd, and that $2\nu(G-A) = |V-A| - \mathrm{odd}(G-A)$. Therefore, $2\nu(G) = (|V-A| - \mathrm{odd}(G-A)) + 2|A| = |V| - (\mathrm{odd}(G-A) - |A|)$. Now, it remains to prove that $A = A(G)$. Suppose not, then there exists $a \in A$ that has no neighbours in X_r. Thus, the components of $G[X_r]$ are all components of $G-(A-\{a\})$, so $\mathrm{odd}(G-(A-\{a\})) \geq \mathrm{odd}(G-A)$. Hence, $|V| - (\mathrm{odd}(G-(A-\{a\})) - |A-\{a\}|) < |V| - (\mathrm{odd}(G-A) - |A|) = \nu(G)$. This is clearly a contradiction, and this completes the proof. ∎

4 Deterministic algorithms

In this section we prove Theorems 1.7 and 1.8; for which we require the following lemmas.

Lemma 4.1 *Let Q_1 and Q_2 be V_r by V_c matrices such that $Q_1 \approx Q_2$, and let $X_r := D^r(Q_1)$. Then, rank $Q_1[X_r, V_c] =$ rank $Q_2[X_r, V_c]$.*

Proof. This is an immediate corollary of Lemma 3.1. ∎

Lemma 4.2 *Let $Q + \tilde{X}$ be an evaluation of a V_r by V_c mixed matrix $X + Q$, and let z be an indeterminate in X that takes the value $\tilde{z}$ in $\tilde{X}$. Then, there exists $a \in \{1, \ldots, |V_r|+1\} - \{\tilde{z}\}$ such that $Q + \tilde{X}(z \to a) \succeq Q + \tilde{X}$.*

Proof. Consider any nonsingular submatrix $Q' + \tilde{X}'$ of $Q + \tilde{X}$. The determinant of $Q' + \tilde{X}'(z \to a)$ is a nonzero linear function in a, and, hence, has exactly one root. Consequently, for *any* submatrix $Q' + \tilde{X}'$ of $Q + \tilde{X}$, we have rank $Q' + \tilde{X}'(z \to a) \geq$ rank $Q' + \tilde{X}'$ for all but at most one choice of a.

Let $\tilde{X}_a$ denote $\tilde{X}(z \to a)$. Suppose that rank $Q + \tilde{X} = |V_r|$. Then $D^r(Q + \tilde{X}) = \emptyset$, so, $Q + \tilde{X}_a \succ Q + \tilde{X}$ if and only if rank $Q + \tilde{X}_a \geq$ rank $Q + \tilde{X}$; so the result holds. Now suppose that rank $Q + \tilde{X} < |V_r|$. Then, $Q + \tilde{X}_a \succeq Q + \tilde{X}$ if and only if rank $(Q + \tilde{X}_a)[V_r - \{i\}, V_c] \geq$ rank $(Q + \tilde{X})[V_r - \{i\}, V_c]$ for all $i \in V_r$. If the indeterminate z is in row j of X then $(Q + \tilde{X}_a)[V_r - \{j\}, V_c] = (Q + \tilde{X})[V_r - \{j\}, V_c]$; for each of the $|V_r| - 1$ other rows, the rank condition excludes at most one possible value for a. Nevertheless, this leaves some choice for $a \in \{1, \ldots, |V_r| + 1\} - \{\tilde{z}\}$ such that $Q + \tilde{X}_a \succeq Q + \tilde{X}$. ∎

Lemma 4.3 *Let $Q + \tilde{X}$ be an evaluation of a V_r by V_c mixed matrix $Q + X$, let $Y_r := D^r(Q + \tilde{X})$ and let $Y_c := V_c - A^c(Q + \tilde{X})$. If there exists an indeterminate z in $X[Y_r, Y_c]$, then there exists $a \in \{1, \ldots, |V_r| + 1\}$ such that $Q + \tilde{X}(z \to a) \succ Q + \tilde{X}$.*

Proof. Suppose that z takes the value $\tilde{z}$ in $\tilde{X}$. By Lemma 4.2, there exists $a \in \{1, \ldots, |V_c|+1\} - \{\tilde{z}\}$ such that $Q + \tilde{X}(z \to a) \succeq Q + \tilde{X}$. Let $\tilde{X}_a$ denote $\tilde{X}(z \to a)$. We may assume that $Q + \tilde{X}_a \approx Q + \tilde{X}$. By Theorem 3.1, $D^r((Q + \tilde{X})[Y_r, V_c]) = Y_r$ and $D_c((Q + \tilde{X})[Y_r, V_c]) = Y_c$. Therefore, by Lemma 3.2, rank $(Q + \tilde{X}_a)[Y_r, V_c] >$ rank $(Q + \tilde{X})[Y_r, V_c]$; contradicting Lemma 4.1. ∎

Proof of Theorem 1.7. Let $Y_r := D^r(Q + \tilde{X})$ and $Y_c := V_c - A^c(Q + \tilde{X})$, and suppose that we cannot improve the independence of $Q + \tilde{X}$ by such perturbations. Therefore, by Lemma 4.3, $X[Y_r, Y_c] = 0$. Thus, rank $Q + X \leq$ rank $Q[Y_r, Y_c] + |V_r - Y_r| + |V_c - Y_c|$. However, by Lemma 3.1, rank $Q + \tilde{X} =$ rank $Q[Y_r, Y_c] + |V_r - Y_r| + |V_c - Y_c| \geq$ rank $Q + X$. Thus rank $Q + \tilde{X} =$ rank $Q + X$ as required. ∎

Now consider Theorem 1.8. We omit the proof of the following Lemma, which is essentially the same as the proof of Lemma 4.2.

