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Space of Fuzzy Measures

(ファジィ測度の空間)

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1 Introduction

In this paper, we deal with fuzzy measures in the sence of Sugeno[14]. That is, a fuzzy measure μ is a nonnegative real valued set function defined on $\sigma-$algebra $\mathcal{X}$ with the properties $\mu(\emptyset)=0$ and $A\subset B\Rightarrow\mu(A)\leq\mu(B)$ for $A, B\in \mathcal{X}$. We consider the space $\mathcal{FM}$ of fuzzy measures, that is, the linear space generated by the set of fuzzy measures. The element of $\mathcal{FM}$ is a non monotonic fuzzy measure [8] of bounded variation. The variation of non monotonic set functions is defined by Aumann and Shapley[1] in the context of game theory. The total variation is a norm in $\mathcal{FM}$. $\mathcal{T}_{BV}$ denotes the topology of the variation norm.

The Choquet integral [3, 7] of a nonnegative measurable function f with respect to a non monotonic fuzzy measure μ is defined by

$$(C)\int fd\mu=\int_{0}^{\infty}\mu(\{x|f(x)\geq a\})da.$$

Fuzzy measure and Choquet integral are basic tools for multicriteria decision making, image processing and recognition [4, 5].

Using the Choquet integral, we introduce the topologies $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{T}_{B+}$ in the space of fuzzy measure $\mathcal{FM}$. The concept of topology is equivalent to the concept of convergence. The convergence of the net of fuzzy measures can be considered in several ways. We discuss the relation and their difference among three types of convergence.

In section 2, we define the space $\mathcal{FM}$ of fuzzy measures and show the preliminary propositions. We also define the variation, two topologies $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{T}_{B+}$.

In section 3, we consider the space $\mathcal{FM}$ and the relation of three convergence. We show that the three convergence are different from each other in the general situation.

In section 4, we consider the space $\mathcal{FM}^+$ of monotone fuzzy measures and its relative topology. Unlike the previous result, we have $\mathcal{T}_{\mathcal{X}} = \mathcal{T}_{B+}$. But it remains that $\mathcal{T}_{\mathcal{X}} \neq \mathcal{T}_{BV}$.

In section 5, we suppose that the universal set X is a finite set. We show that three types of convergence are same in this situation. This means that three topologies coincide, that is , $\mathcal{T}_{B+} = \mathcal{T}_{\mathcal{X}} = \mathcal{T}_{BV}$.

In section 6, we define $0-\alpha$ fuzzy measure generated by $0-\alpha$ necessity measures. We show that every fuzzy measure can be represented by the linear combination of $0-\alpha$ fuzzy measures generated by $0-\alpha$ necessity fuzzy measures.

2 Space of fuzzy measures

In this section, we show some preliminary definitions and propositions.

Definition 2.1. *Let $(X, \mathcal{X})$ be a measurable space. A non monotonic fuzzy measure is a real valued set function on $\mathcal{X}$ with $\mu(\emptyset) = 0$. We say that $(X, \mathcal{X}, \mu)$ is a non monotonic fuzzy measure space when μ is a non monotonic fuzzy measure.*

Definition 2.2. *Let $(X, \mathcal{X}, \mu)$ be a non monotonic fuzzy measure space.*

The positive variation $\mu^+(A)$ *of* μ *on the set* $A \in \mathcal{X}$ *is given by*

$$\mu^+(A) = \sup\{\sum_{i=1}^{n} \max\{\mu(A_i) - \mu(A_{i-1}), 0\}\}$$

where the sup is taken over all non decreasing sequence $\emptyset = A_0 \subset A_{\mathcal{X}} \subset \cdots \subset A_n = A, A_i \in \mathcal{X}, i = 1, 2, \cdots n$, *the negative variation* $\mu^-(A)$ *of* μ *on the set* $A \in \mathcal{X}$ *is given by*

$$\mu^-(A) = \sup\{\sum_{i=1}^{n} \max\{\mu(A_{i-1}) - \mu(A_i), 0\}\}$$

where the sup is taken over all non decreasing sequence $\emptyset = A_0 \subset A_{\mathcal{X}} \subset \cdots \subset A_n = A, A_i \in \mathcal{X}, i = 1, 2, \cdots n$ *and the total variation* $|\mu|(A)$ *of* μ *on the set* $A \in \mathcal{X}$ *is given by*

$$|\mu|(A) = \mu^+(A) + \mu^-(A).$$

We denote the variation $|\mu|(X)$ *by* $\|\mu\|$, *and say that* μ *is of bounded variation if* $\|\mu\| < \infty$.

