Minimax Theorems for Vector-Valued Multifunctions *

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1 Introduction

We present a Ky Fan type inequality of mixed kind for vector-valued multifunctions. We use it for proving our first type minimax theorem for vector-valued multifunctions. It is a generalization of the classical Sion minimax theorem for scalar functions (in the compact case), as well as, a generalization of a theorem of Tanaka for vector-valued functions.

We use a vector-valued variant for multifunctions of Ky Fan type inequality, described in the another presentation of us in this volume, in order to derive our second type minimax theorems for vector-valued multifunctions, which is stronger than the first one and uses a special notion of convexity for multifunctions.

The theory of vector optimization has been intensively developed in recent years, as currently the interest is focused on vector-valued multifunctions. Important parts of this theory are the minimax problems and saddle point problems, which have their one specific features with respect to the real-valued case. For a development of such vector-valued problems we refer to [T1-T5] and references therein. The vector-valued, set-valued case proposes more possibilities for definitions of saddle points. In this paper we prove also a Nash equilibrium theorem for vector-valued multifunctions using scalarization and Ky Fan’s inequality. As a corollary we obtain a loose saddle point theorem for convex-concave multifunctions (with respect to a specified definition). An advantage in our loose saddle point theorems with respect to the existing ones in the literature (see [K-K], [L-V]) is that our conditions are explicit.

*This work is based on research 11740053 supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan and was done while the first named author was a Visiting Professor at the University of Hirosaki, supported by JSPS fellowship and International Grant for Research in 1999 and 2000 at Hirosaki University. He thanks for the warm hospitality of the staff and the students at Hirosaki University.

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2 Scalar and vector-valued Ky Fan type inequality of mixed kind

Proposition 2.1 (Scalar Ky Fan type inequality of mixed kind). Assume that the functions \( f, g : K \times K \to \mathbb{R} \), where \( K \) is a compact convex nonempty subset of topological vector space, satisfy the properties:

(i) \( f(\cdot, y), g(\cdot, y) \) are lower semicontinuous for every \( x, y \in K \);
(ii) \( f(x, \cdot), g(x, \cdot) \) are quasi-concave for every \( x, y \in K \).

(iii) \( \min\{f(x, y), g(x, y)\} \leq 0 \) \( \forall x, y \in K \).

Then there exist \( x_0, y_0 \in K \) such that

\[
\min\{\sup_{y \in K} f(x_0, y), \sup_{x \in K} g(x, y_0)\} \leq 0.
\]

Proof. Define the function

\[
h(\tilde{x}, \tilde{y}, x, y) = \min\{f(\tilde{x}, y), g(x, \tilde{y})\}.
\]

It is easy to see that \( h(\cdot, \cdot, x, y) \) is lower semicontinuous on \( K \times K \) and \( h(\tilde{x}, \tilde{y}, \cdot, \cdot) \) is quasiconvex on \( K \times K \). Applying the classical scalar Ky Fan's inequality (see for instance [A-E]), we obtain the result.

Let \( Y \) be a Banach space, \( C \subset Y \) a closed convex cone with nonempty interior and \( E \) a topological vector space.

Definition 2.2 The multivalued mapping \( F : E \to 2^Y \) is called \( C \)-properly quasiconvex if for every two points \( x_1, x_2 \in X \) and every \( \lambda \in [0, 1] \) we have either

\[
F(\lambda x_1 + (1 - \lambda) x_2) \subset F(x_1) - C \quad \text{or} \quad F(\lambda x_1 + (1 - \lambda) x_2) \subset F(x_2) - C.
\]

If \( -F \) is \( C \)-properly quasiconvex, then \( F \) is called \( C \)-properly quasiconcave, which is equivalent to \( (-C) \)-properly quasiconvex mapping.

Definition 2.3 We shall say that the multifunction \( F : E \to 2^Y \) is \( C \)-lower semicontinuous at \( x_0 \), if for every \( y \in F(x_0) \) and every open \( V \ni 0 \) there exists an open \( U \ni x_0 \) such that \((y + V + C) \cap F(x) = \emptyset \) for every \( x \in U \).

Definition 2.4 The multifunction \( F \) is called \( C \)-upper semicontinuous at \( x_0 \), if for every \( y \in C \cup (-C) \) such that \( F(x_0) \subset y + \text{int}C \), there exists an open \( U \ni x_0 \) such that \( F(x) \subset y + \text{int}C \) for every \( x \in U \).

