Ky Fan's Inequality for Set-Valued Maps with Vector-Valued Images*

PANDO GR. GEORGIEV[†]and TAMAKI TANAKA (田中　環)[‡]

Abstract:　　We consider four variants of Fan's type inequality for vector-valued multifunctions in topological vector spaces with respect to a cone preorder in the target space, when the functions and the cone possess various kinds of semicontinuity and convexity properties. In order to establish these results, firstly we prove a two-function result of Simons directly by the scalar Fan's inequality, after that, by its help we derive a new two-function result, which is the base of our proofs. As a consequence of our Fan's type inequalities we obtain that this new two-function result is equivalent to the scalar Fan's inequality.

Key words:　　Fan's inequality, vector-valued multifunctions, semicontinuous mappings, quasiconvex functions.

1.　Introduction.

This paper is concerned with vector-valued variants of the following type of inequality:
if $f(x, x) \leq 0$ for all x, then
$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq 0,$$

which is equivalent to the famous Fan's minimax inequality (this equivalence was proved by Takahashi [8, Lemma 1] firstly). This inequality is one of the main tools in the nonlinear and convex analysis, equivalent to Brouwer's fixed point theorem, Knaster-Kuratowski-Mazurkiewicz theorem, and so on. As an analytical instrument, in many situations it is more appropriate and applicable than other main theorems in nonlinear analysis. We refer to [2] for various type equivalent theorems in nonlinear analysis.

In this paper we show four kinds of vector-valued Fan's type inequality for multifunctions. One of them (Theorem 3.1) generalizes the main result of Ansari-Yao in [1], namely, the existence result in the so-called Generalized Vector Equilibrium Problem. Any of our Theorems 3.1–3.4 implies the classical Fan inequality, while the main result in [1] does not imply it in its full generality, but only for continuous functions. Our proofs are quite different from that in [1] and are based on the classical scalar Fan inequality. More precisely, in the proofs we use a new result (see Theorem 2.3) which follows from a two-function result of Simons [7, Theorem 1.2] (used in

Fig.1:
$f(x, y^*) \not\subset \operatorname{int} C(x).$

Fig.2:
$f(x, y^*) \cap \operatorname{int} C(x) = \emptyset.$

Fig.3:
$f(x, y^*) \cap (-C(x)) \neq \emptyset.$

Fig.4:
$f(x, y^*) \subset (-C(x)).$

[7] to derive Fan's inequality), which we prove directly by Fan's inequality. For a simple proof of the classical Fan inequality, based on Brouwer's fixed point theorem, we refer to [3] and ??.

Our main tool in this paper (Theorem 2.3) is a slightly more general form of a two-function result of Simons [7, Corollary 1.6] and as a consequence of our results, it implies the classical Fan inequality.

The proofs of the main results (Theorems 4.1–4.4) use Theorem 2.3 for special scalar functions possessing semicontinuity and convexity properties, inherited by the semicontinuity and the convexity properties of the multifunctions. The four types of Fan's inequality can be regarded as generalizations of the classical Fan's inequality by substituting the nonpositivity of of the scalar function ($f(x, y) \leq 0$) by various types of set relations between the images of multifunction and cone; see Figures 1–4.

2. Fan's inequality and a new two-function result.

Theorem 2.1 (Fan). *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \mathbf{R}$ be quasiconcave in its first variable and lower semicontinuous in its second variable. Then*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x).$$

Theorem 2.2 (Simons [7, Theorem 1.2]). *Let Z be a nonempty compact convex subset of a topological vector space, $f : Z \times Z \to \mathbf{R}$ lower semicontinuous in its second variable, $g : Z \times Z \to \mathbf{R}$ quasiconcave in its first variable, and $f \leq g$ on $Z \times Z$. Then*

$$\min_{y \in Z} \sup_{x \in Z} f(x, y) \leq \sup_{z \in Z} g(z, z).$$

Proof. Define the function co f as a quasiconcave envelope of f with respect to the first variable:

$$\operatorname{co} f(x, y) := \sup\{ \min_{i \in \{1, \dots, n\}} f(x_i, y) : x = \sum_{i=1}^{n} \lambda_i x_i, x_i \in Z, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbf{N}\},$$

where $\mathbf{N}$ is the set of the natural numbers. This function satisfies the conditions of Fan's inequality and applying the latter, we obtain the result. ∎

Now we prove our main tool in this paper. Its proof is similar to that of [7, Corollary 1.6].

Theorem 2.3. *Let X be a nonempty compact convex subset of a topological vector space, $a : X \times X \to \mathbf{R}$ lower semicontinuous in its second variable, $b : X \times X \to \mathbf{R}$ quasiconvex in its second variable, and*

$$x, y \in X \text{ and } a(x, y) > 0 \Rightarrow b(y, x) < 0.$$

Suppose that $\inf_{x \in X} b(x, x) \geq 0$. Then there exists $z \in X$ such that $a(x, z) \leq 0$ for all $x \in X$.

