

Nonstandard Representations of Unbounded Self-Adjoint Operators

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1. Introduction

In nonstandard analysis, *standardizations* of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space \mathcal{H} is called the *nonstandard hull* of \mathcal{H} , written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator A on \mathcal{H} with finite norm is naturally defined on $\hat{\mathcal{H}}$. In this paper, the standardization of A shall be called the *standard part* of A , written as \hat{A} . A prominent work of Moore [6] was focused on the case where \mathcal{H} is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of A is not finite, it is not straightforward to give an adequate definition of the standard part of A . Albeverio et al. [4] defined \hat{A} only when \mathcal{H} is hyperfinite-dimensional real Hilbert space and A is an internal positive symmetric operator on \mathcal{H} .

In this paper, we give a definition of \hat{A} for any internal complex Hilbert space \mathcal{H} and for any internal S-bounded self-adjoint operator A on \mathcal{H} , as well as a general consideration on \hat{A} so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a \aleph_1 -saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is \aleph_1 -saturated.

Let $(V, \|\cdot\|)$ be an internal normed linear space. Define the subspaces $\mu(V, \|\cdot\|)$ and $\text{fin}(V, \|\cdot\|)$ of V by

$$\mu(V, \|\cdot\|) = \{\xi \in V \mid \|\xi\| \approx 0\}, \quad \text{fin}(V, \|\cdot\|) = \{\xi \in V \mid \|\xi\| < \infty\}. \quad (1)$$

We often abbreviate them as $\mu(V)$ and $\text{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \text{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $\|\cdot\|$ on \hat{V} by $\|\hat{\xi}\| = \circ\|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i \in \mathbf{N}}$, where $\xi_i \in \text{fin}(V, \|\cdot\|)$, *approximately converges* to $\xi \in V$ in the norm $\|\cdot\|$ if

$$\forall \varepsilon \in \mathbf{R}^+ \exists n \in \mathbf{N} \forall k \in \mathbf{N} [k > n \Rightarrow \|\xi - \xi_k\| < \varepsilon]. \quad (2)$$

A sequence $\{\xi_i\}_{i \in \mathbb{N}}$ approximately converges to $\xi \in V$ if and only if $\{\hat{\xi}_i\}_{i \in \mathbb{N}}$ converges to $\hat{\xi} \in \hat{V}$. A sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V, \|\cdot\|)$, is S - $\|\cdot\|$ -Cauchy if

$$\forall \varepsilon \in \mathbf{R}^+ \exists n \in \mathbf{N} \forall k, l \in \mathbf{N} [k, l > n \Rightarrow \|\xi_k - \xi_l\| < \varepsilon]. \quad (3)$$

A sequence $\{\xi_i\}_{i \in \mathbb{N}}$ is S - $\|\cdot\|$ -Cauchy if and only if the sequence $\{\hat{\xi}_i\}_{i \in \mathbb{N}}$ is Cauchy.

A subset $X \subset \text{fin}(V, \|\cdot\|)$ is S - $\|\cdot\|$ -complete if for any S - $\|\cdot\|$ -Cauchy sequence $\{\xi_i\}_{i \in \mathbb{N}}$, there exists $\xi \in X$ such that $\{\xi_i\}$ approximately converges to ξ in the norm $\|\cdot\|$. The subset X is S - $\|\cdot\|$ -complete if and only if \hat{X} is complete in \hat{V} , where $\hat{X} = \{\hat{\xi} \mid \xi \in X\}$.

The following results, called *the hull completeness theorem*, is a fundamental property of an internal normed space $(V, \|\cdot\|)$. See Hurd and Loeb [3] for detail.

Theorem 2.1. *The subspace $\text{fin}(V)$ is S -complete in $\|\cdot\|$.*

Corollary 2.2. (The Hull Completeness Theorem) *\hat{V} is a Banach space.*

Let \mathcal{H} be an internal Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ an internal bounded linear operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$, called the *standard part* of T , is defined by the relation $\hat{T}\hat{x} = \widehat{Tx}$ for any $x \in \text{fin}(\mathcal{H})$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

3. Several definitions of standard parts

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not S -bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

Lemma 3.1. *Let A be a symmetric operator on a Hilbert space \mathcal{H} . Then, A is self-adjoint if and only if $\text{Rng}(A \pm i) = \mathcal{H}$.*

Let \mathcal{H} be an internal Hilbert space, and A an internal bounded self-adjoint operator on \mathcal{H} . Let $\hat{\mathcal{K}} = \text{Ker}([(A + i)^{-1}]^\wedge)^\perp$. Using the unitarity of $(A + i)(A - i)^{-1}$, we can easily check that $\text{Ker}([(A - i)^{-1}]^\wedge)^\perp = \hat{\mathcal{K}}$.

Proposition 3.2. *There exists the unique (possibly unbounded) self-adjoint operator S on $\hat{\mathcal{K}}$ satisfying*

$$(S + i)^{-1} = [(A + i)^{-1}]^\wedge \upharpoonright \hat{\mathcal{K}}. \quad (4)$$

Proof. We see $\|(A + i)^{-1}\| < \infty$, and $[(A + i)^{-1}]^\wedge$ is an bounded normal operator on $\hat{\mathcal{H}}$. The operator $T := [(A + i)^{-1}]^\wedge \upharpoonright \hat{\mathcal{K}}$ is a bijection from $\hat{\mathcal{K}}$ to $[(A + i)^{-1}]^\wedge \hat{\mathcal{K}}$. Hence the inverse T^{-1} from $[(A + i)^{-1}]^\wedge \hat{\mathcal{K}}$ to $\hat{\mathcal{K}}$ is defined. Clearly the operator $S = T^{-1} - i$ satisfies the equation (4).