We define $\mathcal{FM}^+ := \{\mu | \mu : \mathcal{X} \longrightarrow R^+, \mu \text{ is a fuzzy measure}\}$ $(a\mu)(A) = a(\mu(A))$, $(\mu + \nu)(A) = \mu(A) + \nu(A)$, $(\mu - \nu)(A) = \mu(A) - \nu(A)$ for $\mu, \nu \in \mathcal{FM}^+$, $a \in R$, and $\mathcal{FM} = \{\mu - \nu | \mu, \nu \in \mathcal{FM}^+\}$. Then $\mathcal{FM}^+$ is a positive cone, and $\mathcal{FM}$ is a linear space.

Proposition 2.3. [1] *Let* μ *be a non monotonic fuzzy measure. Then* μ *is of bounded variation if and only if* $\mu \in \mathcal{FM}$.

The variation $\|\cdot\|$ is a norm on $\mathcal{FM}$. We say $\|\cdot\|$ BV-norm. Let (μ_i) be a net in $\mathcal{FM}$. If μ_i converges to μ with respect to BV-norm, we write $\mu_i \longrightarrow^{BV} \mu$.

Definition 2.4. *Let* f *be a nonnegative measurable function. We define the map* $C_f : \mathcal{FM} \longrightarrow R$ *by* $C_f(\mu) = (C) \int f d\mu$. *We define* $C_A = C_{1_A}$ *for* $X \in \mathcal{X}$.

We denote the set of bounded nonnegative measurable functions by B^+.

It is obvious that C_f is a linear map on $\mathcal{FM}$ for all $f \in B^+$.

Definition 2.5. *We shall say that the coarsest topology for which every C_A is continuous for $A \in \mathcal{X}$ is $\mathcal{X}$ - topology for $\mathcal{FM}$, and that the coarsest topology for which every C_f is continuous for $f \in B^+$ is B^+ - topology for $\mathcal{FM}$.*

Let (μ_i) be a net in $\mathcal{FM}$. If μ_i converges to μ with respect to $\mathcal{X}$- topology, we write $\mu_i \longrightarrow^{\mathcal{X}} \mu$. If μ_i converges to μ with respect to B^+- topology, we write $\mu_i \longrightarrow^{B^+} \mu$.

Lemma 2.6. [6] *Let $(\mu_i)_{i \in I}$ be a net in $\mathcal{FM}$.*

(1) $\mu_i \longrightarrow^{\mathcal{X}} \mu$ *if and only if* $\mu_i(A) \longrightarrow \mu(A)$ *for all* $A \in \mathcal{X}$.

(2) $\mu_i \longrightarrow^{B^+} \mu$ *if and only if* $C_f(\mu_i) \longrightarrow C_f(\mu)$ *for all* $f \in B^+$.

3 General theory

In this section, we consider the space $\mathcal{FM}$ and the relations of three type of convergence. The next theorem follows from the definition and Lemma 2.6.

Theorem 3.1. *Let (μ_i) be a net in $\mathcal{FM}$.*

(1) $\mu_i \longrightarrow^{BV} \mu$ *implies* $\mu_i \longrightarrow^{B^+} \mu$.

(2) $\mu_i \longrightarrow^{B^+} \mu$ *implies* $\mu_i \longrightarrow^{\mathcal{X}} \mu$

The converse of (i) is not always true.

Example 1. *Let $X = [0,1]$, λ the Lebesgue measure on X, and $\mathcal{X}$ be the class of Borel subsets of X.*

We define the set function on $\mathcal{X}$ by

$$\mu_n(A) = \begin{cases} n^2 & \text{if } \lambda(A) = \dfrac{k}{n^2} \\ 0 & \text{if otherwise} \end{cases}$$

for $k = 1, 2, 3, \cdots, n$ *and* $A \in \mathcal{X}$.