Theorem 2.5 (Ky Fan type inequality of mixed kind for multifunctions). Suppose that \( E_1 \) and \( E_2 \) are topological vector spaces, \( X \subset E_1 \) is a nonempty convex compact subset, \( K \subset E_2 \) is a nonempty convex compact subset, \( C \) is closed convex strongly pointed cone with nonempty interior in a Banach space \( Y \) and \( F, G : X \times K \to 2^Y \) are multifunctions satisfying the following conditions:

(i) \( G(x, \cdot) \) is \( C \)-quasiconvex for every \( x \in X \), and \( F(\cdot, y) \) is \( C \)-properly quasiconvex for every \( y \in K \);
(ii) \( G(\cdot, y) \) is \(-C\)-lower semicontinuous for every \( y \in K \), and \( F(x, \cdot) \) is \(-C\)-upper semicontinuous for every \( x \in X \).

(iii) for every \( x \in X \), \( y \in K \) we have: either \( G(x, y) \cap (-\text{int}C) = \emptyset \) or \( F(x, y) \not\subseteq -\text{int}C \).

Then there exist \( x_0 \in X \), \( y_0 \in K \) such that for every \( x \in X \), \( y \in K \) we have: either \( G(x_0, y) \cap (-\text{int}C) = \emptyset \) or \( F(x, y_0) \not\subseteq -\text{int}C \).

**Proof.** Define

\[
\varphi((x, y), (x', y')) := \inf\{f(x, y'), g(x', y)\},
\]

where

\[
f(x, y) = -\inf_{k \in B} \sup_{z \in F(x, y)} h(k, x, z),
g(x, y) = -\inf_{k \in B} \inf_{z \in G(x, y)} h(k, x, z)
\]

and \( B \) is an open base of \( C \). Using Lemmas 3.1, 3.3 of [G-T1] we obtain that \( \varphi((\cdot, \cdot), (x', y')) \) is lower semicontinuous for every \( x', y' \in K \), and by Lemmas 3.2, 3.4 in [G-T1] \( \varphi((x, y), (\cdot, \cdot)) \) is quasi-concave for every \( x \in X \), \( y \in K \). We have also \( \varphi((x, y), (x, y)) \leq 0 \) for every \( x, y \in K \). Applying Proposition 2.1 we obtain the result.

We shall denote by \( \sup A \) (resp. \( \inf A \)), where \( A \subset Y \), the set of all efficient points of the set \( \overline{A} \) (the norm closure of \( A \)) with respect to \( C \) (resp. with respect to \(-C\)), i.e.

\[
\sup A = \{a \in \overline{A} : (a + C) \cap A = \{a\}\};
\]

\[
\inf A = \{a \in \overline{A} : (a - C) \cap A = \{a\}\}.
\]

Recall that \( A \) is bounded with respect to \( C \), if the set \( (a + C) \cap A \) is bounded for every \( a \in A \). A classical lemma of R. Phelps [Ph], which is equivalent to Ekeland's variational principle and which we shall use in the sequel, states that \( \sup A \neq \emptyset \) (resp. \( \inf A \neq \emptyset \)), if \( A \) is bounded with respect to \( C \) (resp. with respect to \(-C\)).

We shall say that the multivalued mapping \( F : X \to 2^Y \), where \( X \) is topological space, is bounded with respect to \( C \), if for every \( x \in X \) and every \( y \in F(x) \) the set \( (y + C) \cap F(x) \) is bounded.

### 3 Minimax Theorems

**Theorem 3.1 (Minimax theorem 1).** Suppose that \( E_1 \) and \( E_2 \) are topological vector spaces, \( X \subset E_1 \) is nonempty convex compact subset, \( K \subset E_2 \) is a nonempty convex compact subset, \( C \) is closed convex strongly pointed cone with nonempty interior in a Banach space \( Y \) and \( F, G : X \times K \to 2^Y \) are multifunctions, bounded with respect to \( C \) and \(-C\) respectively, and satisfying the following conditions:

(i) \( G(x, \cdot) \) is \( C\)-quasiconvex for every \( x \in X \), and \(-F(\cdot, y) \) is \( C\)-properly quasiconvex for every \( y \in K \);

(ii) \( G(\cdot, y) \) is \(-C\)-lower semicontinuous for every \( y \in K \), and \( F(x, \cdot) \) is \( C\)-upper semicontinuous for every \( x \in X \).

