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On James and Schäffer constants for Banach spaces

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We introduce James and Schaffer type constants for Banach spaces X, and investigate the relation between these constants and some geometrical properties of Banach spaces.

Let X be a Banach space with dim $X \geqq 2$. Then, geometrical properties of X are determined by its unit ball $B_X = \{ x \in X : \|x\| \leqq 1 \}$ or its unit sphere $S_X = \{ x \in X : \|x\| = 1 \}$. The modulus of convexity of X is a function δ_X : $[0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2 : x, y \in S_X, \|x-y\| = \varepsilon\}$$

In the above definition, it is well-known that S_X may be replaced by B_X. The space X is called uniformly convex (Clarkson [1]) if $\delta_X(\varepsilon)>0$ for all $0 < \varepsilon < 2$, and called uniform non-square (James [5]) if $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$.

James and Schäffer constants:

James constant of X is defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

and Schäffer constant of X is defined by

$$S(X) = \inf\{\max(\|x+y\|, \|x-y\|) : x, y \in S_X\}.$$

Known Facts (cf. [3], [4], [7]):

(1) In the definition of $J(X)$, S_X may be replaced by B_X.

(2) $J(X)S(X) = 2$

(3) X : unif. non-square $\Leftrightarrow J(X)<2 \Leftrightarrow S(X)>1$

(4) Let $1 \leqq p \leqq \infty$, $1/p+1/p' = 1$, $t = \min\{p,p'\}$ and $s = \max\{p,p'\}$. Then, $J(L_p) = 2^{1/t}$ and $S(L_p) = 2^{1/s}$.

(5) $\sqrt{2} \leqq J(X) \leqq 2$ and $1 \leqq S(X) \leqq \sqrt{2}$ for any Banach space X.

(6) If X is a Hilbert space, then $J(X)=\sqrt{2}$, but the converse is not true.

(7) There is a Banach space X such that $J(X) \neq J(X^*)$ ($S(X) \neq S(X^*)$), where X^* is a dual space of X.

(8) $2J(X) - 2 \leqq J(X^*) \leqq J(X)/2 + 1$ for any Banach space X.

New constants of James and Schäffer type :

We denote by $M_t(a,b)$ the power means of order t of the positive real numbers a and b, that is,

$$M_t(a,b) = \{(a^t+b^t)/2\}^{1/t} \quad (t\neq 0) \quad \text{and} \quad M_0(a,b) = (ab)^{1/2}$$

Remark. (1) $M_t(a,b)$ is defined for $a, b \geqq 0$ ($M_t(a,b)=0$ if $t<0$, $ab=0$).

(2) If $t \to -\infty$ ($t \to +\infty$), then $M_t(a,b) \to \min\{a,b\}$ ($M_t(a,b) \to \max\{a,b\}$).

James type constants:

$$J_t(X) = \sup\{ M_t(\|x+y\|, \|x-y\|) : x,y \in S_X \}, \quad -\infty<t<+\infty$$

Schäffer type constants:

$$S_t(X) = \inf\{ M_t(\|x+y\|, \|x-y\|) : x,y \in S_X \}, \quad -\infty<t<+\infty$$

Remark. In the definition of $J_t(X)$, S_X may be replaced by B_X.

Proposition 1. (1) $\sqrt{2} \leqq J(X) \leqq J_t(X) \leqq 2$ for all $t \in (-\infty,+\infty)$, and if $t \geqq 2$, then $J_t(X) \geqq 2^{1-1/t}$.

(2) $J_t(X)$ is non-decreasing on $(-\infty,+\infty)$, $J_t(X) \to 2$ if $t \to +\infty$, and $J_t(X) \to J(X)$ if $t \to -\infty$.

(3) $S_t(X)=0$ if $t \leqq 0$, $S_t(X)=2^{1-1/t}$ if $0<t\leqq 1$, $S_t(X) \leqq 2^{1-1/t}$ for all $t<\infty$, and

$$1 \leqq S_t(X) \leqq S(X) \leqq \sqrt{2} \text{ for all } t \in (1,+\infty).$$

(4) $S_t(X)$ is non-decreasing on $(-\infty,+\infty)$, $S_t(X) \to 1$ if $t \to 1+0$, and $S_t(X) \to S(X)$ if $t \to +\infty$.

Theorem 2. The following assertions are equivalent:

(1) X is uniformly non-square.

(2) $J_t(X)<2$ for all t (some t).

(3) $J(X)<J_t(X)$ for some t.

(4) There exists t_0 such that $J_t(X)$ is strictly increasing on $[t_0,+\infty)$.

(5) $S_t(X)>1$ for all $t>1$ (some $t>1$).

(6) $S(X)>S_t(X)$ for some $t>1$.

Let $1 \leqq p \leqq 2$ and $1/p + 1/p' = 1$. We say that the (p,p')-Clarkson inequality holds in a Banach space X if for any x, y $\in$ X, the inequality

$$(CI_p) \qquad (\|x+y\|^{p'} + \|x-y\|^{p'})^{1/p'} \leqq 2^{1/p'}(\|x\|^p + \|y\|^p)^{1/p}$$

holds.