Then we have

$$\mu_n^+(A) = \begin{cases} n^3 & \text{if } \lambda(A) \geq \dfrac{1}{n} \\ kn^2 & \text{if } \dfrac{k}{n^2} \leq \lambda(A) < \dfrac{k+1}{n^2} \end{cases}$$

and

$$\mu_n^-(A) = \begin{cases} n^2 & \text{if } \lambda(A) > \dfrac{1}{n} \\ kn & \text{if } \dfrac{k}{n^2} < \lambda(A) \leq \dfrac{k+1}{n^2} \\ 0 & \text{if } \lambda(A) = 0 \end{cases}$$

for $k = 0, 1, 2, 3, \cdots, n$ *and* $A \in \mathcal{X}$.

We have $\mu_n \in \mathcal{FM}$. *Let* $A \in \mathcal{X}$. *If* $\lambda(A) = 0$ *then* $\mu_n(A) = 0$ *for every natural number* n. *If* $\lambda(A) > 0$, *there exists a natural number* n_0 *such that* $\lambda(A) > \dfrac{1}{n_0}$. *It follows from the definition of* μ_n *that* $n \geq n_0$ *imply* $\mu_n(A) = 0$. *Therefore we have* $\mu_n(A) \longrightarrow 0$ *as* $n \longrightarrow \infty$ *for all* $A \in \mathcal{X}$. *Define* $\mu(A) = 0$ *for all* $A \in \mathcal{X}$. *We have* $\mu_n \longrightarrow^{\mathcal{X}} \mu$ *as* $n \longrightarrow \infty$. *It is obvious that* $\mu \in \mathcal{FM}$.

Let

$$f(x) = \begin{cases} \dfrac{n}{n+1} & \text{if} \quad \dfrac{1}{(n+1)^2} \leq x < \dfrac{1}{n^2} \\ 0 & \text{if} \quad x = 0 \text{ or } x = 1 \end{cases}$$

for $x \in X$ *and* $n = 1, 2, 3, \cdots$. *It is obvious that* $f \in B^+$. *Let* A_n *denote* $A_n := \{x | f(x) \geq \dfrac{n}{n+1}\}$ *for* $n = 1, 2, 3, \cdots$. *It follows from* $A_n = [0, \dfrac{1}{n^2}]$ *that* $\lambda(A_n) = \dfrac{1}{n^2}$ *and* $\mu_n(A_n) = n^2$. *Suppose that* p *is a prime number, we have* $\mu_p(A_m) = 0$ *for a positive number* m *such that* $m \neq p$.

Then we have

$$C_f(\mu_p) = 1 - \frac{1}{p+1}$$

if p *is a prime number.*

Since $\mu \equiv 0$, *we have* $C_f(\mu) = 0$. *This fact shows that* $\mu_n \longrightarrow^{\mathcal{X}} \mu$ *as* $n \longrightarrow \infty$ *but* $\mu_n \not\longrightarrow^{B^+} \mu$ *as* $n \longrightarrow \infty$.

The converse of (ii) is also not always true.

Example 2. *Let* $X = [0,1]$, $\mathcal{X}$ *be the class of Borel subsets of* X *and* $A_n = (\frac{1}{n+1}, \frac{1}{n})$. *Define the sequence of set functions* $\mu_n : \mathcal{X} \longrightarrow [0,1]$ *by*

$$\mu_n(A) = \begin{cases} 1 & \textit{if } A = A_n \\ 0 & \textit{if otherwise} \end{cases}$$

for $k = 1, 2, 3, \cdots, n$ *and* $A \in \mathcal{X}$.

It follows from Definition 2.2

$$\mu_n^+(A) = \begin{cases} 1 & \textit{if } A_n \subset A \\ 0 & \textit{if otherwise} \end{cases}$$

and

$$\mu_n^-(A) = \begin{cases} 1 & \textit{if } A_n \subset A, A_n \neq A \\ 0 & \textit{if otherwise} \end{cases}$$

for $k = 0, 1, 2, 3, \cdots, n$ *and* $A \in \mathcal{X}$.

Therefore we have $\mu_n \in \mathcal{FM}$ *for all* $n \in N$.

