Quantum Logical Gate Based on Fock Space

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Abstracts:
In usual computer, there exists a restriction of computational speed because of irreversibility of logical gate. In order to avoid this demerit, Fredkin and Toffoli [3] proposed a conservative logical gate. Based on their work, Milburn [4] introduced a physical model of reversible quantum logical gate using beam splittings and a Kerr medium. This model is called FTM (Fredkin - Toffoli - Milburn gate). FTM gate was described by the quantum channel and the efficiency of information transmission of the FTM gate was discussed in [10]. FTM gate is using a photon number state as an input state for control gate. The photon number state might be difficult to realize physically. In this paper, we introduced a new unitary operator related to the Kerr device on symmetric Fock space in order to avoid this difficulty.

Key words: quantum logical gate, channels, beam splittings, FTM gate, Fock space

1. Quantum channels

Let \((B(H_1), \mathcal{S}(H_1))\) and \((B(H_2), \mathcal{S}(H_2))\) be input and output systems, respectively, where \(B(H_k)\) is the set of all bounded linear operators on a separable Hilbert space \(H_k\) and \(\mathcal{S}(H_k)\) is the set of all density operators on \(H_k\) \((k = 1, 2)\). Quantum channel \(\Lambda^*\) is a mapping from \(\mathcal{S}(H_1)\) to \(\mathcal{S}(H_2)\). \(\Lambda^*\) is linear if \(\Lambda^*(\lambda \rho_1 + (1 - \lambda) \rho_2) = \lambda \Lambda^*(\rho_1) + (1 - \lambda) \Lambda^*(\rho_2)\) holds for any \(\rho_1, \rho_2 \in \mathcal{S}(H_1)\).
and any $\lambda \in [0, 1]$. $\Lambda^*$ is completely positive (C.P.) if $\Lambda^*$ is linear and its dual $
abla : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ satisfies
\[
\sum_{i,j=1}^{n} A_i^* \Lambda (\overline{A}_i^* \overline{A}_j) A_j \geq 0
\]
for any $n \in \mathbb{N}$, any $\{\overline{A}_i\} \subset \mathcal{B}(\mathcal{H}_2)$ and any $\{A_i\} \subset \mathcal{B}(\mathcal{H}_1)$, where the dual map $\Lambda$ of $\Lambda^*$ is defined by
\[
tr\Lambda^*(\rho)B = tr\rho \Lambda (B), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_1), \quad \forall B \in \mathcal{B}(\mathcal{H}_2).
\]
(1.1)

Almost all physical transformation can be described by the CP channel [5], [7], [8].

Let $K_1$ and $K_2$ be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [6]: Let $\rho$ be an input state in $\mathfrak{S}(\mathcal{H}_1)$, $\xi$ be a noise state in $\mathfrak{S}(K_1)$.

\[
\begin{array}{cccc}
\mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\
\mathfrak{S}(\mathcal{H}_1 \otimes K_1) & \xrightarrow{\Pi^*} & \mathfrak{S}(\mathcal{H}_2 \otimes K_2)
\end{array}
\]

The above maps $\gamma^*$, $a^*$ are given as
\[
\gamma^* (\rho) = \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \quad (1.2)
\]
\[
a^* (\sigma) = tr_{\mathcal{K}_2} \sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes K_2). \quad (1.3)
\]

The map $\Pi^*$ is a channel from $\mathfrak{S}(\mathcal{H}_1 \otimes K_1)$ to $\mathfrak{S}(\mathcal{H}_2 \otimes K_2)$ determined by physical properties of the device transmitting information. Hence the channel for the above process is given by
\[
\Lambda^* (\rho) \equiv tr_{\mathcal{K}_2} \Pi^* (\rho \otimes \xi) = (a^* \circ \Pi^* \circ \gamma^*) (\rho) \quad (1.4)
\]
for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. Based on this scheme, the noisy quantum channel [9] are constructed as follows:

Noisy quantum channel $\Lambda^*$ with a noise state $\xi$ is defined by
\[
\Lambda^* (\rho) \equiv tr_{\mathcal{K}_2} \Pi^* (\rho \otimes \xi) = tr_{\mathcal{K}_2} V (\rho \otimes \xi) V^*, \quad (1.5)
\]
where $\xi = |m_1\rangle\langle m_1|$ is the $m_1$ photon number state in $\mathfrak{S}(\mathcal{K}_1)$ and $V$ is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ denoted by

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_j^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1 + m_1 - j\rangle,$$

$$C_j^{n_1,m_1} = \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!m_1!j!(n_1+m_1-j)!}}{r!(n_1-j)!(j-r)!(m_1-j+r)!} \alpha^{m_1-j+2r} (-\overline{\beta})^{n_1+j-2r}$$

$K$ and $L$ are constants given by $K = \min\{n_1, j\}$, $L = \max\{m_1-j, 0\}$. In particular for the coherent input state $\rho = |\theta\rangle \langle \theta| \otimes |\kappa\rangle \langle \kappa| \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$, we obtain the output state of $\Pi^*$ by

$$\Pi^* (|\theta\rangle \langle \theta| \otimes |\kappa\rangle \langle \kappa|) = |\alpha\theta + \beta\kappa\rangle \langle \alpha\theta + \beta\kappa| \otimes |-\overline{\beta}\theta + \alpha\kappa\rangle \langle -\overline{\beta}\theta + \alpha\kappa|$$

where $\Pi^*$ is called a generalized beam splitting. When the noise $\xi_0 = |0\rangle\langle 0|$ is given by the vacuum state, $\Lambda^*_0$ is called an attenuation channel [5] and $E^*_0$ (or $\Pi^*_0$) is called a beam splitting. Based on liftings, the beam splitting was studied by Accardi - Ohya [1] and Fichtner - Freudenberg - Libsher [2].