Lemma 4.4 *Let $\tilde{T}$ be an evaluation of a V by V Tutte matrix T, and let z be an indeterminate in T that takes the value $\tilde{z}$ in $\tilde{T}$. Then, there exists $a \in \{1, \ldots, |V|\} - \{\tilde{z}\}$ such that $\tilde{T}(z \to a) \succeq \tilde{T}$.*

We also omit the proof of the following Lemma, which is essentially the same as the proof of Lemma 4.3.

Lemma 4.5 *Let T be the Tutte matrix of a graph $G = (V, E)$, let $\tilde{T}$ be an evaluation of T, let $Y_r := D^r(\tilde{T})$ and let $Y_c := V_c - A^c(\tilde{T})$. If there exists an edge $e = xy \in E$ such that $x \in Y_r$ and $y \in Y_c - Y_r$, then there exists $a \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a) \succ \tilde{T}$.*

Lemma 4.6 *Let T be the Tutte matrix of a graph $G = (V, E)$, let $\tilde{T}$ be an evaluation of T, and let*

$Y_r := D^r(\tilde{T})$. *If there exists an edge* $e = xy \in E$ *such that* $x, y \in Y_r$ *and* x *and* y *are in different series classes of* $M(\tilde{T}[Y_r, V])$ *then there exists* $a \in \{1, \ldots, |V|\}$ *such that* $\tilde{T}(z_e \to a) \succ \tilde{T}$.

Proof. Let $Y_c := V_c - A^c(\tilde{T})$, and suppose that z_e takes the value $\tilde{z}_e$ in $\tilde{T}$. By Lemma 4.4, there exists $a \in \{1, \ldots, |V|\} - \{\tilde{z}_e\}$ such that $\tilde{T}(z_e \to a) \succeq \tilde{T}$. Let $\tilde{T}_a$ denote $\tilde{T}(z \to a)$. We may assume that $\tilde{T}_a \approx \tilde{T}$. Note that $x, y \in Y_r$ and $Y_r \subseteq Y_c$. By Theorem 3.1, $D^r(\tilde{T}[Y_r, V]) = Y_r$ and, since x and y are not in series in $M(\tilde{T}[Y_r, V])$, x is an avoidable column of $\tilde{T}[Y_r, V - \{y\}]$. Therefore, by Lemma 3.2, rank $\tilde{T}_a[Y_r, V - \{y\}] >$ rank $\tilde{T}[Y_r, V - \{y\}]$. However, rank $\tilde{T}[Y_r, V - \{y\}] =$ rank $\tilde{T}[Y_r, V]$, so rank $\tilde{T}_a[Y_r, V_c] >$ rank $\tilde{T}[Y_r, V_c]$; contradicting Lemma 4.1. ∎

Proof of Theorem 1.8. Let $A := A^c(\tilde{T})$, $Y_r := D^r(\tilde{T})$ and $Y_c := V - A$; and suppose that we cannot improve the independence of $\tilde{T}$ by such perturbations. Therefore, by Lemma 4.5, $T[Y_r, Y_c - Y_r] = 0$. Thus rank $\tilde{T}[Y_r, Y_c] =$ rank $\tilde{T}[Y_r]$. Moreover, by Lemma 4.6, for each edge xy of $G[Y_r]$, x and y are in series in $M(\tilde{T}[Y_r, V])$. The elements of A are coloops of $M(\tilde{T}[Y_r, V])$ and the elements in $Y_c - Y_r$ are loops of $M(\tilde{T}[Y_r, V])$. Therefore, for each edge xy of $G[Y_r]$, x and y are in series in $M(\tilde{T}[Y_r])$. Hence, by the transitivity of series–pairs, the series–classes of $M(\tilde{T}[Y_r])$ are determined by the components of $G[Y_r]$. Consequently, rank $\tilde{T}[Y_r] = |Y_r| - \mathrm{odd}(G[Y_r])$. Now, by Lemma 3.1,

$$\begin{aligned}
\text{rank } \tilde{T} &= \text{rank } \tilde{T}[Y_r, Y_c] + |V - Y_r| + |V - Y_c| \\
&= \text{rank } \tilde{T}[Y_r] + |V - Y_r| + |A| \\
&= (|Y_r| - \mathrm{odd}(G[Y_r]) + |V - Y_r| + |A| \\
&\geq (|Y_r| - \mathrm{odd}(G - A)) + |V - Y_r| + |A| \\
&= |V| - (\mathrm{odd}(G - A) - |A|) \\
&\geq \text{rank } T.
\end{aligned}$$

Hence, rank $\tilde{T} =$ rank T, as required. ∎

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