(iii) for every \( x \in X \), \( y \in K \) and every two vectors \( z_1, z_2 \in Y \) satisfying \( z_1 - z_2 \notin C \), we have
either \([G(x, y) - z_1] \cap (-\text{int} C) = \emptyset\), or \(z_2 - F(x, y) \not\subset -\text{int} C\).

Then for every \(z_1\) such that
\[(a)\quad z_1 - \text{int} C \supset \sup \cup_{x \in X} \inf \cup_{y \in K} G(x, y),
\]
and for every \(z_2\) such that
\[(b)\quad z_2 + \text{int} C \supset \inf \cup_{y \in K} \sup \cup_{x \in X} F(x, y),
\]
we have \(z_1 - z_2 \in C\).

Proof. Assume the contrary. By (ii) it follows that \(G(\cdot, y) - z_1\) is \(C\)-lower semicontinuous and \(z_2 - F(x, \cdot)\) is \(C\)-upper semicontinuous. By (i) it follows that \(G(x, \cdot) - z_1\) is \(C\)-quasiconvex and \(z_2 - F(\cdot, y)\) is \(C\)-properly quasiconvex. So, using (iii) we apply Theorem 2.5 and obtain that there exist points \(x_0, y_0\) such that for every \(x \in X, y \in K\) we have:
\[\text{either } (G(x_0, y) - z_1) \cap (-\text{int} C) = \emptyset\]
or \(z_2 - F(x, y_0) \not\subset -\text{int} C\).

Assume that there exists \(x \in X\) such that
\[z_2 - F(x, y_0) \subset -\text{int} C.
\]
Then
\[(G(x_0, y) - z_1) \cap (-\text{int} C) = \emptyset \quad \forall y \in K.
\]
This implies
\[\left( \inf \cup_{y \in K} G(x_0, y) \right) \cap (z_1 - \text{int} C) = \emptyset. \tag{1}
\]
It is easy to see, using Phelps lemma (see [Ph]) that for any set \(S\) which is bounded with respect to \(C\), we have
\[S \subset \sup S - C \tag{2}
\]
So, for \(S = \inf \cup_{y \in K} G(x_0, y)\), by (2) we have (using (a))
\[
\inf \cup_{y \in K} G(x_0, y) \subset \sup \cup_{x \in X} \inf \cup_{y \in K} G(x, y) - C \\
\subset z_1 - \text{int} C - C \\
= z_1 - \text{int} C,
\]
which is a contradiction with (1). Therefore
\[z_2 - F(x, y_0) \not\subset -\text{int} C \quad \forall x \in X.
\]
This implies
\[\sup \cup_{x \in X} F(x, y_0) \not\subset z_2 + \text{int} C \tag{3}
\]
By (b) and (2) we obtain
\[z_2 + \text{int} C = z_2 + \text{int} C + C \]
\[\sup \cup_{y \in K} \sup \cup_{x \in X} F(x, y) + C \\
\sup \cup_{x \in X} F(x, y_0),
\]
which is a contradiction with (3). \(\blacksquare\)

Definition 3.2 A multifunction \(F : E \rightarrow 2^Y\) is called (in the sense of [K-T-H, Definition 3.6])
(a) type-(v) \( C \)-properly quasiconvex if for every two points \( x_1, x_2 \in X \) and every \( \lambda \in [0, 1] \) we have either \( F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C \) or \( F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C \);

(b) type-(iii) \( C \)-properly quasiconvex if for every two points \( x_1, x_2 \in X \) and every \( \lambda \in [0, 1] \) we have either \( F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C \) or \( F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C \).

If \( -F \) is type-(v) [resp. type-(iii)] \( C \)-properly quasiconvex, then \( F \) is said be type-(v) [resp. type-(iii)] \( C \)-properly quasiconcave, which is equivalent to type-(v) [resp. type-(iii)] \( -C \)-properly quasiconvex mapping.

The following theorem is a generalization (in the compact case) of a scalar two-function result of Simon \([S, \text{Theorem 1.4}]\), which in turn is a generalization of Sion's minimax theorem \([S]\).

