On James and Schäffer constants for Banach spaces

We introduce James and Schäffer type constants for Banach spaces $X$, and investigate the relation between these constants and some geometrical properties of Banach spaces.

Let $X$ be a Banach space with $\dim X \geq 2$. Then, geometrical properties of $X$ are determined by its unit ball $B_X = \{ x \in X : \| x \| \leq 1 \}$ or its unit sphere $S_X = \{ x \in X : \| x \| = 1 \}$. The modulus of convexity of $X$ is a function $\delta_X : [0,2] \rightarrow [0,1]$ defined by

$$\delta_X(\epsilon) = \inf \{ 1 - \frac{\| x+y \|}{2} : x, y \in S_X, \| x-y \| = \epsilon \}$$

In the above definition, it is well-known that $S_X$ may be replaced by $B_X$. The space $X$ is called uniformly convex (Clarkson [1]) if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon < 2$, and called uniform non-square (James [5]) if $\delta_X(\epsilon) > 0$ for some $0 < \epsilon < 2$.

James and Schäffer constants:

James constant of $X$ is defined by

$$J(X) = \sup \{ \min( \| x+y \|, \| x-y \|) : x, y \in S_X \}$$

and Schäffer constant of $X$ is defined by

$$S(X) = \inf \{ \max( \| x+y \|, \| x-y \|) : x, y \in S_X \}.$$  

Known Facts (cf. [3], [4], [7]):

1. In the definition of $J(X)$, $S_X$ may be replaced by $B_X$.
2. $J(X)S(X) = 2$
3. $X : \text{unif. non-square} \iff J(X) < 2 \iff S(X) > 1$
4. Let $1 \leq p \leq \infty$, $1/p + 1/p' = 1$, $t = \min\{p, p'\}$ and $s = \max\{p, p'\}$. Then, $J(L_p) = 2^{1/t}$ and $S(L_p) = 2^{1/s}$.
5. $\sqrt{2} \leq J(X) \leq 2$ and $1 \leq S(X) \leq \sqrt{2}$ for any Banach space $X$.
6. If $X$ is a Hilbert space, then $J(X) = \sqrt{2}$, but the converse is not true.
(7) There is a Banach space $X$ such that $J(X) \neq J(X^*) \ (S(X) \neq S(X^*))$,
where $X^*$ is a dual space of $X$.

(8) $2J(X) - 2 \leq J(X^*) \leq J(X)/2 + 1$ for any Banach space $X$.

New constants of James and Schäffer type:

We denote by $M_t(a,b)$ the power means of order $t$ of the positive real numbers $a$ and $b$, that is,

$$M_t(a,b) = \left(\frac{a^t + b^t}{2}\right)^{1/t} \quad (t \neq 0) \quad \text{and} \quad M_0(a,b) = (ab)^{1/2}$$

Remark. (1) $M_t(a,b)$ is defined for $a, b \geq 0 \ (M_t(a,b) = 0$ if $t < 0, ab = 0$).

(2) If $t \to -\infty \ (t \to +\infty)$, then $M_t(a,b) \to \min\{a,b\}$ $(M_t(a,b) \to \max\{a,b\})$.

James type constants:

$$J_t(X) = \sup\{M_t(\|x+y\|, \|x-y\|) : x,y \in S_X\}, \quad -\infty < t < +\infty$$

Schäffer type constants:

$$S_t(X) = \inf\{M_t(\|x+y\|, \|x-y\|) : x,y \in S_X\}, \quad -\infty < t < +\infty$$

Remark. In the definition of $J_t(X)$, $S_X$ may be replaced by $B_X$.

Proposition 1. (1) $\sqrt{2} \leq J(X) \leq J_t(X) \leq 2$ for all $t \in (-\infty, +\infty)$, and if $t \geq 2$, then $J_t(X) \geq 2^{1-t}$.  

(2) $J_t(X)$ is non-decreasing on $(-\infty, +\infty)$, $J_t(X) \to 2$ if $t \to +\infty$, and $J_t(X) \to J(X)$ if $t \to -\infty$.

(3) $S_t(X) = 0$ if $t \leq 0$, $S_t(X) = 2^{1-t}$ if $0 < t \leq 1$, $S_t(X) \leq 2^{1-t}$ for all $t < \infty$, and $1 \leq S_t(X) \leq S(X) \leq \sqrt{2}$ for all $t \in (1, +\infty)$.

(4) $S_t(X)$ is non-decreasing on $(-\infty, +\infty)$, $S_t(X) \to 1$ if $t \to 1+$, and $S_t(X) \to S(X)$ if $t \to +\infty$.

Theorem 2. The following assertions are equivalent:

(1) $X$ is uniformly non-square.

(2) $J_t(X) < 2$ for all $t \ (\text{some } t)$.

(3) $J(X) < J_t(X)$ for some $t$.

(4) There exists $t_0$ such that $J_t(X)$ is strictly increasing on $[t_0, +\infty)$.

(5) $S_t(X) > 1$ for all $t > 1 \ (\text{some } t > 1)$.

(6) $S(X) > S_t(X)$ for some $t > 1$. 
Let $1 \leq p \leq 2$ and $1/p + 1/p' = 1$. We say that the $(p,p')$-Clarkson inequality holds in a Banach space $X$ if for any $x$, $y \in X$, the inequality

$$(\text{CI}_p) \quad (\|x + y\|^{p'} + \|x - y\|^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|^{p'} + \|y\|^{p'})^{1/p}$$

holds.