Proof. The proof is straightforward from Theorem 2.2 by defining $f(x, y) = 1$ if $a(x, y) > 0$ and $f(x, y) = 0$ otherwise; $g(x, y) = 1$ if $b(y, x) < 0$ and $g(x, y) = 0$ otherwise. ∎

3. Definitions and auxiliary results.

Further let E and Y be topological vector spaces and $F, C : E \to 2^Y$ two multivalued mappings and let for every $x \in E$, $C(x)$ be a closed convex cone with nonempty interior. We introduce two types of cone-semicontinuity for set-valued mappings, which are regarded as extensions of the ordinary lower semicontinuity for real-valued functions; see [5].

Denote $B(x) = (\operatorname{int} C(x)) \cap (2S \setminus \overline{S})$ (an open base of $\operatorname{int} C(x)$), where S is a neighborhood of 0 in Y, and define the function $h(k, x, y) = \inf\{t : y \in tk - C(x)\}$, Note that $h(k, x, \cdot)$ is positively homogeneous and subadditive for every fixed $x \in E$ and $k \in \operatorname{int} C(x)$. Moreover, we use the following notations $h(k, y) = \inf\{t : y \in tk - C\}$, and $B = C \cap (2S \setminus \overline{S})$, where C is a convex closed cone and S is a neighborhood of 0 in Y. Note again that $h(k, \cdot)$ is positively homogeneous and subadditive for every fixed $k \in \operatorname{int} C$.

Firstly, we prove some inherited properties from cone-semicontinuity.

Definition 3.1. Let $\hat{x} \in E$. The multifunction F is $C(\hat{x})$-*upper semicontinuous* at x_0, if for every $y \in C(\hat{x}) \cup (-C(\hat{x}))$ such that $F(x_0) \subset y + \operatorname{int} C(\hat{x})$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \operatorname{int} C(\hat{x})$ for every $x \in U$. If Y is a Banach space, we shall say that F is $(-C)^c$-*upper semicontinuous* at x_0, if for any $\varepsilon > 0$ and $k \in C$ such that $(k + \varepsilon B_Y - C) \cap F(x_0) = \emptyset$, there exists $\delta > 0$ such that $(k + \varepsilon B_Y - C) \cap F(x) = \emptyset$ for every $x \in B(x_0; \delta)$.

Definition 3.2. Let $\hat{x} \in E$. The multifunction F is $C(\hat{x})$-*lower semicontinuous* at x_0, if for every open V such that $F(x_0) \cap V \neq \emptyset$, there exists an open $U \ni x_0$ such that $F(x) \cap (V + \operatorname{int} C(\hat{x})) \neq \emptyset$ for every $x \in U$. If Y is a Banach space, we shall say that F is $C(\hat{x})$-*lower semicontinuous* at x_0, if for any $\varepsilon > 0$ and $y_0 \in F(x_0)$ there exists an open $U \ni x_0$ such that $F(x) \cap (y_0 + \varepsilon B_Y + C(\hat{x})) \neq \emptyset$ for every $x \in U$, where B_Y denotes the open unit ball in Y.

Remark 3.1. In the two definitions above, the corresponding notions for single-valued function are equivalent to the ordinary one of lower semicontinuity for real-valued function whenever $Y = \mathbb{R}$ and $C = [0, \infty)$. When the cone $C(\hat{x})$ consists only of the zero of the space, the notion in Definition 3.2 coincides with that of lower semicontinuous set-valued mapping. Moreover, it is equivalent to the cone-lower semicontinuity defined in [5], based on the fact of $V + \operatorname{int} C(\hat{x}) = V + C(\hat{x})$; see [9, Theorem 2.2].

Proposition 3.1 *If for some $x_0 \in E$, $A \subset \operatorname{int} C(x_0)$ is a compact subset and multivalued mapping $W(\cdot) := Y \setminus \{\operatorname{int} C(\cdot)\}$ has a closed graph, then there exists an open set $U \ni x_0$ such that $A \subset C(x)$ for every $x \in U$. In particular C is lower semicontinuous .*

Proof. Assume the contrary. Then there exists a net $\{x_i\}$ converging to x_0 such that for every i there exists $a_i \in A \setminus C(x_i)$. Since A is compact, we may assume that $a_i \to a \in A$. Since W has a closed graph, it follows that $a \in W(x_0)$, which is a contradiction. ∎

Lemma 3.1. *Suppose that multifunction $W : E \to 2^Y$ defined as $W(x) = Y \setminus \operatorname{int} C(x)$ has a closed graph. If the multifunction F is $(-C(x))$-upper semicontinuous at x for each $x \in E$, then the function $\varphi_1|_X$ (the restriction of*

$$\varphi_1(x) := \inf_{k \in B(x)} \sup_{y \in F(x)} h(k, x, y)$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P);

(P) for every $x \in X$ there exists an open $U \ni x$ such that the set $F(U \cap X)$ is precompact in Y, that is, $\overline{F(U \cap X)}$ is compact.

If the mapping C is constant-valued, then φ_1 is upper semicontinuous.

