1. Introduction

Let $C$ be a nonempty closed convex subset of a real Banach space $E$. Then a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. For any $x \in C$, the $\omega$-limit set of $x$ is defined by

$$\omega(x) = \{ z \in C : z = \lim_{i \to \infty} T^{n_i}x \text{ with } n_i \to \infty \text{ as } i \to \infty \}.$$ 

Similarly, the $\omega$-limit set of $x$ for a one-parameter semigroup $S$ on $C$ is defined by

$$\omega(S, x) = \{ z \in C : z = \lim_{i \to \infty} T(s_i)x \text{ with } s_i \to \infty \text{ as } i \to \infty \}.$$ 

Edelstein [10] obtained the following nonlinear ergodic theorem for nonexpansive mappings with compact domains in a strictly convex Banach space:

**Theorem 1.1** (Edelstein). Let $C$ be a nonempty compact convex subset of a strictly convex Banach space and let $T$ be a nonexpansive mapping of $C$ into itself. Let $x \in C$. Then, for any $\xi \in \overline{co}\omega(x)$, the Cesàro mean $S_n(\xi) = (1/n) \sum_{k=0}^{n-1} T^k \xi$ converges strongly to a fixed point of $T$, where $\overline{co}A$ is the closure of the convex hull of $A$.

Dafermos and Slemrod [9] also obtained the following theorem:

**Theorem 1.2** (Dafermos and Slemrod). Let $C$ be a nonempty compact convex subset of a strictly convex Banach space and let $S = \{ T(t) : 0 \leq t < \infty \}$ be a one-parameter nonexpansive semigroup on $C$. Let $x \in C$. Then, for any $\xi \in \overline{co}\omega(S, x)$, $(1/t) \int_{0}^{t} T(s)\xi ds$ converges strongly to a common fixed point of $T(t), t \in \mathbb{R}^+.$

On the other hand, the first nonlinear weak ergodic theorem for nonexpansive mappings with bounded domains was established in the framework of a Hilbert space by Baillon [5]. Bruck [7] extended Baillon's theorem in [5] to a uniformly convex Banach space whose norm is Fréchet differentiable. Brézis and Browder [6] also proved a nonlinear strong ergodic theorem for nonexpansive mappings of odd-type in a Hilbert space (see also Reich [15]).

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The purpose of this paper is to study nonlinear strong ergodic theorems for families of nonexpansive mappings with compact domains in a strictly convex Banach space. In Section 2, we give an improved result of Edelstein’s theorem in [10] by using Bruck [7, 8] and [1, 2]. In Section 3, we give a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup. In Section 4, we study nonlinear strong ergodic properties for commutative semigroups of nonexpansive mappings in a strictly convex Banach space.

2. THEOREM FOR NONEXPANSIVE MAPPINGS

Throughout this paper, we assume that a Banach space $E$ is real. We denote by $E^*$ the dual space of $E$ and by $\mathbb{N}$ the set of all positive integers. In addition, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset $A$ of $E$, $\overline{A}$ and $\text{co} A$ mean the closer of $A$, the convex hull of $A$ and the closure of the convex hull of $A$, respectively. We write $x_n \to x$ (or $\lim_{n \to \infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly to $x$.

A Banach space $E$ is said to be strictly convex if $\|x + y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$. Throughout this paper, we assume that $E$ is a strictly convex Banach space.

In this section, we give a nonlinear strong ergodic theorem for nonexpansive mappings with compact domains in a strictly convex Banach space. The following Lemma will be useful for us.

**Lemma 2.1** ([2]). Let $C$ be a nonempty compact convex subset of $E$ and let $T$ be a nonexpansive mapping of $C$ into itself. Let $x \in C$ and $n \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists $l_0 = l_0(n, \varepsilon) \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^{l+k+m} x - T^k \left( \frac{1}{n} \sum_{l=0}^{n-1} T^{l+m} x \right) \right\| < \varepsilon$$

for every $m \geq l_0$.

Using Lemma 2.1, we can prove the following lemma.

**Lemma 2.2** ([2]). Let $C$ be a nonempty compact convex subset of $E$ and let $T$ be a nonexpansive mapping of $C$ into itself. Let $x \in C$. Then, there exists a sequence $\{t_n\}$ in $\mathbb{N}$ such that for each $z \in F(T)$,

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+t_n} x - z \right\|$$

exists.
Remark 2.3 ([2]). In Lemma 2.2, take a sequence \( \{i'_n\} \) in \( \mathbb{N} \) such that \( i'_n \geq i_n \) for each \( n \in \mathbb{N} \). Then, we can see that
\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n'} x - z \right\| = \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\|
\]
for every \( z \in \mathcal{F}(T) \).