We will show that S is symmetric. Let $x_1, x_2 \in \text{Dom}(S)$ ($= [(A + i)^{-1}]^\wedge \hat{\mathcal{K}}$). Then, we can show that there exist $\xi_i \in x_i$ such that $A\xi_i \in Sx_i$ ($i = 1, 2$) as follows. There

are $y_i \in \hat{\mathcal{K}}$ and $\eta_i \in \mathcal{H}$ such that $(S+i)^{-1}y_i = [(A+i)^{-1}]^\wedge y_i = x_i$ and $\eta_i \in y_i$. Let $\xi_i = (A+i)^{-1}\eta_i$. Then $\xi_i \in x_i$ and $(A+i)\xi_i = \eta_i \in y_i = (S+i)x_i$. Hence $A\xi_i \in Sx_i$. Thus, $\langle x_1, Sx_2 \rangle = {}^\circ\langle \xi_1, A\xi_2 \rangle = {}^\circ\langle A\xi_1, \xi_2 \rangle = \langle Sx_1, x_2 \rangle$. Therefore, S is symmetric.

To prove the self-adjointness, it is sufficient to show $\text{Rng}(S+i) = \text{Rng}(S-i) = \hat{\mathcal{K}}$ by Lemma 3.1. Clearly $\text{Rng}(S+i) = \text{Rng}(T^{-1}) = \hat{\mathcal{K}}$. Let $x \in \text{Dom}(S)$, $\xi \in x$ and $A\xi \in Sx$. Then we have

$$\left(\frac{A-i}{A+i}\right)^\wedge (S+i)x = \left(\frac{A-i}{A+i}(A+i)\xi\right)^\wedge = (S-i)x. \quad (5)$$

Thus, by the equation (4) with $\text{Ker}([(A-i)^{-1}]^\wedge)^\perp = \hat{\mathcal{K}}$, we have

$$(S-i)^{-1} = [(A-i)^{-1}]^\wedge | \hat{\mathcal{K}}. \quad (6)$$

Therefore, we can show $\text{Rng}(S-i) = \hat{\mathcal{K}}$ in the similar way to the proof of $\text{Rng}(S+i) = \hat{\mathcal{K}}$. The uniqueness of S is clear. *QED*

Definition 3.3. Under the condition of Proposition 3.2, define the self-adjoint operator $\text{st}_1(A)$ on $\hat{\mathcal{K}}$ by $(\text{st}_1(A) + i)^{-1} = [(A+i)^{-1}]^\wedge | \hat{\mathcal{K}}$.

The operator $\text{st}_1(A)$ is called the *standard part* of A . We see that $\text{st}_1(A) = \hat{A}$ when A is S-bounded.

Definition 3.4. Let A be an internal bounded operator on \mathcal{H} , an internal Hilbert space. Define $\text{fin}(A) \subseteq \mathcal{H}$ by

$$\text{fin}(A) = \{\xi \in \text{fin}\mathcal{H} \mid A\xi \in \text{fin}\mathcal{H}\}. \quad (7)$$

Definition 3.5. Let A be an internal bounded self-adjoint operator on \mathcal{H} . Let $\hat{\mathcal{K}}$ be the closure of the subspace $[\text{fin}(A)]^\wedge = \{\hat{\xi} \mid \xi \in \text{fin}(A)\}$ of $\hat{\mathcal{H}}$. Define the self-adjoint operator $\text{st}_2(A)$ on $\hat{\mathcal{K}}$ by

$$e^{it\text{st}_2(A)} = e^{\widehat{itA}} | \hat{\mathcal{K}}. \quad t \in \mathbf{R}. \quad (8)$$

We see that $\{e^{\widehat{itA}} | \hat{\mathcal{K}}\}_{t \in \mathbf{R}}$ is one-parameter unitary group, since $\hat{\mathcal{K}}$ is invariant under $e^{\widehat{itA}}$ for all $t \in \mathbf{R}$. We also see that it is strongly continuous as follows. Let $\xi \in \text{fin}(A)$. Then, we have $\|(*d/dt)e^{itA}\xi\| = \|ie^{itA}A\xi\| < \infty$, where $*d/dt$ is the internal differentiation. This implies that $e^{\widehat{itA}}\hat{\xi}$ is continuous with respect to $t \in \mathbf{R}$. Thus, $e^{\widehat{itA}}$ is strongly continuous on $[\text{fin}(A)]^\wedge$. Hence by Stone's theorem, $\text{st}_2(A)$ is uniquely defined.

If A is S-bounded, $\text{st}_2(A)$ coincides with \hat{A} defined in Section 2. This is seen from the following:

Proposition 3.6. Let A be an internal S-bounded self-adjoint operator. Then,

$$e^{\widehat{itA}} = e^{it\hat{A}}, \quad (9)$$

for all $t \in \mathbf{R}$.

Proof. For any infinitesimal $\epsilon \in {}^*\mathbf{R}_0^+$,

$$\epsilon^{-1}(e^{i\epsilon A} - I) \approx iA, \quad (10)$$

holds, because

$$\begin{aligned} \|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| &= \|\epsilon^{-1} \sum_{\nu=2}^{*\infty} (i\epsilon A)^\nu / \nu!\| \leq \epsilon^{-1} \sum_{\nu=2}^{*\infty} (\epsilon \|A\|)^\nu / \nu! \\ &= \epsilon^{-1}(e^{\epsilon \|A\|} - 1) - \|A\| \approx 0. \end{aligned}$$

Thus, by the permanence principle,

$$\forall \delta \in \mathbf{R}_+, \exists \epsilon \in \mathbf{R}_+, |t| < \epsilon \Rightarrow \|t^{-1}(e^{itA} - I) - iA\| < \delta. \quad (11)$$

Hence, we have

$$\lim_{\epsilon \rightarrow 0} \|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| = 0. \quad (12)$$

Thus we have $(d/dt)e^{itA}|_{t=0} = i\hat{A}$, where d/dt is the usual differentiation. Because $(e^{itA})_{t \in \mathbf{R}}$ is one-parameter unitary group, it follows that $e^{itA} = e^{it\hat{A}}$. *QED*

Let $E(\cdot)$ be an internal projection-valued measure on ${}^*\mathbf{R}$, i.e., for each internal Borel set $\Omega \subseteq {}^*\mathbf{R}$, $E(\Omega)$ is an orthogonal projection on \mathcal{H} such that

- (1) $E(\emptyset) = 0$, $E({}^*\mathbf{R}) = I$
- (2) If $\Omega = \bigcup_{n=1}^{*\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then $E(\Omega) = \text{s-lim}_{N \rightarrow * \infty} \sum_{n=1}^N E(\Omega_n)$
- (3) $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$.