Proof. Assume that (F, X) has property (P). Let $\varepsilon > 0$ and $x_0 \in X$ be given. By the definition of φ_1 there exists $k_0 \in B(x_0)$ such that

$$\sup_{y \in F(x_0)} h(k_0, x_0, y) < \varphi_1(x_0) + \varepsilon.$$

Since $\sup_{y \in F(x_0)} h(k_0, x_0, y) = \inf\{t : F(x_0) \subset tk_0 - C(x_0)\}$, we can take

$$\inf\{t : F(x_0) \subset tk_0 - C(x_0)\} < t_0 < \varphi_1(x_0) + \varepsilon.$$

Since F is $(-C(x_0))$-upper semicontinuous at x_0, there exists an open $U \ni x_0$ such that

$$F(x) \subset t_0 k_0 - \operatorname{int} C(x_0) \quad \text{for every } x \in U.$$

By Proposition 3.1 and property (P), for $t_0 < t' < \varphi_1(x_0) + \varepsilon$, there exists an open $U_1 \subset U$ such that

$$F(x) \subset t'k_0 - intC(x) \quad \text{and} \quad k_0 \in B(x) \qquad \text{for every } x \in U_1 \cap X.$$

Then

$$
\begin{aligned}
\varphi_1(x) &= \inf_{k \in B(x)} \sup_{y \in F(x)} h(k, x, y) \\
&\le \sup_{y \in t'k_0 - C(x)} h(k_0, x, y) \\
&= t'h(k_0, x, k_0) + \sup_{y \in -C(x)} h(k_0, x, y) \\
&\le t' \\
&\le \varphi_1(x_0) + \varepsilon.
\end{aligned}
$$

The proof of the second statement (when C is constant-valued) is similar, but in this case there is no need to use Proposition 3.1 and property (P). ∎

Lemma 3.2. *Suppose that the multifunction F is $-C(x)$-lower semicontinuous for each $x \in E$ and the multifunction $W : E \to 2^Y$ defined by $W(x) = Y \setminus \operatorname{int} C(x)$ has a closed graph. Then the function $\varphi_2|_X$ (the restriction of*

$$\varphi_2(x) := \inf_{k \in B(x)} \inf_{y \in F(x)} h(k, x, y)$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P). If the mapping C is constant-valued, then φ_2 is upper semicontinuous.

Proof. Let $\varepsilon > 0$ and $x_0 \in E$ be given. By the definition of φ_2, for $t_0 \in (\varphi_2(x_0), \varphi_2(x_0) + \varepsilon)$ there exists $k_0 \in B(x_0), k_0 \in \operatorname{int} C(x_0)$, and $z_0 \in F(x_0)$ such that $z_0 - t_0 k_0 \in -\operatorname{int} C(x_0)$. By Proposition 3.1, there exists an open set $U_1 \ni x_0$ such that

$$z_0 - t_0 k_0 \in -\operatorname{int} C(x) \quad \text{and} \quad k_0 \in \operatorname{int} C(x) \quad \text{for every } x \in U_1.$$

Therefore

$$h(k_0, x, z_0) \le t_0 \qquad \text{for every } x \in U_1. \tag{3.1}$$

Let $\gamma < \varepsilon/2$. By $(-C(x_0))$-lower semicontinuity of F, there exists an open set $U_2 \subset U_1, x_0 \in U_2$ such that

$$G(x) := F(x) \cap [z_0 + \gamma k_0 - \operatorname{int} C(x_0)] \neq \emptyset \qquad \text{for every } x \in U_2. \tag{3.2}$$

Hence

$$G(U_2 \cap X) \subset z_0 + \gamma k_0 - \operatorname{int} C(x_0)$$

and

$$\overline{G(U_2 \cap X)} \subset z_0 + 2\gamma k_0 - \text{int}\, C(x_0).$$

By Proposition 3.1 there exists an open $U_3 \subset U_2, U_3 \ni x_0$ such that

$$\overline{G(U_2 \cap X)} \subset z_0 + 2\gamma k_0 - \text{int}\, C(x) \quad \text{for every } x \in U_3.$$

This implies

$$F(x) \cap (z_0 + 2\gamma k_0 - \text{int}\, C(x)) \neq \emptyset \quad \text{for every } x \in U_3 \cap X.$$

Take $x \in U_3 \cap X$ and $y_x \in F(x) \cap (z_0 + 2\gamma k_0 - \text{int}\, C(x))$. Therefore $y_x = z_0 + 2\gamma k_0 + c_x$, where $c_x \in -\text{int}\, C(x)$. We obtain

$$
\begin{aligned}
\varphi_2(x_0) + \varepsilon \;&\geq\; t_0 \\
&\geq\; h(k_0, x, z_0) \quad \text{(by (3.1))} \\
&=\; h(k_0, x, y - 2\gamma k_0 - c_x) \\
&\geq\; h(k_0, x, y) - h(k_0, x, 2\gamma k_0) - h(k_0, x, c_x) \quad \text{(by subadditivity of } h(k_0, x, \cdot)) \\
&\geq\; h(k_0, x, y) - 2\gamma \\
&\geq\; \varphi_2(x) - \varepsilon.
\end{aligned}
$$

Hence

$$\varphi_2(x_0) + 2\varepsilon \geq \varphi_2(x) \quad \text{for every } x \in U_3 \cap X.$$

The proof of the second statement (when C is constant-valued) is similar, but in this case there is no need to use Proposition 3.1 and property (P). ∎