2. Quantum logical gate on symmetric Fock space

Recently, we reformulate a quantum channel for the FTM gate and we rigorously study the conservation of information for FTM gate [10]. However, it might be difficult to realize the photon number state $|n\rangle \langle n|$ for the input of the Kerr medium physically.

In this section, we reformulate beam splittings on symmetric Fock space and we introduce a new operator on this space instead of the Kerr medium. We discuss the mathematical formulation of quantum logical gate by means of beam splittings and the new operator.

Let $G$ be a complete separable metric space and $\mathcal{G}$ be a Borel $\sigma$-algebra of $G$. $v$ is called a locally finite diffuse measure on the measurable space $(G, \mathcal{G})$ if $v$ satisfies the conditions (1) $v(K) < \infty$ for bounded $K \in \mathcal{G}$ and (2) $v(\{x\}) = 0$ for any $x \in G$. We denote the set of all finite integer-valued measures $\varphi$ on $(G, \mathcal{G})$ by $M$. For a set $K \in \mathcal{G}$ and a natural number $n \in \mathbb{N}$, we put the set of $\varphi$ satisfying $\varphi(K) = n$ as

$$M_{K,n} \equiv \{\varphi \in M; \varphi(K) = n\}.$$
Let $\mathfrak{M}$ be a $\sigma$-algebra generated by $M_{K,n}$. $F$ is the $\sigma$-finite measure on $(M, \mathfrak{M})$ defined by
\[ F(Y) \equiv 1_Y(\varphi_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_M 1_Y(\sum_{j=1}^{n} \delta_{x_j}) v^n(dx_1 \cdots dx_n), \]
where $1_Y$ is the characteristic function of a set $Y$, $\varphi_0$ is an empty configuration in $M$ and $\delta_{x_j}$ is a Dirac measure in $x_j$. $\mathcal{M} \equiv L^2(M, \mathfrak{M}, F)$ is called a (symmetric) Fock space. We define an exponental vector $\exp_g : M \rightarrow \mathbb{C}$ generated by a given function $g : G \rightarrow \mathbb{C}$ such that
\[ \exp_g(\varphi) \equiv \left\{ \begin{array}{ll}
1 & (\varphi = \varphi_0), \\
\prod_{x \in \varphi} g(x) & (\varphi \neq \varphi_0),
\end{array} \right. \quad (\varphi \in M). \]

2.1. Generalized beam splittings on Fock space

Let $\alpha, \beta$ be measurable mappings from $G$ to $\mathbb{C}$ satisfying
\[ |\alpha(x)|^2 + |\beta(x)|^2 = 1, \quad x \in G. \]
We introduce an unitary operator $V_{\alpha,\beta} : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{M}$ defined by
\[ (V_{\alpha,\beta} \Phi)(\varphi_1, \varphi_2) \equiv \sum_{\hat{\varphi}_1 \leq \varphi_1} \sum_{\hat{\varphi}_2 \leq \varphi_2} \exp_{\alpha}(\hat{\varphi}_1) \exp_{\beta}(\varphi_1 - \hat{\varphi}_1) \exp_{-\beta}(\hat{\varphi}_2) \exp_{\alpha}(\varphi_2 - \hat{\varphi}_2) \times \Phi(\hat{\varphi}_1 + \hat{\varphi}_2, \varphi_1 + \varphi_2 - \hat{\varphi}_1 - \hat{\varphi}_2) \]
for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in M$. Let $A \equiv \mathcal{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{M}$ and $\mathfrak{S}(A)$ be the set of all normal states on $A$. $\mathcal{E}_{\alpha,\beta} : A \otimes A \rightarrow A \otimes A$ defined by
\[ \mathcal{E}_{\alpha,\beta}(C) \equiv V_{\alpha,\beta}^* C V_{\alpha,\beta}, \quad \forall C \in A \otimes A \]
is the lifting in the sense of Accardi and Ohya [1] and the dual map $\mathcal{E}_{\alpha,\beta}^*$ of $\mathcal{E}_{\alpha,\beta}$ given by
\[ \mathcal{E}_{\alpha,\beta}^*(\omega)(\bullet) \equiv \omega(\mathcal{E}_{\alpha,\beta}(\bullet)), \quad \forall \omega \in \mathfrak{S}(A \otimes A) \]
is the CP channel from $\mathfrak{S}(A \otimes A)$ to $\mathfrak{S}(A \otimes A)$. Using the exponental vectors, one can denote a coherent state $\theta^f$ by
\[ \theta^f(A) \equiv \langle \exp_f, A \exp_f \rangle e^{-\|f\|^2}, \quad \forall f \in L^2(G, \nu), \forall A \in A. \]
In particular, for the input coherent states $\eta_0 \otimes \omega_0 = \theta^f \otimes \theta^g$, two output states $\omega_1 (\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta} ((\bullet) \otimes I))$ and $\eta_1 (\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta} (I \otimes (\bullet)))$ are obtained by

$$\omega_1 = \theta^{\alpha f + \beta g}, \quad \eta_1 = \dot{\theta}^{-\overline{\beta} f + \overline{\alpha} g}.$$  