For $r \in {}^*\mathbf{R}$, let $\mathcal{H}_r = \text{Rng}(E(-r, r))$, the range of $E((-r, r))$. Let $D(E) = \bigcup_{r \in \mathbf{R}^+} \mathcal{H}_r \cap \text{fin}\mathcal{H}$. $D(E)$ is called the *standardization domain* of $E(\cdot)$. Clearly, $\widehat{D(E)}^{\perp\perp} = (\bigcup_{r \in \mathbf{R}^+} \mathcal{H}_r)^{\perp\perp}$.

For $a \in \mathbf{R}$, define the orthogonal projection $\hat{E}_{\text{st}}(-\infty, a]$ by

$$\hat{E}_{\text{st}}(-\infty, a] = \sup\{\hat{E}(-K, a + \epsilon) | \hat{D}(E)^{\perp\perp} \upharpoonright K, \epsilon \in \mathbf{R}^+\} \quad (13)$$

$$= \text{s-lim}_{n \rightarrow \infty} \hat{E}(-n, a + \frac{1}{n}) | \hat{D}(E)^{\perp\perp}. \quad (14)$$

Then we see

$$\text{s-lim}_{a \rightarrow -\infty} \hat{E}_{\text{st}}(-\infty, a] = 0 \quad (15)$$

$$\text{s-lim}_{\epsilon \downarrow 0} \hat{E}_{\text{st}}(-\infty, a + \epsilon] = \hat{E}_{\text{st}}(-\infty, a] \quad (16)$$

$$a < b \Rightarrow \hat{E}_{\text{st}}(-\infty, a] \leq \hat{E}_{\text{st}}(-\infty, b]. \quad (17)$$

Hence, $\hat{E}_{\text{st}}(-\infty, \cdot]$ defines a projection-valued measure on \mathbf{R} .

Definition 3.7. For any internal bounded self-adjoint operator A , define the self-adjoint operator $\text{st}_3(A)$ on $\widehat{D(E)}^{\perp\perp}$ by

$$\text{st}_3(A) = \int \lambda d\hat{E}_{\text{st}}(\lambda). \quad (18)$$

Proposition 3.8. *Let A be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of A . Then*

$$\hat{D}(E)^{\perp\perp} = \widehat{\text{fin}(A)}^{\perp\perp}. \quad (19)$$

Proof. $\hat{D}(E)^{\perp\perp} \subseteq \widehat{\text{fin}(A)}^{\perp\perp}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \widehat{\text{fin}(A)}^{\perp\perp}$, it is sufficient to show that for any $\hat{x} \in \widehat{\text{fin}(A)}^{\perp}$ there is a sequence $\hat{x}_n \in \hat{D}(E)$ ($n \in \mathbf{N}$) such that $\hat{x}_n \rightarrow \hat{x}$. Let $x_n = E(-n, n)x$ ($n \in {}^*\mathbf{N}$). Notice that $\|A(x - x_n)\| \geq n\|x - x_n\|$. Suppose $\|x - x_n\| > \epsilon$ for all $n \in \mathbf{N}$. By the permanence principle, there is $N \in {}^*\mathbf{N}_\infty$ such that $\|x - x_N\| > \epsilon$. Hence, $\|A(x - x_N)\| \geq N\|x - x_N\| > N\epsilon \sim \infty$. This contradicts $\|A(x - x_N)\| \leq \|Ax\| < \infty$. *QED*

Theorem 3.9. *Let A be an internal bounded self-adjoint operator. Then,*

$$\text{st}_2(A) = \int \lambda d\hat{E}_{\text{st}}(\lambda), \quad (20)$$

and hence $\text{st}_2(A) = \text{st}_3(A)$.

Proof. It is sufficient to show

$$\langle \hat{x}, \exp(it \text{st}_2(A))\hat{x} \rangle = \int e^{it\lambda} \langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle \quad (21)$$

for all $\hat{x} \in \widehat{\text{fin}(A)}^{\perp\perp}$. Define the internal Borel measure μ by $\mu(d\lambda) = \langle x, E(d\lambda)x \rangle$. Let $L\mu$ denote the Loeb measure of μ , and $L'\mu$ the Borel measure on \mathbf{R} defined by $L'\mu(\Omega) = L\mu(\text{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\text{st}^{-1}[\Omega]$ is $L\mu$ -measurable for any Borel set $\Omega \subseteq \mathbf{R}$). We also see that $L\mu$ is supported by $\text{fin } {}^*\mathbf{R}$, since $L\mu({}^*\mathbf{R} \setminus \text{fin } {}^*\mathbf{R}) \leq \circ \langle x, E({}^*\mathbf{R} \setminus (-n, n))x \rangle = \circ \|(1 - E(-n, n))x\|^2 \leq (1/n^2) \circ \|Ax\|^2$ for all $n \in \mathbf{N}$. Therefore

$$\begin{aligned} \langle \hat{x}, \exp(it \text{st}_2(A))\hat{x} \rangle &= \langle \hat{x}, e^{itA}\hat{x} \rangle \\ &= \circ \langle x, e^{itA}x \rangle \\ &= \circ \int_{{}^*\mathbf{R}} e^{it\lambda} d\mu(\lambda) \\ &= \int_{{}^*\mathbf{R}} \circ e^{it\lambda} dL\mu(\lambda) \\ &= \int_{\mathbf{R}} e^{it\lambda} dL'\mu(\lambda). \end{aligned}$$