Lemma 3.3. *Suppose that Y is a Banach space and the multifunction $F : E \to 2^Y$ is $(-C)^c$-upper semicontinuous and locally bounded (it means that for every point $x_0 \in E$ there exists an open set $U \ni x_0$ and $p > 0$ such that $F(x) \subset pB_Y$ for every $x \in U$, where B_Y denotes the open unit ball in Y). Suppose that the multifunction C has a closed graph and the cone $C(x)$ has a compact base $B(x) = (2\overline{B_Y} \setminus B_Y) \cap C(x)$ for every x. Then the function φ_2 is lower semicontinuous.*

Proof. Firstly we shall prove that the function $g(k, x) := \inf_{y \in F(x)} h(k, x, y)$ is lower semicontinuous. It is easy to see that

$$g(k, x) = \inf\{t : (tk - C(x)) \cap F(x) \neq \emptyset\}$$

(if $(tk - C(x)) \cap F(x) = \emptyset$ for every t, we put $g(k, x) = +\infty$). Take $(k_0, x_0) \in Y \times E$ and let $\{x_i\}, \{k_i\}$ be sequences such that $x_i \to x_0$ and $k_i \to k_0$. Let $\liminf h(k_i, x_i) = l$. There exists a subsequence $\{(k_{i_n}, x_{i_n})\}$ of $\{(k_i, x_i)\}$ such that $k_{i_n} \to k_0 \in B(x_0)$ and $l = \lim g(k_{i_n}, x_{i_n})$. Assume that $l < g(k_0, x_0)$. Then there exists $\varepsilon > 0$ such that

$$l + \varepsilon < g(k_0, x_0) - \varepsilon. \tag{3.3}$$

By the definition of g, there exists

$$y_i \in F(x_i) \cap [(g(k_i, x_i) + \varepsilon)k_i - C(x_i)] \quad \forall i \in \mathbf{N}.$$

Hence

$$y_i = [g(k_i, x_i) + \varepsilon]k_i - c_i \tag{3.4}$$

for some $c_i \in C(x_i)$. By the locally boundedness of F and from the compactness of $B(x_0)$, we obtain that the sequence $\{c_i\}$ is precompact. Then by (3.4), passing to limits and using the fact that C has a closed graph, we obtain

$$\lim y_i = y_0 = (l + \varepsilon)k_0 - c_0, \tag{3.5}$$

where $c_0 \in C(x_0)$. Since $F(x_0)$ is bounded and $B(x_0)$ is compact, the distance between the sets $F(x_0)$ and $[g(k_0, x_0) - \varepsilon]k_0 - C(x_0)$ is positive, so there exists $\alpha > 0$ such that

$$([g(k_0, x_0) - \varepsilon]k_0 + \alpha B_Y - C(x_0)) \cap F(x_0) = \emptyset.$$

By the $(-C)^c$-upper semicontinuity of C we obtain that for some index $i_0 \in \mathbf{N}$,

$$y_i \notin [g(k_0, x_0) - \varepsilon]k_0 + \alpha B_Y - C(x_0) \quad \forall i > i_0.$$

Hence passing to limit, by (3.3) we obtain $y_0 \notin [l + \varepsilon]k_0 - C(x_0)$, which is a contradiction with (3.5). So we proved the lower semicontinuity of g at (k_0, x_0). Now, we apply Proposition 3.1.21 in [2] and finish the proof. ∎

Lemma 3.4. *Suppose that Y is a Banach space and the multifunction $F : E \to 2^Y$ is $C(x)$-lower semicontinuous for each $x \in E$ and locally bounded. Suppose that the multifunction C has a closed graph and the cone $C(x)$ has a compact base $B(x) = (2\overline{B_Y} \setminus B_Y) \cap C(x)$ for every x. Then the function φ_1 is lower semicontinuous.*

Proof. Firstly we shall prove that the function $g(k, x) := \sup_{y \in F(x)} h(k, x, y)$ is lower semicontinuous. Take (k_0, x_0) and let $\{x_i\}, \{k_i\}$ be sequences such that $x_i \to x_0$ and $k_i \to k_0$. Let $\varepsilon > 0$ be given. There exists $y_0 \in F(x_0)$ such that

$$h(k_0, x_0, y_0) > g(k_0, x_0) - \varepsilon. \tag{3.6}$$

Since F is C-lower semicontinuous, for $\beta > 0$ there exists index i_0 such that

$$F(x_i) \cap [y_0 + \beta B_Y + C(x_0)] \neq \emptyset \quad \forall i > i_0.$$

Take $y_i \in F(x_i) \cap [y_0 + \beta B_Y + C(x_0)]$. Hence

$$y_i = y_0 + \beta b + c_i, \tag{3.7}$$

where $c_i \in C(x_0)$ and $b \in B_Y$. Since $y_i \in [h(k_i, x_i, y_i) + \varepsilon]k_i - C(x_i)$, we have $y_i \in [g(k_i, x_i) + \varepsilon]k_i - C(x_i)$, and hence

$$-y_0 - \beta b - c_i + [g(k_i, x_i) + \varepsilon]k_i \in C(x_i). \tag{3.8}$$

By the locally boundedness of F, from (3.7) and the compactness of $B(x_0)$, we obtain that the sequence $\{c_i\}$ is precompact. Let $\liminf h(k_i, x_i, y_0) = l$. Without loss of generality (taking subsequences) we may suppose that $k_i \to k_0 \in B(x_0)$ and $l = \lim g(k_i, x_i)$. Then by (3.8), passing to limits and using the assumption that C has a closed graph, we obtain $y_0 + \beta b \in (l + \varepsilon)k_0 - C(x_0)$. Hence by (3.6), $g(k_0, x_0) - \varepsilon \leq h(k_0, x_0, y_0) \leq l + \varepsilon + \alpha$, where $\alpha = h(k_0, x_0, -\beta b)$. Since $\varepsilon > 0, \beta$ are arbitrarily small (therefore α is arbitrarily small too, by continuity of $h(k_0, x_0, \cdot)$), we obtain $h(k_0, x_0, y_0) \leq l$. This proves lower semicontinuity of g at (k_0, x_0). Now, we apply Proposition 3.1.21 in [2] and finish the proof. ∎