Let $f \in B^+$. *Suppose that there exists* $a > 0$ *and* $n \in N$ *such that* $A_n = \{x|f(x) \geq a\}$. *Let* $m > n$. *Suppose that there exists* $b > 0$ *such that* $A_m = \{x|f(x) \geq b\}$, *then we have* $\{x|f(x) \geq a\} \subset \{x|f(x) \geq b\}$ *or* $\{x|f(x) \geq b\} \subset \{x|f(x) \geq a\}$, *and* $A_n \cap A_m = \emptyset$. *This is contradictory. Therefore we have*

$$(C) \int f d\mu_m = 0$$

for all $m > n$. *This means* $\mu_n \longrightarrow^{B+} 0$. *On the other hand, we have* $||\mu_n|| = 2$ *for all* $n \in N$. *That is,* $\mu \not\longrightarrow^{BV} 0$.

4 Space of monotone fuzzy measure

In this section, we consider the space of monotone fuzzy measure $\mathcal{FM}^+$ and three type of its relative topology. Unlike the previous result, the convergence with respect to $\mathcal{X}$ coincide with the convergence with respect to B^+.

Theorem 4.1. [9] *Let (μ_i) be a net in $\mathcal{FM}^+$ and consider the relative topology to $\mathcal{FM}^+$. Then $\mu_i \longrightarrow^{\mathcal{X}} \mu$ implies $\mu_i \longrightarrow^{B^+} \mu$.*

Even if we restrict the topology to $\mathcal{FM}^+$, the convergence with respect to BV is not always coincide with the convergence with respecct to $\mathcal{X}$ (therefore to B^+).

Example 3. *Let $X = [0,1]$, $\mathcal{X}$ be the class of Borel subsets of X and $A_n = [0, \frac{n-1}{n}]$. Define the sequence of set functions $\mu_n : \mathcal{X} \longrightarrow [0,1]$ by*

$$\mu_n(A) = \begin{cases} 1 & \textit{if } A_n subset A \\ 0 & \textit{if otherwise} \end{cases}$$

It is obvious from the defition that $\mu_n \in \mathcal{FM}$ for all $n \in N$. Define the fuzzy measure μ in $\mathcal{X}$ by

$$\mu(A) = \begin{cases} 1 & \textit{if } [0,1) \subset A \\ 0 & \textit{if otherwise} \end{cases}$$

Then we have $\mu_n \longrightarrow^{\mathcal{X}} \mu$, since $[0,1) = \bigcup_{n=0}^{\infty} A_n$. On the other hand, we have $||\mu_n - \mu|| = 2$ for all $n \in N$, that is, $\mu_n \not\longrightarrow^{\mathcal{X}} \mu$.

5 Finite case

Suppose that $\mu_i \longrightarrow^{\mathcal{X}} \mu$. If X is a finite set, there exists a real number $M > 0$ such that $2^{|X|} < M$. For every $X \in \mathcal{X}$ and $\epsilon > 0$, there exists $j_0 \in J$ such that $j \geq j_0$ implies

$|\mu_j(A)-\mu(A)| < \dfrac{\epsilon}{2M}$. Define $n_j := \sum_{A\neq B} |(\mu_j(A)-\mu(A))-(\mu_j(B)-\mu(B))|$. It is obvious $n_j < \epsilon$. It follows from Definition 2.2 that $||\mu_j - \mu|| \leq n_j$. We have $\mu_j \longrightarrow^{BV} \mu$. Therefore three topologies coincide.

Theorem 5.1. *Suppose that X is a finite set. Let (μ_i) be a net in $\mathcal{FM}$ and $\mu \in \mathcal{FM}$. Then $\mu_i \longrightarrow^{\mathcal{X}} \mu$ implies $\mu_i \longrightarrow^{BV} \mu$.*

Remark. n_j in the above proof may be replaced by Banzhaf value $B(\mu_j)$ [2]. In fact, it is obvious that $n_j \longrightarrow 0$ if and only if $B(\mu_j - \mu) \longrightarrow 0$.

6 Extreme point of fuzzy measure space

First, we define a convex hull and an extreme point in a general vector space.