On the other hand, for $a, b \in \mathbf{R}$ with $a < b$,

$$\begin{aligned} L'\mu(a, b) &= L\mu\left(\bigcup_{\epsilon \in \mathbf{R}^+} (a + \epsilon, b - \epsilon)\right) \\ &= \lim_{\epsilon \downarrow 0} \circ \langle x, E(a + \epsilon, b - \epsilon)x \rangle \\ &= \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle \\ &= \langle \hat{x}, \text{s-lim}_{\epsilon \downarrow 0} \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle \\ &= \langle \hat{x}, \hat{E}_{\text{st}}(a, b)\hat{x} \rangle. \end{aligned}$$

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\text{st}}(\Omega)\hat{x} \rangle$ for any Borel set $\Omega \subseteq \mathbf{R}$. *QED*

Let $C \in \mathbf{R}$ be a positive constant, and h be an internal Borel function from ${}^*\mathbf{R}$ to ${}^*\mathbf{C}$ satisfying the following properties:

$$\begin{aligned} h(x) \approx h(y) \quad \text{iff} \quad x \approx y \quad \text{for all } x, y \text{ with } |x|, |y| < \infty, \\ |h(x)| < C \quad \text{for all } x \in {}^*\mathbf{R}. \end{aligned}$$

Define the function $\hat{h} : \mathbf{R} \rightarrow \mathbf{C}$ by

$$\hat{h}(x) = {}^\circ h(x),$$

for $x \in \mathbf{R}$. We see that \hat{h} is injective and continuous. Let A be an internal bounded self-adjoint operator. Notice that $h(A)$ is an S-bounded internal normal operator.

Theorem 3.10. *There exists the unique self-adjoint operator B on $\text{fin}(A)^{\perp\perp}$ such that*

$$\hat{h}(B) = \widehat{h(A)}|_{\text{fin}(A)^{\perp\perp}}. \quad (22)$$

Moreover, B equals to $\text{st}_3(A)$.

Proof. By the argument similar to the proof of Theorem 3.9, we can show

$$\begin{aligned} \langle \hat{x}, \widehat{h(A)}\hat{x} \rangle &= \int_{\mathbf{R}} \hat{h}(\lambda) dL'\mu(\lambda) \\ &= \int_{\mathbf{R}} \hat{h}(\lambda) \langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle \end{aligned}$$

for any $\hat{x} \in \text{fin}(A)^{\perp\perp}$. Thus,

$$\widehat{h(A)}|_{\text{fin}(A)^{\perp\perp}} = \int_{\mathbf{R}} \hat{h}(\lambda) d\hat{E}_{\text{st}}(\lambda).$$

Because \hat{h} is injective, the unique self-adjoint operator B satisfying (22) is $\text{st}_3(A) = \int_{\mathbf{R}} \lambda d\hat{E}_{\text{st}}(\lambda)$. QED

Corollary 3.11. *Definition 3.3, 3.5 and 3.7 are equivalent, that is, $\text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.*

Proof. Let $h(x) = 1/(x + i)$. QED

In section 2, \hat{A} is defined only when A is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where A is an internal bounded self-adjoint operator which is not S-bounded; $\hat{A} := \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

Definition 3.12. *Let A be an internal linear operator on an internal Hilbert space \mathcal{H} . Let D be an (external) subspace of $\text{fin}\mathcal{H}$. A is standardizable on D if $D \subset \text{fin}(A)$ and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator \hat{A}_D with domain $\hat{D} = \{\hat{x} | x \in D\}$, called the standard part of A on D , by*

$$\hat{A}_D \hat{x} = \widehat{Ax}, \quad x \in D. \quad (23)$$

Clearly, A is standardizable on D if and only if $D \subset \text{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

Lemma 3.13. *An internal bounded operator A is standardizable on $\text{fin}(A^*A)$.*

Proof. First, we prove $\text{fin}(A^*A) \subset \text{fin}(A)$ as follows. Suppose that $\xi \in \text{fin}(A)$. Let $E(\cdot)$ be the internal spectral projection-valued measure of the self-adjoint operator A^*A . Then, $\|A\xi\|^2 = \langle \xi, A^*A\xi \rangle = \langle \xi, E[0, 1]A^*A\xi \rangle + \langle \xi, (I - E[0, 1])A^*A\xi \rangle \leq \langle \xi, E[0, 1]A^*A\xi \rangle + \langle \xi, (I - E[0, 1])(A^*A)^2\xi \rangle \leq \langle \xi, E[0, 1]A^*A\xi \rangle + \|A^*A\xi\|^2 < \infty$. Thus, $\xi \in \text{fin}(A)$. Second, suppose $x \approx 0$ and $\|A^*Ax\| < \infty$. Then, $\|Ax\|^2 = \langle x, A^*Ax \rangle \leq \|x\| \|A^*Ax\| \approx 0$. QED

Corollary 3.14. *If $D \subseteq \text{fin}\mathcal{H}$ is invariant under A and A^* , A is standardizable on D .*

The operator B in the above proof is called a *hyperfinite extension* of A [6].

We use the following lemma in the proof of Theorem 3.16.