Next, we show some inherited properties from cone-quasiconvexity.

Definition 3.3. A multifunction $F : E \to 2^Y$ is called *C-quasiconvex*, if the set $\{x \in E : F(x) \cap (a - C) \neq \emptyset\}$ is convex for every $a \in Y$. If $-F$ is C-quasiconvex, then F is said to be *C-quasiconcave*, which is equivalent to $(-C)$-quasiconvex mapping.

Remark 3.2. The above definition is exactly that of *Ferro type (-1)-quasiconvex* mapping in [6, Definition 3.5].

Definition 3.4. A multifunction $F : E \to 2^Y$ is called (in the sense of [6, Definition 3.6])

(a) *type*-(iii) *C-properly quasiconvex* if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0,1]$ we have either $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$.

(b) *type*-(v) *C-properly quasiconvex* if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0,1]$ we have either $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$;

If $-F$ is type-(iii) [resp. type-(v)] C-properly quasiconvex, then F is said be type-(iii) [resp. type-(v)] C-properly quasiconcave, which is equivalent to type-(iii) [resp. type-(v)] $(-C)$-properly quasiconvex mapping.

Remark 3.3. The convexity of (b) above is exactly that of *C-quasiconvex-like* multifunction in [1].

Lemma 3.5. *If the multifunction $F : E \to 2^Y$ is type-(v) C-properly quasiconvex, then the function*

$$\psi_1(x) := \inf_{k \in B} \sup_{y \in F(x)} h(k, y)$$

is quasiconvex.

Proof. By definition, for every $\lambda \in [0,1]$ and every $x_1, x_2 \in X$ we have: either $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$. Assume that $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$. Then

$$
\begin{aligned}
\psi_1(\lambda x_1 + (1 - \lambda)x_2) \quad &:= \quad \inf_{k \in B} \sup\{h(k,y) : y \in F(\lambda x_1 + (1 - \lambda)x_2)\} \\
&\leq \quad \inf_{k \in B} \sup\{h(k,y) : y \in F(x_1) - C\} \\
&= \quad \inf_{k \in B} \sup_{\substack{y \in F(x_1) \\ c \in C}} h(k, y - c) \\
&\leq \quad \inf_{k \in B} \sup_{\substack{y \in F(x_1) \\ c \in C}} (h(k,y) + h(k,-c)) \quad \text{(by subadditivity of } h(k,\cdot)) \\
&\leq \quad \psi_1(x_1) \\
&\leq \quad \max\{\psi_1(x_1), \psi_1(x_2)\}.
\end{aligned}
$$

Analogously we proceed in the second case, when $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$. ∎

Lemma 3.6. *If F is C-quasiconvex, then for every $k \in B$ the function*

$$\psi_2(x; k) := \inf\{h(k,y) : y \in F(x)\}$$

is quasiconvex.

Proof. By the definition of ψ_k, for every $\varepsilon > 0$ and $x_1, x_2 \in E$ there exist $z_i \in F(x_i), t_i \in \mathbf{R}$ such that

$$z_i - t_i k \in -C, \tag{3.9}$$

and

$$t_i < \psi_k(x_i) + \varepsilon, i = 1, 2. \tag{3.10}$$

Since $s_1 k - C \subset s_2 k - C$ for $s_1 \leq s_2$, by (3.9), we have $z_i \in t_i k - C \subset \max\{t_1, t_2\}k - C$. Hence, by the C-quasiconvexity of F, for every $\lambda \in [0,1]$ there exists $y \in F(\lambda x_1 + (1 - \lambda)x_2)$ such that $y \in \max\{t_1, t_2\}k - C$, which means

$$
\begin{aligned}
h(k,y) \quad &\leq \quad \max\{t_1, t_2\} \\
&< \quad \max\{\psi_k(x_1), \psi_k(x_2)\} + \varepsilon
\end{aligned}
$$

(by 3.10) and since, the definition, we have

$$\psi_k(\lambda x_1 + (1 - \lambda)x_2) = \inf\{h(k, y) : y \in F(\lambda x_1 + (1 - \lambda)x_2)\},$$

and $\varepsilon > 0$ is arbitrarily small, we obtain $\psi_k(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{\psi_k(x_1), \psi_k(x_2)\}$. ∎

Lemma 3.7. *If the multifunction $F : E \to 2^Y$ is type-(v) C-properly quasiconcave, then the function $\psi_2(x; k)$ is quasiconcave, where $k \in \operatorname{int} C$.*