Remark. Let $1 \leq p \leq 2$.

(1) $(\text{CI}_p)$ holds in $L_p$ and $L_{p'}$ (Clarkson [1]).

(2) $(\text{CI}_p)$ holds in $X$ if and only if it holds in $X^*$; if $(\text{CI}_p)$ holds in $X$, then $(\text{CI}_t)$ holds in $X$ for any $t \in [1,p]$; and if $(\text{CI}_p)$ holds in $X$, then $(\text{CI}_t)$ holds in $L_r(X)$, where $1 \leq r \leq \infty$ and $t = \min\{p,r,r'\}$ (Takahashi and Kato [9]).

A Banach space $Y$ is said to be finitely representable (f.r.) in a Banach space $X$ if for any $\lambda > 1$ and for any finite dimensional subspace $E$ of $Y$ there is a finite dimensional subspace $F$ of $X$ with $\dim E = \dim F$ such that the Banach-Mazur distance $d(E,F) \leq \lambda$.

Proposition 3. If $Y$ is f.r. in $X$, then $J_t(Y) \leq J_t(X)$ and $S_t(Y) \geq S_t(X)$ for any $t$.

Theorem 4. Let $1 < p \leq 2$ and suppose that the $(p,p')$-Clarkson inequality holds in $X$.

(1) $J_t(X) = 2^{1-1/t}$ for $t \geq p'$, and $S_t(X) = 2^{1-1/t}$ for $0 < t \leq p$.

(2) If $\mathcal{E}_p$ (or $\mathcal{E}_{p'}$) is finitely representable (f.r.) in $X$, then $J_t(X) = 2^{1/p}$ for $t \leq p'$, and $S_t(X) = 2^{1/p'}$ for $t \geq p$.

Corollary 1. (1) $J_t(H) = \sqrt{2}$ if $t \leq 2$, $J_t(H) = 2^{1-1/t}$ if $t \geq 2$, $S_t(X) = 2^{1-1/t}$ if $0 < t \leq 2$, and $S_t(X) = \sqrt{2}$ if $t \geq 2$, where $H$ is a Hilbert space.

(2) $J_t(L_p) = 2^{1/r}$ if $t \leq r'$, $J_t(L_p) = 2^{1-1/t}$ if $t \geq r'$, $S_t(L_p) = 2^{1-1/t}$ if $0 < t \leq r$, and $S_t(L_p) = 2^{1/r'}$ if $t \geq r$, where $r = \min\{p,p'\}$.

(3) Let $X = L_p(L_q)$, and $r = \min\{p,p',q,q'\}$. Then $J_t(X) = 2^{1/r}$ if $t \leq r'$, $J_t(X) = 2^{1-1/t}$ if $t > r'$, $S_t(X) = 2^{1-1/t}$, and $S_t(X) = 2^{1/r'}$ if $t \geq r$. 
Corollary 2. Let $X = L_p(L_q)$, $1 < p, q < \infty$. Then, $J(X) = 2^{1/\tau}$ and $S(X) = 2^{1/\tau'}$, where $\tau = \min\{p, p', q, q'\}$.

Remark. As already mentioned, for any Banach space $X$, it holds $J(X)S(X) = 2$, $J_t(X) \to J(X)$ if $t \to -\infty$, and $S_t(X) \to S(X)$ if $t \to +\infty$. By Corollary 1, we know that for various Banach spaces $X$, $J_t(X)S_t(X) = 2$, where $1 < t < \infty$ and $1/t + 1/t' = 1$. Note that for any $t (1 < t < \infty)$, there is a Banach space $X$ such that $J_t(X)S_t(X) \neq 2$.

Now we give a characterization of a Hilbert space. As mentioned before, if $X$ is a Hilbert space, then $J(X) = \sqrt{2}$; but the converse is not true.

Theorem 5. A Banach space $X$ is isometric to a Hilbert space if and only if $J_2(X) = \sqrt{2}$.

Remark. Let $C_NJ(X) \text{ denote the von Neumann-Jordan constant of } X \text{ (Clarkson [2]). Then it is easy to see that } \sqrt{2} \leq J_2(X) \leq \sqrt{2C_NJ(X)} \text{ for any Banach space } X. \text{ Hence, Theorem 5 generalizes a result of Jordan and von Neumann [6], which asserts that } X \text{ is a Hilbert space if and only if } C_NJ(X) = 1.$

Proposition 6. Let $X$ be a Banach space. If there is $t \in [2, \infty)$ such that $J_t(X) = 2^{1-1/t}$, then $X$ is uniformly convex.

Remark. For any Banach space $X$, we have $J_t(X) \geq 2^{1-1/t}$ for all $t \geq 2$ (see, Proposition 1). It can be shown that for any $\varepsilon > 0$, there is a Banach space $X$ which is not uniformly convex such that $J_t(X) < 2^{1-1/t} + \varepsilon$.

Theorem 7. (1) For any Banach space $X$, $J_1(X) = J_1(X^*)$.
(2) For any $t < 1$, there is a Banach space $X$ such that $J_t(X) \neq J_t(X^*)$.

Corollary 3. $X$: uniformly non-square $\iff X^*$: uniformly non-square
References