Lemma 3.15. *Let A be a symmetric operator with domain $D \subset \mathcal{H}$, a Hilbert space. Let $D_1 \subset D$ be a dense linear subset of \mathcal{H} and suppose that $A|_{D_1}$ is essentially self-adjoint. Then, A is essentially self-adjoint and $\overline{A} = \overline{A|_{D_1}}$.*

Theorem 3.16. *Let A be an internal self-adjoint operator on \mathcal{H} , and $E(\cdot)$ the projector-valued spectral measure of A . Then,*

$$\hat{A} = \overline{\hat{A}_{D(E)}} = \overline{\hat{A}_{\text{fin}(A^2)}} \quad (24)$$

Proof. We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson's analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that \hat{A} is an extension of $\hat{A}_{D(E)}$. If $E(-r, r)\xi = \xi$ ($r \in \mathbf{R}^+$, $\xi \in \mathcal{H}$), then $\hat{E}_{\text{st}}(-s, s)\hat{\xi} = \hat{\xi}$ ($s \in \mathbf{R}^+$, $s > r$). Thus, $\hat{A}_D\hat{\xi} = \overline{\hat{A}\xi} = \overline{[\int_{-s}^s \lambda dE(\lambda)]\hat{\xi}} = \int_{-s}^s \lambda d\hat{E}_{\text{st}}(\lambda)\hat{\xi} = \int \lambda d\hat{E}_{\text{st}}(\lambda)\hat{\xi} = \text{st}_3(A)\hat{\xi} = \hat{A}\hat{\xi}$. Therefore $\hat{A} = \overline{\hat{A}_{D(E)}}$. $\hat{A}_{D(E)} = \overline{\hat{A}_{\text{fin}(A^2)}}$ follows from $D(E) \subseteq \text{fin}(A^2)$ and Lemma 3.15. QED

4. The domain of \hat{A}

Definition 4.1. *For an internal bounded self-adjoint operator A on \mathcal{H} , define $D(A)$ by*

$$D(A) = \{\xi \in \text{fin}\mathcal{H} \mid \text{for all } t \in \mathbf{R}_0^+, e^{-t|A|}A\xi \approx A\xi \in \text{fin}\mathcal{H}\}.$$

Clearly, $D(A)$ is a subspace of \mathcal{H} .

Proposition 4.2. *An internal bounded self-adjoint operator A is standardizable on $D(A)$.*

Proof. Let $\xi \in D(A)$ and $\|\xi\| \approx 0$. We can easily check $\|e^{-t|A|}A\| < \infty$ for all $t > 0$, $t \not\approx 0$. Hence, ${}^\circ\|A\xi\| \leq {}^\circ\|e^{-t|A|}A\xi\| + {}^\circ\|(1 - e^{-t|A|})A\xi\|$. By the S-boundedness of $e^{-t|A|}A$, the first term equals 0, and by the definition of $D(A)$, the second term equals 0. Thus we have ${}^\circ\|A\xi\| = 0$. *QED*

The following lemmas are easily shown.

Lemma 4.3. *Let $f : {}^*\mathbf{N} \rightarrow {}^*\mathbf{R}^+$ be internal and increasing. If $f(M) < \infty$ for some $M \sim \infty$, then*

$$\lim_{n \rightarrow \infty} {}^\circ f(n) < \infty.$$

Lemma 4.4. *Under the same condition to Lemma 4.3, there is $K \sim \infty$ such that for all $L \sim \infty$,*

$$f(K) \approx f(L) \quad \text{if } L \leq K.$$

Proposition 4.5. *Let $\xi \in \text{fin}(\mathcal{H})$. For sufficiently large $t \approx 0$,*

$$e^{-t|A|}\xi \in D(A). \quad (25)$$

Proof. Applying Lemma 4.4 to $f(n) = \|e^{-|A|/n}A\xi\|$, we find that for sufficiently small $K \sim \infty$ and $L \sim \infty$, $e^{-|A|/K}A\xi \approx e^{-|A|/L}A\xi$. Thus, for sufficiently large $s \approx 0$ and $t \approx 0$, $e^{-s|A|}A\xi \approx e^{-t|A|}A\xi$. Hence for all $x \approx 0$, $x > 0$,

$$e^{-x|A|}Ae^{-t|A|}\xi = e^{-(x+t)|A|}A\xi \approx Ae^{-t|A|}\xi.$$

Therefore, $e^{-t|A|}\xi \in D(A)$. *QED*

Theorem 4.6. *Let $E(\cdot)$ be the spectral resolution of A and $E_K = E(-K, K)$ for $K \in {}^*\mathbf{R}^+$. For any $\xi \in \text{fin}(A)$,*

$$\xi \in D(A) \quad \text{iff} \quad A\xi \approx E_K A\xi \quad \text{for all } K \sim \infty. \quad (26)$$

Remark. The right-hand condition is equivalent to

$$\lim_{\substack{K \rightarrow \infty \\ K \in \mathbf{R}}} {}^\circ\|(I - E_K)A\xi\| = 0. \quad (27)$$

Proof. Suppose that $\xi \in \text{fin}(A)$ and $A(I - E_K)\xi \approx 0$ for all $K \sim \infty$. For any $t \approx 0$, there exists a $K \sim \infty$ such that $tK \approx 0$. Thus,

$$\begin{aligned} \|e^{-t|A|}A\xi - A\xi\|^2 &\approx \|e^{-t|A|}E_K A\xi - E_K A\xi\|^2 \\ &= \left\| \int_{-K}^K e^{-t|\lambda|} \lambda - \lambda dE(\lambda) \xi \right\|^2 \\ &= \int_{-K}^K |(e^{-t|\lambda|} - 1)\lambda|^2 \|dE(\lambda)\xi\|^2 \\ &\leq \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \int_{-K}^K \lambda^2 \|dE(\lambda)\xi\|^2 \\ &= \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \|E_K A\xi\|^2 \\ &\approx 0. \end{aligned}$$

Hence $\xi \in D(A)$.