**Theorem 3.3 (Minimax theorem II).** Suppose that \( E_1 \) and \( E_2 \) are topological vector spaces, \( X \subset E_1 \) is a nonempty convex compact subset, \( K \subset E_2 \) is a nonempty convex compact subset, \( C \) is closed convex strongly pointed cone with nonempty interior in a Banach space \( Y \) and \( F, G : X \times K \rightarrow 2^Y \) are multifunctions, bounded with respect to \( C \) and \( -C \) respectively, such that the set \( \cup_{y \in K} \sup_{x \in X} F(x, y) \) is bounded with respect to \( -C \) and the set \( \cup_{x \in X} \inf_{y \in K} G(x, y) \) is bounded with respect to \( C \). Suppose that \( F \) and \( G \) satisfy the following conditions:

(i) \( G(x, \cdot) \) is type-(iii) \( C \)-properly quasiconvex on \( K \) for every \( x \in X \); and \( F(\cdot, y) \) is type-(iii) \( C \)-properly quasiconcave on \( K \) for every \( y \in K \);

(ii) \( G(\cdot, y) \) is \( -C \)-lower semicontinuous for every \( y \in K \), and \( F(x, \cdot) \) is \( C \)-lower semicontinuous for every \( x \in X \).

(iii) \( F(x, y) - G(x, y) \subset -C \) for every \( x \in X, y \in K \).

Then there exist two points

\[
Z_1 = \sup_{x \in X} \inf_{y \in K} G(x, y)
\]

and

\[
Z_2 = \inf_{y \in K} \sup_{x \in X} F(x, y)
\]

such that \( Z_1 - Z_2 \in C \).

For the proof of this theorem we need the following result.

**Theorem 3.4 ([G-T] Theorem 4.4).** Let \( K \) be a nonempty convex subset of a topological vector space \( E \), \( Y \) a Banach space, and \( F : K \times K \rightarrow 2^Y \) a multifunction. Assume that

1. \( C : K \rightarrow 2^Y \) is a multifunction with a closed graph such that \( C(x) \) is closed convex cone with compact base \( B(x) = (2B_Y \setminus B_Y) \cap C(x) \) for every \( x \);

2. for every \( x, y \in K \), \( F(\cdot, y) \) is \( C(x) \)-lower semicontinuous and locally bounded;

3. there exists a multifunction \( G : K \times K \rightarrow 2^Y \) such that

(a) for every \( x \in K \), \( G(x, x) \subset -C(x) \),

(b) \( F(x, y) \not\subset -C(x) \) implies \( G(x, y) \not\subset -C(x) \),
(c) $G(x, \cdot)$ is type-(iii) $C(x)$-properly quasiconcave on $K$ for every $x \in K$;

4. there exists a nonempty compact convex subset $D$ of $K$ such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x, y) \subset -C(x)$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \subset -C(x), \text{ for all } y \in K\}$$

is a nonempty and compact subset of $D$.

**Proof of Theorem 3.3.** Define the mapping $H : X \times K \times X \times K \to 2^Y$ by

$$H(\tilde{x}, \tilde{y}, x, y) = F(x, \tilde{y}) - G(\tilde{x}, y).$$

Applying Theorem 3.4 for $H$ we obtain that there exists $x_0, y_0$ such that

$$H(x_0, y_0, x, y) \subset -C \quad \forall x \in X, \forall y \in K,$n

whence

$$\sup \bigcup_{x \in X} F(x, y_0) - \inf \bigcup_{y \in K} G(x_0, y) \subset -C.$$ (4)

By (2) we obtain

$$\sup \bigcup_{x \in X} F(x, y_0) \subset \inf \bigcup_{y \in K} \sup \bigcup_{x \in X} F(x, y) + C \quad \text{and} \quad \inf \bigcup_{y \in K} G(x_0, y) \subset \sup \bigcup_{x \in X} \inf \bigcup_{y \in K} G(x, y) - C.$$

Therefore, by (4) there exist

$$z_1 \in \sup \bigcup_{x \in X} \inf_{y \in K} G(x, y), c_1 \in C$$

and

$$z_2 \in \inf \bigcup_{y \in K} \sup \bigcup_{x \in X} F(x, y), c_2 \in C$$

such that

$$z_2 + c_2 - (z_1 - c_1) \in -C,$$

which implies

$$z_1 - z_2 \in C + c_1 + c_2 \subset C. \quad \blacksquare$$

4 Nash equilibrium and loose saddle point theorems

**Definition 4.1** The multifunction $F : E \ni X \rightarrow 2^X$, where $X$ is a convex nonempty subset, is called $C$-convex, if for every $x, y \in X, \lambda \in [0, 1], u \in \lambda F(x) + (1 - \lambda)F(y)$ there exists $v \in F(\lambda x + (1 - \lambda)y)$ such that $u - v \in C$. If $F$ is $-C$-convex, then $F$ is called $C$-concave.