The following lemma plays an important role in the proof of Theorem 2.5.

Lemma 2.4 ([2]). Let \( C \) be a nonempty compact convex subset of \( E \). Then,
\[
\lim_{n \to \infty} \sup_{T \in N(C)} \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^{i} y - T \left( \frac{1}{n} \sum_{i=0}^{n-1} T^{i} y \right) \right\| = 0,
\]
where \( N(C) \) denotes the set of all nonexpansive mappings of \( C \) into itself.

Using Lemma 2.2, 2.4 and Remark 2.3, we can prove a nonlinear strong ergodic theorem for nonexpansive mappings (see [2]).

Theorem 2.5 ([2]). Let \( X \) be a nonempty closed convex subset of \( E \). Let \( T \) be a nonexpansive mapping of \( X \) into itself such that \( T(X) \subset K \) for some compact subset \( K \) of \( X \) and let \( x \in X \). Then, \( (1/n) \sum_{i=0}^{n-1} T^{i} x \) converges strongly to a fixed point of \( T \) uniformly in \( h \in \mathbb{N} \cup \{0\} \).

3. THEOREM FOR A ONE-PARAMETER NONEXPANSIVE SEMIGROUP

In this section, we give a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup with compact domains in a strictly convex Banach space.

A family \( S = \{T(s) : 0 \leq s < \infty\} \) of mappings of \( C \) into itself is called a one-parameter nonexpansive semigroup on \( C \) if it satisfies the following conditions:

(i) \( T(0)x = x \) for all \( x \in C \);
(ii) \( T(s + t) = T(s)T(t) \) for all \( s, t \in \mathbb{R}^+ \);
(iii) \( \|T(s)x - T(t)y\| \leq \|x - y\| \) for all \( x, y \in C \) and \( s, t \in \mathbb{R}^+ \);
(iv) for each \( x \in C \), \( s \mapsto T(s)x \) is continuous.

We denote by \( \mathcal{F}(S) \) the set of common fixed points of \( T(t), t \in \mathbb{R}^+ \), that is, \( \mathcal{F}(S) = \bigcap_{0 \leq t < \infty} \mathcal{F}(T(t)) \).

The following lemma will be useful for us.

Lemma 3.1 ([3]). Let \( C \) be a nonempty compact convex subset of \( E \) and let \( S = \{T(s) : 0 \leq s < \infty\} \) be a one-parameter nonexpansive semigroup on \( C \). Let \( x \in C \) and \( t > 0 \).
Then, for any $\varepsilon > 0$, there exists $p_t = p_t(\varepsilon) \in \mathbb{R}^+$ such that
\[
\sup_{h \in \mathbb{R}^+} \left\| \frac{1}{t} \int_0^t T(h + p + \tau)xd\tau - T(h) \left( \frac{1}{t} \int_0^t T(p + \tau)xd\tau \right) \right\| < \varepsilon
\]
for every $p \geq p_t$.

Using Lemma 3.1, we can show the following lemma.

**Lemma 3.2** ([3]). Let $C$ be a nonempty compact convex subset of $E$ and let $S = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on $C$. Let $x \in C$. Then, there exists a net $\{p_t\}$ in $\mathbb{R}^+$ such that for each $z \in F(S)$,
\[
\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)xd\tau - z \right\|
\]
exists.

**Remark 3.3** ([3]). In Lemma 3.2, take a net $\{p_t'\}$ in $\mathbb{R}^+$ such that $p_t' \geq p_t$ for each $t > 0$. Then, we can see
\[
\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)xd\tau - z \right\| = \lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t')xd\tau - z \right\|
\]
for every $z \in F(S)$.

The following lemma plays an important role in the proof of Theorem 3.5.

**Lemma 3.4** ([3]). Let $C$ be a nonempty compact convex subset of $E$ and let $S = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on $C$. Then, for any $h \in \mathbb{R}^+$,
\[
\lim_{t \to \infty} \sup_{y \in C} \left\| \frac{1}{t} \int_0^t T(s)yd\tau - T(h) \left( \frac{1}{t} \int_0^t T(s)yd\tau \right) \right\| = 0.
\]

Using Lemmas 3.2, 3.4 and Remark 3.3, we can show a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup (see [3]).