Remark. Let $1 \leqq p \leqq 2$.

(1) (CI_p) holds in L_p and $L_{p'}$ (Clarkson [1]).

(2) (CI_p) holds in X if and only if it holds in X^*; if (CI_p) holds in X, then (CI_t) holds in X for any $t \in [1,p]$; and if (CI_p) holds in X, then (CI_t) holds in $L_r(X)$, where $1 \leqq r \leqq \infty$ and $t = \min\{p,r,r'\}$ (Takahashi and Kato [9]).

A Banach space Y is said to be finitely representable (f.r.) in a Banach space X if for any $\lambda > 1$ and for any finite dimensional subspace F of Y there is a finite dimensional subspace E of X with dim E = dim F such that the Banach-Mazur distance $d(E,F) \leqq \lambda$.

Proposition 3. If Y is f.r. in X, then $J_t(Y) \leqq J_t(X)$ and $S_t(Y) \geqq S_t(X)$ for any t.

Theorem 4. Let $1 < p \leqq 2$ and suppose that the (p,p')-Clarkson inequality holds in X.

(1) $J_t(X) = 2^{1-1/t}$ for $t \geqq p'$, and $S_t(X) = 2^{1-1/t}$ for $0 < t \leqq p$.

(2) If ℓ_p (or $\ell_{p'}$) is finitely representable (f.r.) in X, then $J_t(X) = 2^{1/p}$ for $t \leqq p'$, and $S_t(X) = 2^{1/p'}$ for $t \geqq p$.

Corollary 1. (1) $J_t(H) = \sqrt{2}$ if $t \leqq 2$, $J_t(H) = 2^{1-1/t}$ if $t \geqq 2$, $S_t(X) = 2^{1-1/t}$ if $0 < t \leqq 2$, and $S_t(X) = \sqrt{2}$ if $t \geqq 2$, where H is a Hilbert space.

(2) $J_t(L_p) = 2^{1/r}$ if $t \leqq r'$, $J_t(L_p) = 2^{1-1/t}$ if $t \geqq r'$, $S_t(L_p) = 2^{1-1/t}$ if $0 < t \leqq r$, and $S_t(L_p) = 2^{1/r'}$ if $t \geqq r$, where $r = \min\{p,p'\}$.

(3) Let $X = L_p(L_q)$, and $r = \min\{p,p',q,q'\}$. Then $J_t(X) = 2^{1/r}$ if $t \leqq r'$, $J_t(X) = 2^{1-1/t}$ if $t > r'$, $S_t(X) = 2^{1-1/t}$, and $S_t(X) = 2^{1/r'}$ if $t \geqq r$.

Corollary 2. Let $X = L_p(L_q)$, $1 < p, q < \infty$. Then, $J(X) = 2^{1/r}$ and $S(X) = 2^{1/r'}$, where $r = \min\{p, p', q, q'\}$.

Remark. As already mentioned, for any Banach space X, it holds $J(X)S(X) = 2$, $J_t(X) \to J(X)$ if $t \to -\infty$, and $S_t(X) \to S(X)$ if $t \to +\infty$. By Corollary 1, we know that for various Banach spaces X, $J_t(X)S_{t'}(X) = 2$, where $1<t<\infty$ and $1/t + 1/t' = 1$. Note that for any t ($1<t<\infty$), there is a Banach space X such that $J_t(X)S_t(X) \neq 2$.

Now we give a characterization of a Hilbert space. As mentioned before, if X is a Hilbert space, then $J(X) = \sqrt{2}$; but the converse is not true.

Theorem 5. A Banach space X is isometric to a Hilbert space if and only if $J_2(X) = \sqrt{2}$.

Remark. Let $C_{NJ}(X)$ denote the von Neumann-Jordan constant of X (Clarkson [2]). Then it is easy to see that $\sqrt{2} \leqq J_2(X) \leqq \sqrt{2C_{NJ}(X)}$ for any Banach space X. Hence, Theorem 5 generalizes a result of Jordan and von Neumann [6], which asserts that X is a Hilbert space if and only if $C_{NJ}(X) = 1$.

Proposition 6. Let X be a Banach space. If there is $t \in [2, \infty)$ such that $J_t(X) = 2^{1-1/t}$, then X is uniformly convex.

Remark. For any Banach space X, we have $J_t(X) \geqq 2^{1-1/t}$ for all $t \geqq 2$ (see, Proposition 1). It can be shown that for any $\varepsilon > 0$, there is a Banach space X which is not uniformly convex such that $J_t(X) < 2^{1-1/t} + \varepsilon$.

Theorem 7. (1) For any Banach space X, $J_1(X) = J_1(X^*)$.

(2) For any $t < 1$, there is a Banach space X such that $J_t(X) \neq J_t(X^*)$.

Corollary 3. X : uniformly non-square $\Leftrightarrow$ X^* : uniformly non-square

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