Proof. By definition, for every $\lambda \in [0, 1]$ and every $x_1, x_2 \in X$ we have either $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) + C$ or $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) + C$. Assume that $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) + C$. Then

$$
\begin{aligned}
\psi_1(\lambda x_1 + (1 - \lambda)x_2; k) &= \inf\{h(k, y) : y \in F(\lambda x_1 + (1 - \lambda)x_2)\} \\
&\geq \inf\{h(k, y + c) : y \in F(x_1), c \in C\} \\
&\geq \inf\{h(k, y) - h(k, -c) : y \in F(x_1), c \in C\} \\
&\geq \inf\{h(k, y) : y \in F(x_1)\} \\
&= \psi_1(x_1; k) \\
&\geq \min\{\psi_1(x_1; k), \psi_1(x_2; k)\}.
\end{aligned}
$$

Analogicaly we proceed in the second case, when $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) + C$. ∎

Lemma 3.8. *If the multifunction $F : E \to 2^Y$ is type-(iii) C-properly quasiconcave, then the function*

$$\psi_1(x; k) := \sup\{h(k, y) : y \in F(x)\}$$

is quasiconcave, where $k \in \operatorname{int} C$.

Proof. By definition, for every $\lambda \in [0, 1]$ and every $x_1, x_2 \in X$ we have either $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) - C$ or $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) - C$.

Assume that $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) - C$. Then

$$
\begin{aligned}
\psi_2(x_1; k) &= \sup\{h(k, y) : y \in F(x)\} \\
&\leq \sup\{h(k, y - c) : y \in F(\lambda x_1 + (1 - \lambda)x_2), c \in C\} \\
&\leq \sup\{h(k, y) + h(k, -c) : y \in F(\lambda x_1 + (1 - \lambda)x_2), c \in C\} \\
&\leq \sup\{h(k, y) : y \in F(\lambda x_1 + (1 - \lambda)x_2)\} \\
&= \psi_2(\lambda x_1 + (1 - \lambda)x_2; k),
\end{aligned}
$$

and hence $\min\{\psi_2(x_1; k), \psi_2(x_2; k)\} \leq \psi_2(\lambda x_1 + (1 - \lambda)x_2; k)$.

Analogicaly we proceed in the second case, when $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) - C$. ∎

4. Set-valued Fan's inequalities.

Now we state the main results in this paper. The following theorem is a generalization of that in [1]. The main difference between our result and that in [1] is the condition (iii), but it allows us to recover the classical Fan inequality, when Y is the real line. The result in [1] recovers it only for continuous functions.

Theorem 4.1 *Let K be a nonempty convex subset of a topological vector space E, Y be a topological vector space. Let $F : K \times K \to 2^Y$ be a multifunction. Assume that*

(i) $C : K \to 2^Y$ *is a multifunction such that for every $x \in K, C(x)$ is a closed convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$;*

(ii) $W : K \to 2^Y$ *is a multifunction defined as* $W(x) = Y \setminus \operatorname{int} C(x)$, *and the graph of* W *is closed in* $K \times Y$;

(iii) *for every* $x, y \in K$, $F(\cdot, y)$ *is* $C(x)$-*upper semicontinuous at* x *with closed values on* K *and if the mapping* C *is not constant-valued, then the mapping* $F(\cdot, y)$ *maps the compact subsets of* K *into precompact subsets of* Y;

(iv) *there exists a multifunction* $G : K \times K \to 2^Y$ *such that*

(a) *for every* $x \in K$, $G(x, x) \not\subset \operatorname{int} C(x)$,

(b) *for every* $x, y \in K$, $F(x, y) \subset \operatorname{int} C(x)$ *implies* $G(x, y) \subset \operatorname{int} C(x)$,

(c) $G(x, \cdot)$ *is type-*(v) $C(x)$-*properly quasiconcave on* K *for every* $x \in X$,

(d) $G(x, y)$ *is compact, if* $G(x, y) \subset \operatorname{int} C(x)$;

(v) *there exists a nonempty compact convex subset* D *of* K *such that for every* $x \in K \setminus D$, *there exists* $y \in D$ *with* $F(x, y) \subset \operatorname{int} C(x)$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \not\subset \operatorname{int} C(x), \text{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof. Put

$$a(x, y) := - \inf_{k \in B(y)} \sup_{z \in -F(y, x)} h(k, y, z), \quad b(x, y) := \inf_{k \in B(x)} \sup_{z \in -G(x, y)} h(k, x, z).$$

It is easy to check that

$$a(x, y) > 0 \quad \text{if and only if } F(y, x) \subset \operatorname{int} C(y)$$

by using the compactness of $\overline{F(x, y)}$, and also $b(y, x) < 0$ if $G(y, x) \subset \operatorname{int} C(y)$ by using condition (d), and then $a(x, x) \leq 0$ and $b(x, x) \geq 0$.

Denote

$$S_y := \{x \in D : F(x, y) \not\subset \operatorname{int} C(x)\}. \tag{4.1}$$

Since $a(y, \cdot)$ is lower semicontinuous (by Lemma 3.1), the set S_y is closed. Let Y_0 be a finite subset of K. Denote by Z the closed convex hull of $Y_0 \cup D$. Obviously Z is compact and convex. Lemmas 3.1, 3.5 and condition (iv) (b) show that the conditions of Theorem 2.3 are satisfied.