**Theorem 3.5** ([3]). Let $C$ be a nonempty compact convex subset of $E$. Let $S = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on $C$ and let $x \in C$. Then, $(1/t) \int_0^t T(\tau + h)xd\tau$ converges strongly to a common fixed point of $T(t), t \in \mathbb{R}^+$ uniformly in $h \in \mathbb{R}^+$. In this case, if $Qx = \lim_{t \to \infty} (1/t) \int_0^t T(s)yd\tau$ for each $x \in C$, then $Q$ is a nonexpansive mapping of $C$ onto $F(S)$ such that $QT(q) = T(q)Q = Q$ for every $q \in \mathbb{R}^+$ and $Qx \in \overline{co}\{T(s)x : 0 \leq s < \infty\}$ for every $x \in C$.

4. **Theorem for Commutative Semigroups**

In this section, we establish our main strong mean ergodic theorem for commutative semigroups with compact domains in a strictly convex Banach space. Throughout the rest of this paper, we assume that $S$ is a commutative semigroup with identity unless other specified. In this case, $(S, \leq)$ is a directed system when the binary relation $\leq$ on $S$ is defined by $a \leq b$ if and only if there is $c \in S$ with $a + c = b$. 

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*Note: The content is a continuation of a mathematical proof and should be read in context with the rest of the text.*

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Let $B(S)$ be the Banach space of all bounded real-valued functions on $S$ with the supremum norm. Then, for each $s \in S$ and $g \in B(S)$, we can define $r_sg \in B(S)$ by $(r_sg)(t) = g(t + s)$ for all $t \in S$. We also denote by $r^*_g$ the conjugate operator of $r_g$. Let $D$ be a subspace of $B(S)$ and let $\mu$ be an element of $D^*$. Then, we denote by $\mu(g)$ the value of $\mu$ at $g \in D$. Sometimes, $\mu(g)$ will also be denoted by $\mu_t(g(t))$ or $\int g(t)d\mu(t)$. When $D$ contains 1, a linear functional $\mu$ on $D$ is called a mean on $D$ if $\|\mu\| = \mu(1) = 1$. Further, let $D$ be $r_s$-invariant, i.e., $r_s(D) \subset D$ for every $s \in S$. Then, a mean $\mu$ on $D$ is said to be invariant if $\mu(r_sg) = \mu(g)$ for all $s \in S$ and $g \in D$. For $s \in S$, we can define the point evaluation $\delta_s$ by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a finite mean on $S$. A finite mean $\mu$ on $S$ is also a mean on any subspace $D$ of $B(S)$ containing 1.

The following definition which was introduced by Takahashi [17] is crucial in the non-linear ergodic theory for abstract semigroups (see also [11]). Let $f$ be a function of $S$ into $E$ such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Let $D$ be a subspace of $B(S)$ containing 1 and $r_s$-invariant for every $s \in S$. Assume that for each $x^* \in E^*$, the function $t \mapsto \langle f(t), x^* \rangle$ is in $D$. Then, for any $\mu \in D^*$ there exists a unique element $f_\mu \in E$ such that

$$ \langle f_\mu, x^* \rangle = \int \langle f(t), x^* \rangle d\mu(t) $$

for all $x^* \in E^*$. If $\mu$ is a mean on $D$, then $f_\mu$ is contained in $\overline{\text{co}}\{f(t) : t \in S\}$ (for example, see [12, 13, 17]). Sometimes, $f_\mu$ will be denoted by $\int f(t)d\mu(t)$.

Let $C$ be a subset of a Banach space $E$. Then, a family $S = \{T(s) : s \in S\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(i) $T(s + t) = T(s)T(t)$ for all $s, t \in S$;
(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(S)$ the set of common fixed points of $T(t), t \in S$, that is, $F(S) = \bigcap_{t \in S} F(T(t))$. If $C$ is a compact convex subset of strictly convex Banach space $E$ and $S$ is commutative, then we know that $F(S)$ is nonempty. Let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$ such that for each $x \in C$, $\{T(t)x : t \in S\}$ is contained in a weakly compact, convex subset of $C$. Let $D$ be a subspace of $B(S)$ containing 1 with the property that the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of $D$ for each $x \in C$ and $x^* \in E^*$, and let $\mu$ be a mean on $D$. Following [16], we also write $T_\mu x$ instead of $\int T(t)x d\mu(t)$ for $x \in C$. We remark that $T_\mu$ is a nonexpansive mapping of $C$ onto itself and $T_\mu x = x$ for each $x \in F(S)$.

The following lemma will be useful for us (see Lemmas 2.1 and 3.1).