Conversely, suppose $\xi \in D(A) (\subset \text{fin}(A))$. Applying Lemma 4.4 to $f(n) = \|E_n A\xi\|$, we see that for sufficiently small $K \sim \infty$ and $L \sim \infty$ ($L \leq K$),

$$\|E_L A\xi\| \approx \|E_K A\xi\|.$$

Thus, $(E_K - E_L)A\xi \approx 0$, since $\|E_L A\xi - E_K A\xi\|^2 = \|E_K A\xi\|^2 - \|E_L A\xi\|^2 \approx 0$. Let $t \in \mathbf{R}_0^+$ satisfy $tK \sim \infty$ so that

$$\begin{aligned} & \|E_K A\xi - e^{-t|A|} A\xi\| \\ &= \left\| \int_{-K}^K \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi - \int_{(-\infty, -K) \cup (K, \infty)} e^{-t|\lambda|} \lambda dE(\lambda)\xi \right\| \\ &\leq \left\| \int_{-K}^K \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi \right\| + e^{-tK} \|A\xi\| \\ &\approx \left\| \int_{-K}^K \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi \right\|. \end{aligned}$$

Let $L \sim \infty$ satisfy $tL \approx 0$, so that the above

$$\begin{aligned} &\leq \left\| \int_{-L}^L \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi \right\| + \left\| \int_{(-K, K) \setminus (-L, L)} \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi \right\| \\ &\leq |1 - e^{-tL}| \|A\xi\| + \|(E_K - E_L)A\xi\| \\ &\approx 0. \end{aligned}$$

Thus, for sufficiently small $K \sim \infty$ and for any $t \approx 0$ such that $tK \sim \infty$,

$$E_K A\xi \approx e^{-t|A|} A\xi \approx A\xi.$$

Since $\|A\xi - E_K A\xi\| \geq \|A\xi - E_{K'} A\xi\| > 0$ if $K < K'$, we have $E_{K'} A\xi \approx A\xi$ holds for any $K' \sim \infty$. *QED*

Proposition 4.7. *Let $\xi \in \text{fin}(A)$. Then, $E_K \xi \in D(A)$ for sufficiently small $K \sim \infty$.*

Proof. Applying Lemma 4.4 to $f(n) = \|E_n A\xi\|$, we find that for sufficiently small $K, L \sim \infty$, $E_K A\xi \approx E_L A\xi$. Thus, if $L \sim \infty$, $L \leq K$, then $\|(1 - E_L)E_K A\xi\| = \|(E_K - E_L)A\xi\| \approx 0$. If $L > K$, clearly $(1 - E_L)E_K A\xi = 0$. Hence for all $L \sim \infty$, $E_K A\xi \approx E_L E_K A\xi$. Thus $E_K \xi \in D(A)$ by Theorem 4.6. *QED*

Corollary 4.8. $[\text{fin}(A)]^\wedge = [D(A)]^\wedge$, i.e., if $\xi \in \text{fin}(A)$, then there is $\eta \in D(A)$ such that $\eta \approx \xi$.

Example We have seen that the following relations hold:

$$\text{fin}(A^2) \subset D(A) \subset \text{fin}(A) \subset \text{fin}\mathcal{H},$$

$$[\text{fin}(A^2)]^\wedge \subset [D(A)]^\wedge = [\text{fin}(A)]^\wedge \subset \hat{\mathcal{H}},$$

$$[\text{fin}(A^2)]^{\perp\perp} = [D(A)]^{\perp\perp} = [\text{fin}(A)]^{\perp\perp} \subset \hat{\mathcal{H}}.$$

An example of A such that $\text{fin}(A) \setminus D(A) \neq \emptyset$ is given as follows. Let ν be an infinite hypernatural number, and $\mathcal{H} = {}^*\mathbf{C}^\nu$, ν -dimensional internal Hilbert space. Define the internal self-adjoint operator A on \mathcal{H} by $A(x_1, x_2, \dots, x_\nu) = (x_1, 2x_2, \dots, \nu x_\nu)$. Let $\xi = (0, 0, \dots, 0, \nu^{-1})$. Then we see $\xi \in \text{fin}(A) \setminus D(A)$ from Theorem 4.6.

We also find $D(A) \setminus \text{fin}(A^2) \neq \emptyset$; let $\eta = (1^{-2}, 2^{-2}, \dots, \nu^{-2})$, then we easily see $\eta \in D(A) \setminus \text{fin}(A^2)$. Moreover we find $\hat{\eta} \in [D(A)]^\wedge \setminus [\text{fin}(A^2)]^\wedge$. In fact, if $\eta' \approx \eta$, then $\circ\|A^2\eta'\| \geq \lim_{n \in \mathbf{N}} \circ\|A^2 E_n \eta'\| = \lim_{n \in \mathbf{N}} \circ\|A^2 E_n \eta\| = \lim_{n \in \mathbf{N}} \circ\sqrt{n} = \infty$. Thus, we have $\hat{\eta} \notin [\text{fin}(A^2)]^\wedge$ by Theorem 4.6.

Theorem 4.9. *Let $\xi \in \text{fin}(A)$, then*

$$\xi \in D(A) \quad \text{iff} \quad \lim_{\substack{t \downarrow 0 \\ t \neq 0}} \left(\frac{e^{-t|A|} - 1}{t} \xi \right)^\wedge = -|A|\xi. \quad (28)$$

Proof. Suppose that the right-hand side does not hold. In other words, suppose that

$$\exists \varepsilon \in \mathbf{R}^+ \forall n \in \mathbf{N} \exists t \in {}^*\mathbf{R}, 0 < t < \frac{1}{n} \wedge \left\| \left(\frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \varepsilon. \quad (29)$$

By permanence,

$$\exists \varepsilon \in \mathbf{R}^+ \exists N \in {}^*\mathbf{N}_\infty \exists t \in {}^*\mathbf{R}, 0 < t < \frac{1}{n} \wedge \left\| \left(\frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \varepsilon. \quad (30)$$

That is, there is positive infinitesimal t such that $t^{-1}(e^{-t|A|} - 1)\xi \not\approx -|A|\xi$.