Let $k^0 \in \text{int} C$ be fixed. Define the functions

$$h(x) = \inf \{t \in \mathbb{R} : x \in tk^0 - C\},$$

$$\varphi(x) = \inf h(F(x)),$$

$$\psi(x) = \sup h(F(x)).$$
It is easy to see that $h$ is continuous and sublinear (see [Tam1], [Tam2]).

**Lemma 4.2** Let the multifunction $F : E \supset X \rightarrow 2^Z$ be $C$-convex. Then the function $\varphi$ is convex.

**Proof.** Let $x_1, x_2 \in X$. By definition of $\varphi$ and $h$, for every $\varepsilon > 0$ there exist $z_i \in F(x_i), t_i \in \mathbb{R}, i = 1, 2$ such that

$$z_i - t_i k^0 \in -C$$

and

$$t_i < \varphi(x_i) + \varepsilon.$$  

By definition of $C$-convex multifunction,

$$\exists v \in F(\lambda x_1 + (1 - \lambda) x_2) : \lambda z_1 + (1 - \lambda) z_2 \in v + C.$$  

By (5) we have

$$-C \ni \lambda (z_1 - t_1 k^0) + (1 - \lambda) (z_2 - t_2 k^0) = \lambda z_1 + (1 - \lambda) z_2 - (\lambda t_1 + (1 - \lambda) t_2) k^0.$$  

By (6) and (7) we have

$$v \in \lambda z_1 + (1 - \lambda) z_2 - C$$

$$\subseteq (\lambda t_1 + (1 - \lambda) t_2) k^0 - C - C$$

$$= (\lambda t_1 + (1 - \lambda) t_2) k^0 - C.$$  

Hence

$$h(v) \leq \lambda t_1 + (1 - \lambda) t_2$$

$$< \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2) + 2\varepsilon.$$  

Therefore

$$\varphi(\lambda x_1 + (1 - \lambda) x_2) := \inf_{z \in F(\lambda x_1 + (1 - \lambda) x_2)} h(z) \leq \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2) + 2\varepsilon.$$  

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\varphi(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2).$$  

**Definition 4.3** The multifunction $F : E \rightarrow 2^Z$ will be called $(C, k^0)$-upper semicontinuous at $x_0$, if for every $\varepsilon > 0$ there exists an open $U \ni x_0$ such that

$$[(\varphi(x_0) - \varepsilon) k^0 - C] \cap F(x) = \emptyset \quad \forall x \in U.$$  

**Lemma 4.4** If $F$ is $-C$-lower semicontinuous, then $\varphi$ is upper semicontinuous.

**Proof.** Let $x_0 \in E, \varepsilon > 0$ be fixed and $y_0 \in F(x_0)$ be such that

$$h(y_0) < \inf h(F(x_0)) + \varepsilon.$$
By continuity of \( h \), there exists an open \( V \ni 0 \) such that
\[
h(v) < \varepsilon \quad \forall v \in V.
\]
By definition of \(-C\)-lower semicontinuity, there exists an open \( U \ni x_0 \) such that
\[
F(x) \cap (y_0 + V - C) \neq \emptyset \quad \forall x \in U.
\]
Let \( y \in F(x) \cap (y_0 + V - C) \). Then \( y = y_0 + v - c \) for some \( v \in V, c \in C \) and we can write
\[
\varphi(x) = \inf_{y \in F(x)} h(y')
\leq h(y)
\leq h(y_0) + h(v) + h(-c) \quad \text{(by sublinearity of } h)\]
\[
\leq \varphi(x_0) + 2\varepsilon.
\]

**Lemma 4.5** If \( F \) is \((C, k^0)\)-upper semicontinuous, then \( \varphi \) is lower semicontinuous.

**Proof.** Let \( x_0 \in E, y \in F(x_0) \) and \( x \in U \), where \( U \) is given by the definition of \((C, k^0)\)-upper semicontinuity of \( F \) at \( x_0 \). Let \( z \in F(x) \). Then by definition we have:
\[
0 \leq \inf\{t : z - tk^0 \in (\varphi(x_0) - \varepsilon)k^0 - C\}
\leq \inf\{t : z - (t + \varphi(x_0) - \varepsilon)k^0 \in -C\}
\leq \varepsilon - \varphi(x_0) + \inf\{t : z - tk^0 \in -C\}
\leq \varepsilon - \varphi(x_0) + h(z).
\]

Hence \( \varphi(x_0) \leq h(z) + \varepsilon \), and \( z \in F(x) \) is arbitrary, this implies \( \varphi(x_0) \leq \varphi(x) + \varepsilon. \bullet \)

Below we prove a Nash equilibrium type theorem and a loose saddle point theorem. The proofs are based on scalarization via the previous lemmas and on the Ky Fan inequality.