Now we apply Theorem 2.3 and obtain a point $z \in Z$ such that $a(y, z) \leq 0$ for every $y \in Z$, which means

$$F(z, y) \not\subset \operatorname{int} C(z) \quad \text{for every} \quad y \in Z. \tag{4.2}$$

The conditions (v) and (4.2) imply that $z \in D$. Relation (4.1) implies that $\cap\{S_y : y \in Y_0\} \neq \emptyset$. So we proved that the family $\{S_y : y \in K\}$ has finite intersection property. Since D is compact, $\cap\{S_y : y \in K\} \neq \emptyset$, which means that there exists $x_0 \in K$ such that $F(x_0, y) \not\subset \operatorname{int} C(x_0)$ for every $y \in K$. So we proved that S is nonempty, and since S is a closed subset of D, the proof is completed. ∎

Theorem 4.2. *Let* K *be a nonempty convex subset of a topological vector space* E, Y *a topological vector space, and* $F : K \times K \to 2^Y$ *a multifunction. Assume that*

(i) $C : K \to 2^Y$ *is a multifunction such that for every* $x \in K, C(x)$ *is a closed convex cone in* Y *with* $\operatorname{int} C(x) \neq \emptyset$;

(ii) $W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus \operatorname{int} C(x)$, for every $x \in K$ such that the graph of W is closed in $K \times Y$;

(iii) for every $x, y \in K$, $F(\cdot, y)$ is $C(x)$-lower semicontinuous with closed values on K and if the mapping C is not constant-valued, then the mapping $F(\cdot, y)$, for every $y \in K$, maps the compact subsets of K into precompact subsets of Y;

(iv) there exists a multifunction $G : K \times K \to 2^Y$ such that

    (a) for every $x \in K$, $G(x, x) \cap \operatorname{int} C(x) = \emptyset$,

    (b) for every $x, y \in K$, $F(x, y) \cap \operatorname{int} C(x) \neq \emptyset$ implies $G(x, y) \cap \operatorname{int} C(x) \neq \emptyset$,

    (c) $G(x, \cdot)$ is $C(x)$-quasiconcave on K for every $x \in K$;

(v) there exists a nonempty compact convex subset D of K such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x, y) \cap \operatorname{int} C(x) \neq \emptyset$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \cap (\operatorname{int} C(x)) = \emptyset, \text{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof. Put

$$a(x, y) := -\inf_{k \in B(y)} \inf_{z \in -F(y,x)} h(k, y, z), \quad b(x, y) := \inf_{z \in -G(x,y)} h(k(x), x, z),$$

where the function k is any fixed selection of the multivalued mapping $x \mapsto \operatorname{int} C(x)$, i.e., $k(x) \in \operatorname{int} C(x)$ for every $x \in K$. It is easy to check that

$$a(x, y) > 0 \quad \text{if and only if } F(y, x) \cap (\operatorname{int} C(y)) \neq \emptyset,$$

$$b(y, x) < 0 \quad \text{if and only if } G(y, x) \cap (\operatorname{int} C(y)) \neq \emptyset,$$

$$a(x, x) \leq 0, \quad b(x, x) \geq 0.$$

Lemmas 3.2, 3.6 and condition (iv) (b) show that the conditions of Theorem 2.3 are satisfied. Further the proof is the same as that of Theorem 4.1, but in this case $S_y := \{x \in D : F(x, y) \cap (\operatorname{int} C(x)) = \emptyset\}$. ∎

Theorem 4.3. *Let K be a nonempty convex subset of a topological vector space E, Y a Banach space, and $F : K \times K \to 2^Y$ a multifunction. Assume that*

(i) $C : K \to 2^Y$ is a multifunction with a closed graph and $C(x)$ is a closed convex cone with a compact base $B(x) = (2\overline{B_Y} \setminus B_Y) \cap C(x)$ for every x;

(ii) for every $y \in K$, $F(\cdot, y)$ is $(-C)^c$-upper semicontinuous and locally bounded;

(iii) there exists a multifunction $G : K \times K \to 2^Y$ such that

    (a) for every $x \in K$, $G(x, x) \cap (-C(x)) \neq \emptyset$,

    (b) for every $x, y \in K$, $F(x, y) \cap (-C(x)) = \emptyset$ implies $G(x, y) \cap (-C(x)) = \emptyset$,

    (c) $G(x, \cdot)$ is type-(v) $C(x)$-properly quasiconcave on K for every $x \in K$;

(iv) there exists a nonempty compact convex subset D of K such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x, y) \cap (-C(x)) = \emptyset$.