Thus, for some $\eta \in \text{fin}(\mathcal{H})$,

$$\Re \left\langle \eta, \frac{e^{-t|A|} - 1}{t} \xi \right\rangle \not\approx \Re \langle \eta, -|A|\xi \rangle.$$

Let $f(t) = \Re \langle \eta, e^{-t|A|} \xi \rangle$. By the mean value theorem, for some $s \in {}^*\mathbf{R}$ with $0 < s < t$,

$$f'(s) = \frac{f(t) - f(0)}{t} = \Re \left\langle \eta, \frac{e^{-t|A|} - 1}{t} \xi \right\rangle \not\approx \Re \langle \eta, -|A|\xi \rangle.$$

Therefore, by the definition of $D(A)$, we have $\xi \in \text{fin}(A) \setminus D(A)$.

Conversely, suppose $\xi \in \text{fin}(A) \setminus D(A)$. Then, there is positive infinitesimal t_0 satisfying $e^{-t_0|A|}|A|\xi \not\approx |A|\xi$. Let $\eta = (|A| - e^{t_0|A|})\xi \in \text{fin}(\mathcal{H})$. Then this is equivalent to

$$\langle \eta, e^{-t_0|A|}|A|\xi \rangle \not\approx \langle \eta, |A|\xi \rangle. \quad (31)$$

Let $f(x) = \langle \eta, e^{-x|A|} \xi \rangle$ ($x \in {}^*\mathbf{R}^+$). We see that f' is increasing and $-\infty < f' < 0$, and hence f is decreasing and $0 < f < \infty$. The relation (31) is equivalent to

$$f'(t_0) \not\approx f'(0), \quad (32)$$

We have $f(x) \geq f'(t_0)(x - t_0) + f(t_0)$. Thus we have

$$0 > \frac{f(x) - f(0)}{x} \geq \frac{f'(t_0)(x - t_0) + f(t_0) - f(0)}{x}. \quad (33)$$

Let $F(x) = [f'(t_0)(x - t_0) + f(t_0) - f(0)]/x$, then for $c \in {}^*\mathbf{R}^+$,

$$F(ct_0) = f'(t_0) \left(1 - \frac{1}{c}\right) + \frac{1}{c} \frac{f(t_0) - f(0)}{t_0}. \quad (34)$$

By the mean value theorem and $-\infty < f'(x) < 0$, we have $|(f(x) - f(0))/x| < \infty$. Hence $F(ct_0) \approx f'(t_0)$ for all $c \sim \infty$. Thus, by (32) and (33),

$$0 > \frac{f(ct_0) - f(0)}{ct_0} \geq F(ct_0) \not\approx f'(0), \quad (35)$$

for all $c \sim \infty$. Thus there is $\varepsilon \in \mathbf{R}^+$ such that for sufficiently large $x \approx 0$, $\frac{f(x) - f(0)}{x} - f'(0) > \varepsilon$. By the permanence principle, for sufficiently small $x \in \mathbf{R}^+$, $\frac{f(x) - f(0)}{x} - f'(0) > \varepsilon$. We can check the relations

$$\left\langle \eta, \left(\frac{e^{-x|A|} - 1}{x} \right) \xi \right\rangle = \frac{f(x) - f(0)}{x}, \quad \langle \eta, |A|\xi \rangle = -f'(0), \quad \frac{e^{-x|A|} - 1}{x} > -|A|,$$

for $x > 0$. Therefore, using the increasingness of $(e^{-x|A|} - 1)/x$, x , we have

$$\lim_{\substack{x \downarrow 0 \\ x \neq 0}} \left\langle \eta, \frac{e^{-x|A|} - 1}{x} \xi \right\rangle \neq \langle \eta, -|A|\xi \rangle.$$

QED

Theorem 4.10. *Let A be an internal bounded self-adjoint operator. Then, $\hat{A} = \hat{A}_{D(A)}$.*

Proof. By Theorem 3.16 and Lemma 3.15, it suffices to show that $\hat{A}_{D(A)}$ is a closed extension of $\hat{A}_{\text{fin}(A^2)}$. If $\xi \in \text{fin}(A^2)$, for any $K \sim \infty$, $\|(1 - E_K)A\xi\| \leq \frac{1}{K} \|(1 - E_K)A^2\xi\| \leq \frac{1}{K} \|A^2\xi\| \approx 0$. Hence $\xi \in D(A)$, and hence $\hat{A}_{D(A)}$ is an extension of $\hat{A}_{\text{fin}(A^2)}$.

To prove that $\hat{A}_{D(A)}$ is closed, it suffices to show that $\hat{D}(A) = [D(A)]^\wedge$ is complete in the norm $\|\cdot\|_A$ defined by $\|\hat{\xi}\|_A = \|\xi\| + \|\hat{A}\xi\|$. Define the internal norm $\|\cdot\|_A$ on \mathcal{H} by $\|\xi\|_A = \|\xi\| + \|A\xi\|$. We can check $\|\hat{\xi}\|_A = \|\xi\|_A$ for $\xi \in D(A)$.

By Theorem 2.1, $\text{fin}(A)$ is S- $\|\cdot\|_A$ -complete. Hence, if the sequence $\{\xi_i\}_{i \in \mathbf{N}} \subset D(A)$ ($\subset \text{fin}(A)$) is S- $\|\cdot\|_A$ -Cauchy, then there is $\xi \in \text{fin}(A)$ such that $\{\xi_i\}$ approximately converges to ξ in the norm $\|\cdot\|_A$. This ξ is shown to be in $D(A)$ as follows. Regarding Theorem 4.6, and $\xi_i \in D(A)$ ($i < \infty$), this relation leads to $\| (I - E_K)A\xi_\nu \| = \lim_{i \rightarrow \infty} \| (I - E_K)A\xi_i \| = 0$, for any $K \sim \infty$. Therefore, from Theorem 4.6, we have $\xi \in D(A)$ and hence any Cauchy sequence in $\widehat{D(A)}$ converges in $\widehat{D(A)}$ in the norm $\|\cdot\|_A$. *QED*