Let \( E_1, E_2 \) be topological vector spaces, \( Z \) be a Banach space, \( X \subset E_1, Y \subset E_2 \) be convex compact nonempty subsets and \( C_i \subset Z \) be closed convex cones with nonempty interiors, \( k_i^0 \in \text{int}C_i, i = 1, 2. \)

**Theorem 4.6** (Nash equilibrium). Let the multifunctions \( F_i : X \times Y \rightarrow 2^Z \) be \((C_i, k_i^0)\)-upper semicontinuous. Assume that \( F_1(\cdot, y) \) is \( C_i \)-convex for every \( y \in Y \), \( F_1(x, \cdot) \) is \(-C_i\)-lower semicontinuous for every \( x \in X \), \( F_2(x, \cdot) \) is \( C_2 \)-convex for every \( x \in X \) and \( F_2(\cdot, y) \) is \(-C_2\)-lower semicontinuous for every \( y \in Y \). Then there exists a Nash equilibrium, \((x_0, y_0) \in X \times Y \), which means
\[
F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0))k_1^0 - \text{int}C_1] = \emptyset \quad \forall x \in X,
\]
\[
F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0))k_2^0 - \text{int}C_2] = \emptyset \quad \forall y \in Y.
\]

**Proof.** Define
\[
f(x, y, \overline{x}, \overline{y}) = \inf h(F_1(x, y)) - \inf h(F_1(\overline{x}, y)) + \inf h(F_2(x, y)) - \inf h(F_2(x, \overline{y}))
\]

By Lemma 4.2, \( f(x, y, \cdot, \cdot) \) is concave for every \( x \in X, y \in Y \) and by Lemmas 4.4, 4.5, \( f(\cdot, \cdot, \overline{x}, \overline{y}) \) is lower semicontinuous for every \( \overline{x} \in X, \overline{y} \in Y \). By Ky Fan's inequality (see [A-E, Theorem 6.3.5]) there exists \((x_0, y_0) \in X \times Y \) such that
\[
\sup_{(\overline{x}, \overline{y}) \in X \times Y} f(x_0, y_0, \overline{x}, \overline{y}) \leq 0
\]
Putting $\overline{y} = y_0$ we obtain
\[
\inf h(F_1(x_0, y_0)) \leq \inf h(F_1(x, y_0)) \ orall x \in X,
\]
and putting $\overline{x} = x_0$ we obtain
\[
\inf h(F_2(x_0, y_0)) \leq \inf h(F_2(x, y)) \ orall y \in Y.
\]
But (8) implies
\[
F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0))k_1^0 \cap \text{int}C_1] = \emptyset
\]
and (9) implies
\[
F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0))k_2^0 \cap \text{int}C_2] = \emptyset,
\]
which finishes the proof.
\[\blacksquare\]

In the special case when $F_1 = -F_2$ and $C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$, we obtain the following loose saddle point theorem.

**Theorem 4.7 (Loose saddle point theorem).** Suppose that the multifunction $F : X \times Y \to 2^Z$ have compact images and is $(C, k^0)$-lower semicontinuous and $(-C, -k^0)$-lower semicontinuous, $F(\cdot, y), y \in Y$ is $C$-convex and $C$-lower semicontinuous, $F(x, \cdot), x \in X$ is $C$-concave and $-C$-lower semicontinuous. Then there exists a loose saddle point $(x_0, y_0) \in X \times Y$, namely there exist $z_1, z_2 \in F(x_0, y_0)$, such that
\[
(z_1 - \text{int}C) \cap F(x, y_0) = \emptyset \ \forall x \in X,
\]
\[
(z_2 + \text{int}C) \cap F(x_0, y) = \emptyset \ \forall y \in Y.
\]

**REFERENCES**