Definition 6.1. *Let E be a vector space and $A \subset E$.*

We define the convex hull $c(A)$ by

$$c(A) = \cap\{Y | A \subset Y, Y \textit{ is a convex set}\}.$$

We say that $x \in X$ is an extreme point of X if $x = \lambda x_1 + (1-\lambda)x_2; x_1, x_2 \in X, 0 \leq \lambda \leq 1$ implies $x_1 = x_2 = x$. We denote the set of extreme points of A by $\mathcal{E}(A)$.

It is obvious that $\mathcal{FM}^\alpha$ is a convex set. In fact we have $\lambda\mu_1(X) + (1-\lambda)\mu_2(X) = \alpha$ for $\mu_1, \mu_2 \in \mathcal{FM}^\alpha, 0 \leq \lambda \leq 1$.

Definition 6.2. *Let $\alpha > 0$.*

(1) *We say that $\mu \in \mathcal{FM}^\alpha$ is $0-\alpha$ fuzzy measure if $\mu(A) = 0$ or $\mu(A) = \alpha$ for all $A \in \mathcal{B}$. We denote the set of $0-\alpha$ fuzzy measures by $\mathcal{FM}_0^\alpha$. That is,*

$$\mathcal{FM}_0^\alpha = \{\mu | \mu \in \mathcal{FM}^+, \mu : \mathcal{B} \longrightarrow \{0, \alpha\}\}.$$

(2) *Let $B \in \mathcal{B}$. We say that $N_B \in \mathcal{FM}_0^\alpha$ is $0-\alpha$ necessity measure if*

$$N_B(A) = \begin{cases} \alpha & B \subset A \\ 0 & o.w. \end{cases}.$$

(3) *Let $B \in \mathcal{B}$ and $\mathcal{C} \subset \mathcal{B}$. $0-\alpha$ fuzzy measure $\mathcal{N}_\mathcal{C}$ of $\mathcal{C}$ generated by $0-\alpha$ fuzzy measure is defined by*

$$\mathcal{N}_\mathcal{C} = \sup_{B \in \mathcal{C}} N_B$$

where N_B is a $0-\alpha$ necessity measure.

The next proposition follows from Definition 6.2

Proposition 6.3. *Let $B \in \mathcal{B}$, $\mathcal{C} \subset \mathcal{B}$ and $\alpha > 0$.*

(1) *$0-\alpha$ fuzzy measure is $0-\alpha$ fuzzy measure generated by $0-\alpha$ necessity fuzzy measures. That is, $\mathcal{FM}_0^\alpha = \{\mathcal{N}_\mathcal{C} | \mathcal{C} \subset \mathcal{B}\}$.*

(2) *$0-\alpha$ fuzzy measure is an extreme point of $\alpha-$ fuzzy measure. Conversely, an extreme point of $\alpha-$ fuzzy measure is $0-\alpha$ fuzzy measure. That is, $\mathcal{E}(\mathcal{FM}^\alpha) = \mathcal{FM}_0^\alpha$.*

Applying Klein-Milman's theorem [13], we have the next theorem.

Theorem 6.4. *Let $\alpha > 0$.*

(1) *$\mathcal{FM}^\alpha = cl(c(\mathcal{FM}_0^\alpha))$.*

(2) *If $|\mathcal{B}| < \infty$, $\mathcal{FM}^\alpha = c(\mathcal{FM}_0^\alpha)$.*

Corollary 6.5. (Representation of fuzzy measures)

Suppose that $|\mathcal{B}| < \infty$. For every $\mu \in \mathcal{FM}^\alpha$ there exist $a_1, a_2, \cdots a_m \geq 0$ $(a_1 + a_2 + \cdots + a_m = 1)$ and $\mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_m \subset \mathcal{B}$ such that $\mu = \sum_{i=1}^m a_i \mathcal{N}_{\mathcal{C}_i}$.

Remark In the case of $\alpha = 1$ a $0 - \alpha$ fuzzy measure is sometimes called a logical fuzzy measure. Radojević [11, 12] gives a logical interpretation to a discrete fuzzy measure. In his theory, the relations between any fuzzy measure and fuzzy logical measures are important. Radojević's proposition (Proposition 2 in [11]) is one of the special case of Theorem 6.4.

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