Theorem 4.11. *The domain $D(A)$ is maximal. That is, if $D(A) \subset S \subset \text{fin}(\mathcal{H})$ and A is standardizable on S , then $S = D(A)$.*

Proof. Suppose that $D(A) \subset S \subset \text{fin}(\mathcal{H})$ and that A is standardizable on S . Let $\eta \in S$. By Corollary 4.8 and $\eta \in \text{fin}(A)$, there is $\xi \in D(A)$ such that $\xi \approx \eta$. By the definition of $D(A)$ and the standardizability on S , for all positive infinitesimal t , $e^{-t|A|}A\eta \approx e^{-t|A|}A\xi \approx A\xi \approx A\eta$, since $\|e^{-t|A|}\| \leq 1$. Thus, $\eta \in D(A)$. *QED*

Proposition 4.12. *Let A be an internal positive operator on \mathcal{H} . Then, for any $\eta \in \text{fin}(A^{\frac{1}{2}})$,*

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \sim \infty} \circ \langle \eta, E_\alpha A\eta \rangle. \quad (36)$$

Proof. Suppose $\eta \approx \xi$. If $\alpha < \infty$, $\langle \eta, E_\alpha A\eta \rangle \approx \langle \xi, E_\alpha A\xi \rangle \leq \langle \xi, A\xi \rangle$, that is,

$$\forall \varepsilon \in \mathbf{R}^+, \forall \alpha < \infty, \quad \langle \eta, E_\alpha A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon,$$

Thus, by the permanence principle,

$$\forall \varepsilon \in \mathbf{R}^+, \exists K \sim \infty, \forall \alpha \leq K, \quad \langle \eta, E_\alpha A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.$$

By saturation,

$$\exists K \sim \infty, \forall \varepsilon \in \mathbf{R}^+, \forall \alpha \leq K, \quad \langle \eta, E_\alpha A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.$$

Hence we have

$$\exists K \sim \infty, \quad \circ \langle \eta, E_K A\eta \rangle \leq \circ \langle \xi, A\xi \rangle.$$

It follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \geq \inf_{\alpha \sim \infty} \circ \langle \eta, E_\alpha A\eta \rangle$.

On the other hand, we see that for all $\alpha \sim \infty$, $\|\eta - E_\alpha \eta\|^2 \leq \alpha^{-1} \|A^{\frac{1}{2}}(\eta - E_\alpha \eta)\|^2 \leq \alpha^{-1} \|A^{\frac{1}{2}}\eta\|^2 \approx 0$. Hence,

$$\forall \alpha \sim \infty, \quad \inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \circ \langle E_\alpha \eta, A E_\alpha \eta \rangle = \circ \langle \eta, E_\alpha A\eta \rangle.$$

Thus it follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \inf_{\alpha \sim \infty} \circ \langle \eta, E_\alpha A\eta \rangle$. *QED*

Proposition 4.13. *Let A be an internal positive operator and $\eta \in \text{fin}(A)$. Then,*

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle. \quad (37)$$

Proof. From Proposition 4.12, we see $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \sim \infty} \circ \langle \eta, E_\alpha A\eta \rangle$. By Theorem 4.10 and Proposition 4.7, for sufficiently small $\alpha \sim \infty$, $\circ \langle \eta, E_\alpha A\eta \rangle = \circ \langle E_\alpha \eta, A E_\alpha \eta \rangle = \langle \widehat{E_\alpha \eta}, \widehat{A E_\alpha \eta} \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle$. *QED*

Definition 4.14. *Let A be a internal bounded positive operator, and $D \subset \text{fin}(A^{\frac{1}{2}})$. The sesquilinear form $\langle \cdot, A \cdot \rangle$ is standardizable on D if $\langle \xi_1, A\eta_1 \rangle \approx \langle \xi_2, A\eta_2 \rangle$ for all $\xi_1, \xi_2, \eta_1, \eta_2 \in D$ with $\xi_1 \approx \xi_2$ and $\eta_1 \approx \eta_2$.*

Proposition 4.15. *Let D be a subspace of $\text{fin}(\mathcal{H})$ and $A \geq 0$. Then, $\langle \cdot, A \cdot \rangle$ is standardizable on D if and only if $A^{\frac{1}{2}}$ is standardizable on D .*

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on D . Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = \|A^{\frac{1}{2}}\xi\|^2 \approx \|A^{\frac{1}{2}}\eta\|^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A \cdot \rangle$ is standardizable on D . Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $\|A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta\|^2 = \|A^{\frac{1}{2}}(\xi - \eta)\|^2 = \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. *QED*

Corollary 4.16. *The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A \cdot \rangle$, and ${}^\circ\langle \xi, A\eta \rangle = \langle \widehat{A^{\frac{1}{2}}\xi}, \widehat{A^{\frac{1}{2}}\eta} \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.*

References

- [1] A. Robinson. *Non-Standard Analysis* (North-Holland, Amsterdam, 1966).
- [2] K. D. Stroyan and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals* (Academic Press, New York, 1976).
- [3] A. E. Hurd, and P. A. Loeb, *An Introduction to Nonstandard Real Analysis* (Academic Press, Orlando, 1985).
- [4] S. Albeverio, J. Fenstad, R. Høegh-Krohn, and T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics* (Academic Press, Orlando, 1986).
- [5] C. W. Henson and L. Moore, "Nonstandard analysis and the theory of Banach spaces," In *Nonstandard Analysis—Recent Developments*, (A. E. Hurd ed.), Lecture Notes in Mathematics, Vol. 983. Springer, Berlin, 1983.
- [6] L. C. Moore, JR., "Hyperfinite extensions of bounded operators on a Hilbert space," *Trans. Am. Math. Soc.* 218, 285–295 (1976).
- [7] C. C. Chang and H. J. Keisler, *Model Theory*, 3rd edition, North-Holland, Amsterdam, 1990.
- [8] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I, Functional Analysis*, (Academic Press, New York, 1972).