$\mathcal{E}_{\alpha,\beta}^*$ is called a generalized beam splitting on Fock space because it also holds the same properties satisfied by the generated beam splitting $\Pi^*$ in Section 1.

Now we introduce a self-adjoint unitary operator $\tilde{U}$, which denotes a new device instead of the Kerr medium, defined by

$$\tilde{U} (\Phi) (\varphi_1, \varphi_2) \equiv (-1)^{||\varphi_1|| ||\varphi_2||} \Phi (\varphi_1, \varphi_2)$$  

for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in G$, where $||\varphi_k|| \equiv \varphi_k (G)$ ($k = 1, 2$). For the input state $\omega_1 \otimes \kappa \equiv \theta^f \otimes \frac{1}{||\psi||^2} \langle \psi, \bullet \psi \rangle$, the output state $\omega_2$ of new device is

$$\omega_2 (A) \equiv \omega_1 \otimes \kappa (\tilde{U} (A \otimes I) \tilde{U}) = \frac{1}{||\psi||^2} \int_M F (d\varphi) |\psi (\varphi)|^2 \theta \left(\frac{-1}{||\psi||^2} f \right) (A)$$  

for any $A \in \mathcal{A}$, $\psi \in \mathcal{M}$ ($\psi \neq 0$) and $f \in L^2 (G, \nu)$ . If $\kappa$ is given by the vacuum state $\theta^0$, then the output state $\omega_2$ is equal to $\omega_1$ and if $\kappa$ is given by one particle state, that is, $\kappa = \frac{1}{||\psi||^2} \langle \psi, \bullet \psi \rangle$ with $\psi \mid_{M_1}$ (where $M_1$ is the set of one-particle states), then $\omega_2$ is obtained by $\theta^{-f}$. Let $M_o$ (resp. $M_e$) be the set of $\varphi \in M$ which satisfies that $||\varphi||$ is odd (resp. even) and $M$ be the union of $M_o$ and $M_e$. The output states $\omega_2$ of the new device is written by

$$\omega_2 (A) = \lambda_1 \theta^{-f} (A) + \lambda_2 \theta^f (A) \quad \forall A \in \mathcal{A},$$

where $\lambda_1$ and $\lambda_2$ are given by

$$\begin{align*}
\lambda_1 &= \frac{1}{||\psi||^2} \int_{M_o} F (d\varphi) |\psi (\varphi)|^2, \\
\lambda_2 &= \frac{1}{||\psi||^2} \int_{M_e} F (d\varphi) |\psi (\varphi)|^2.
\end{align*}$$

Two output states $\omega_3 (\bullet) \equiv \omega_2 \otimes \eta_2 (\mathcal{E}_{\alpha_2,\beta_2} ((\bullet) \otimes I))$ and $\eta_3 (\bullet) \equiv \omega_2 \otimes \eta_2 (\mathcal{E}_{\alpha_2,\beta_2} (I \otimes (\bullet)))$ of the total logical gate including two beam splittings $\mathcal{E}_{\alpha_k,\beta_k}^*$ with $||\alpha_k||^2 + ||\beta_k||^2 = 1$ ($k = 1, 2$) and the new device instead of Kerr medium are obtained by

$$\begin{align*}
\omega_3 &= \lambda_1 \theta^{\alpha_2 (- (\alpha_1 f + \beta_1 g) + \beta_2 (- \beta_1 f + \alpha_1 g))} + \lambda_2 \theta^{\alpha_2 (\alpha_1 f + \beta_1 g) + \beta_2 (- \beta_1 f + \alpha_1 g)}, \\
\eta_3 &= \lambda_1 \theta^{- \beta_2 (- (\alpha_1 f + \beta_1 g) + \alpha_2 (- \beta_1 f + \alpha_1 g))} + \lambda_2 \theta^{- \beta_2 (\alpha_1 f + \beta_1 g) + \alpha_2 (- \beta_1 f + \alpha_1 g)}.
\end{align*}$$
where $\omega_2 = \lambda_1 \theta^{-(\alpha_1 f + \beta_1 g)} + \lambda_2 \theta^{\alpha_1 f + \beta_1 g}$ and $\eta_2 = \eta_1 = \theta^{-\overline{\beta}_1 f + \overline{\alpha}_1 g}$.

Based on the above settings, we could show that new logical gate performs the complete truth table. The further development of our study will be appear in [11].

References