Then, the solutions set

$$S = \{x \in K : F(x,y) \cap (-C(x)) \neq \emptyset, \text{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof. Put

$$a(x,y) := \inf_{k \in B(y)} \inf_{z \in F(y,x)} h(k,y,z), \quad b(x,y) := - \inf_{k \in B(x)} \inf_{z \in G(x,y)} h(k,x,z).$$

It is easy to check that

$$a(x,y) \leq 0 \quad \text{if and only if } F(y,x) \cap (-C(y)) \neq \emptyset,$$

$$b(y,x) \geq 0 \quad \text{if and only if } G(y,x) \cap (-C(y)) \neq \emptyset,$$

$$a(x,x) \leq 0 \quad \text{and} \quad b(x,x) \geq 0.$$

Lemmas 3.3, 3.7 and condition (iii) (b) show that the conditions of Theorem 2.3 are satisfied. Denote $S_y := \{x \in D : F(x,y) \cap (-C(x)) \neq \emptyset\}$. Since $a(y,\cdot)$ is lower semicontinuous (by Lemma 3.3), the set S_y is closed. Let Y be a finite subset of K. Denote by Z the intersection of K and the linear hull of $Y \cup D$. Obviously Z is compact and convex. Now we apply Theorem 2.3 and obtain a point $z \in Z$ such that

$$a(y,z) \leq 0 \qquad \text{for every } y \in Z \tag{4.3}$$

which means

$$F(z,y) \cap (-C(x)) \neq \emptyset \qquad \text{for every } y \in Z. \tag{4.4}$$

Assumption (iv) and condition (4.4) imply that $z \in D$, and condition (4.4) implies also $\cap\{S_y : y \in Y\} \neq \emptyset$. So the family $\{S_y : y \in K\}$ has finite intersection property. Since D is compact, $\cap\{S_y : y \in K\} \neq \emptyset$, which completes the proof. ∎

Theorem 4.4. *Let K be a nonempty convex subset of a topological vector space E, Y a Banach space, and $F : K \times K \to 2^Y$ a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction with a closed graph such that $C(x)$ is a closed convex cone with a compact base $B(x) = (2\overline{B}_Y \setminus B_Y) \cap C(x)$ for every x;*

(ii) *for every $x,y \in K$, $F(\cdot,y)$ is $C(x)$-lower semicontinuous and locally bounded;*

(iii) *there exists a multifunction $G : K \times K \to 2^Y$ such that*

    (a) *for every $x \in K, G(x,x) \subset -C(x)$,*

    (b) *for every $x,y \in K$, $F(x,y) \not\subset -C(x)$ implies $G(x,y) \not\subset -C(x)$,*

    (c) *$G(x,\cdot)$ is type-(iii) $C(x)$-properly quasiconcave on K for every $x \in K$;*

(iv) *there exists a nonempty compact convex subset D of K such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x,y) \not\subset -C(x)$.*

Then, the solutions set

$$S = \{x \in K : F(x,y) \subset -C(x), \text{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof. Put

$$a(x,y) := \inf_{k \in B(y)} \sup_{z \in F(y,x)} h(k,y,z), \quad b(x,y) := - \inf_{k \in B(x)} \sup_{z \in G(x,y)} h(k,x,z).$$

It is easy to check that

$$a(x,y) \le 0 \quad \text{if and only if } F(y,x) \subset -C(y),$$
$$b(y,x) \ge 0 \quad \text{if and only if } G(y,x) \subset -C(y),$$
$$a(x,x) \le 0 \quad \text{and} \quad b(x,x) \ge 0.$$

Lemmas 3.4, 3.8 and condition (iii) (b) show that the conditions of Theorem 2.3 are satisfied. Further the proof is the same as that of Theorem 4.3, but in this case $S_y := \{x \in D : F(x,y) \subset -C(x)\}$. ∎

5. Conclusions.

We have presented four type generalizations of the scalar Fan's inequality in the following setting:

(i) set-valued maps with vector-valued images instead of scalar functions;

(ii) two-function type instead of single function type;

(iii) parametric ordering structure instead of fixed ordering structure;

(iv) complete extensions including the result of [1].

As a corollary from any of Theorems 4.1–4.4, we obtain that Theorem 2.3 implies the scalar Fan inequality.

References

[1] Q. H. Ansari and J.-C. Yao (1999). *An existence result for the generalized vector equilibrium problem*, Appl. Math. Lett., **12**, 53–56.

[2] J.-P. Aubin and I. Ekeland (1984). *Applied Nonlinear Analysis*, Wiley Interscience, New York.

[3] J.-P. Aubin and H. Frankowska (1990). *Set-Valued Analysis*, Birkhäuser, Boston.

[4] P. G. Georgiev and T. Tanaka (2000). *Vector-valued set-valued variants of Ky Fan's inequality*, J. Nonlinear and Convex Analysis, **1(3)**, 245–254.

[5] Y. Kimura, K. Tanaka, and T. Tanaka (1999). *On semicontinuity of set-valued maps and marginal functions*, 181–188 in Nonlinear Anlysis and Convex Analysis —Proceedings of the International Conference (W. Takahashi and T. Tanaka, eds.), World Scientific, Singapore.

[6] D. Kuroiwa, T. Tanaka, and T.X.D. Ha (1997). *On cone convexity of set-valued maps*, Nonlinear Analysis, TMA, **30(3)**, 1487–1496.

[7] S. Simons (1986). *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Proc. Symp. Pure Math., **45 (2)**, 377–392.

[8] W. Takahashi (1976). *Nonlinear variational inequalities and fixed point theorems*, JMSJ, **28 (1)**, 168–181.

[9] T. Tanaka and D. Kuroiwa (1994). *Another observation on conditions assuring* $\text{int } A + B = \text{int } (A + B)$, Appl. Math. Lett., **7 (1)**, 19–22.