**Lemma 4.1** ([4]). Let $C$ be a nonempty compact convex subset of $E$ and let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. Let $x \in C$. Then, for any finite mean $\mu$ on $S$
and $\epsilon > 0$, there exists $w_0 = w_0(\mu, \epsilon) \in S$ such that

\[
\left\| \int T(h + s + w)xd\mu(s) - T(h) \left( \int (s + w)xd\mu(s) \right) \right\| < \epsilon
\]

for every $h \in S$ and $w \geq w_0$.

Using Lemma 4.1, we can prove the following lemma (see Lemmas 2.2 and 3.2).

**Lemma 4.2** ([4]). Let $C$ be a nonempty compact convex subset of $E$ and let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. Let $x \in C$ and let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on $S$ such that

\[
\lim_{\alpha} \|\mu_\alpha - r_{*}^* \mu_\alpha\| = 0 \quad \text{and} \quad \lim_{\beta} \|\lambda_\beta - r_{*}^* \lambda_\beta\| = 0 \quad \text{for every} \quad t \in S.
\]

Then, there exist nets $\{p_\alpha : \alpha \in I\}$ and $\{q_\beta : \beta \in J\}$ in $S$ such that for any $z \in F(S),

\[
\lim_{\alpha} \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\| = \lim_{\beta} \left\| \int T(q_\beta + t)xd\lambda_\beta(t) - z \right\|. \quad (1)
\]

**Remark 4.3** ([4]). In Lemma 4.2, take nets $\{p_\alpha' : \alpha \in I\}$ and $\{q_\beta' : \beta \in J\}$ in $S$ such that $p_\alpha' \geq p_\alpha$ and $q_\beta' \geq q_\beta$. Then, we can see

\[
\lim_{\alpha} \left\| \int T(p_\alpha' + t)xd\mu_\alpha(t) - z \right\| = \lim_{\beta} \left\| \int T(q_\beta' + t)xd\lambda_\beta(t) - z \right\|
\]

for every $z \in F(S)$.

The following lemma plays an important role in the proof of Lemma 4.5 (see Lemmas 2.4 and 3.4).

**Lemma 4.4** ([4]). Let $C$ be a nonempty compact convex subset of $E$, let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$ and let $x \in C$. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on $S$ such that

\[
\lim_{\alpha} \|\mu_\alpha - r_{*}^* \mu_\alpha\| = 0 \quad \text{for every} \quad t \in S. \quad (\ast)
\]

Then, for any $\epsilon > 0$ and $t \in S$, there exists $\alpha_0(\epsilon, t) \in I$ such that

\[
\left\| \int (s + p)xd\mu_\alpha(s) - T(t) \left( \int (s + p)xd\mu_\alpha(s) \right) \right\| < \epsilon
\]

for all $\alpha \geq \alpha_0(\epsilon, t)$ and $p \in S$.

Using Lemmas 4.2, 4.4 and Remark 4.3, we can show the following lemma which is crucial to prove the main theorem (Theorem 4.6).

**Lemma 4.5** ([4]). Let $X$ be a nonempty closed convex subset of $E$ and let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $X$. Assume $\bigcup_{t \in S} T(t)(X) \subset K$ for some compact subset $K$ of $X$. Let $D$ be a subspace of $B(S)$ such that $1 \in D$, $D$ is $r_s$-invariant for each
s ∈ S and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of $D$ for each $x \in X$ and $x^* \in E^*$. Let \( \{\mu_\alpha : \alpha \in I\} \) be a net of finite means on $S$ such that
\[
\lim_\alpha \|\mu_\alpha - r^*_s \mu_\alpha\| = 0 \quad \text{for every } s \in S.
\]
Then, for any $x \in X$, $\int T(p + t) xd\mu_\alpha(t)$ converges strongly to a common fixed point $y_0$ of $T(t), t \in S$ uniformly in $p \in S$. Furthermore, $y_0$ is independent of $\{\mu_\alpha : \alpha \in I\}$ and for any invariant mean $\mu$ on $D$, $y_0 = T_\mu x = \int T(t) xd\mu(t)$.

**Sketch of proof.** Let $x \in X$. From Mazur’s theorem, $C = \overline{co}(\{x\} \cup \bigcup_{t \in S} T(t)(X))$ is a compact subset of $X$. We see that $C = \overline{co}(\{x\} \cup \bigcup_{t \in S} T(t)(X))$ is convex and invariant under $T(t), t \in S$. Thus, we may assume that $S = \{T(t) : t \in S\}$ is a nonexpansive semigroup on a compact convex subset of $X$.

Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on $S$ such that
\[
\lim_\alpha \|\mu_\alpha - r^*_t \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r^*_t \lambda_\beta\| = 0
\]
for each $t \in S$. By Lemma 4.2, we can take a net $\{p_\alpha\}$ in $S$ such that for any $z \in F(S)$,
\[
\lim_\alpha \left\| \int T(p_\alpha + t) xd\mu_\alpha(t) - z \right\| = 0
\]
exists. Let $\{\Phi_\alpha\} = \{\int T(p_\alpha + t) xd\mu_\alpha(t) : \alpha \in I\}$. Then, we first prove that $\Phi_\alpha$ converges strongly to a common fixed point of $T(t), t \in S$. From the compactness, $\{\Phi_\alpha\}$ must contain a subnet which converges strongly to a point. So, let $\{\Phi_{\alpha,}\}$ be a subnet of $\{\Phi_\alpha\}$ such that $\lim_{\alpha,} \Phi_{\alpha,} = y_0$. Using Lemma 4.4, we can show that $y_0$ is a common fixed points of $T(t), t \in S$. So, from (2), we have
\[
\lim_\alpha \|\Phi_\alpha - y_0\| = \lim_\gamma \|\Phi_{\alpha,} - y_0\| = 0.
\]
This implies that $\Phi_\alpha \rightarrow y_0$.

Next we prove that $\int T(h + t) xd\mu_\alpha(t)$ converges strongly to $y_0 \in F(S)$ uniformly in $h$. In the above argument, take a net $\{p_\alpha' : \alpha \in I\}$ in $S$ such that $p_\alpha' \geq p_\alpha$ for each $\alpha \in I$. Then, repeating the above argument, we see that $\Phi_\alpha' = \int T(p_\alpha' + t) xd\mu_\alpha(t)$ converges strongly to a common fixed point $y_1$ of $T(t), t \in S$. By Remark 4.3, we can show $y_0 = y_1 \in F(S)$. Since $\{p_\alpha'\}$ is an arbitrary net in $S$ such that $p_\alpha' \geq p_\alpha$ for each $\alpha \in I$, we have that $\int T(h + p_\alpha + t) xd\mu_\alpha(t)$ converges strongly to $y_0$ uniformly in $h \in S$. Hence, we can show that $\int T(h + t) xd\lambda_\beta(t)$ converges strongly to $y_0$ uniformly in $h \in S$. Since $\{\lambda_\beta : \beta \in J\}$ and $\{\mu_\alpha : \alpha \in I\}$ are arbitrary nets of finite means on $S$ such that
\[
\lim_\beta \|\lambda_\beta - r^*_t \lambda_\beta\| = 0 \quad \text{and} \quad \lim_\beta \|\mu_\alpha - r^*_t \mu_\alpha\| = 0,
\]
for every $t \in S$, we see that such an element $y_0$ of $F(S)$ is independent of $\{\lambda_\beta : \beta \in J\}$ and $\{\mu_\alpha : \alpha \in I\}$. Further, we can prove that for any invariant mean $\mu$ on $D$, $y_0 = T_\mu x$. □
Let $D$ be a subspace of $B(S)$ containing 1 and $r_s$-invariant for every $s \in S$. Then, a net $\{\mu_\alpha : \alpha \in I\}$ of linear functionals on $D$ is called strongly regular [11] if it satisfies the following conditions:

(a) $\sup_\alpha ||\mu_\alpha|| < +\infty$;
(b) $\lim_\alpha \mu_\alpha(1) = 1$;
(c) $\lim_\alpha ||\mu_\alpha - r_s^*\mu_\alpha|| = 0$ for every $s \in S$.

Now, we can show a nonlinear strong ergodic theorem for commutative semigroups.

**Theorem 4.6** ([4]). Let $X$ be a nonempty a closed convex subset of $E$ and let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $X$. Assume $\bigcup_{t \in S} T(t)(X) \subset K$ for some compact subset $K$ of $X$. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on $D$ and let $x \in X$. Then, $\int T(h+t)xd\lambda_\alpha(t)$ converges strongly to a common fixed point $y_0$ of $T(t), t \in S$ uniformly in $h \in S$. Further, such an element $y_0$ of $F(S)$ is independent of $\{\lambda_\alpha\}$ and for any invariant mean $\mu$ on $D$, $y_0 = T_{\mu}x = \int T(t)xd\mu(t)$. In this case, putting $Qx = \lim_\alpha \int T(t)xd\lambda_\alpha(t)$ for each $x \in X$, $Q$ is a nonexpansive mapping of $X$ onto $F(S)$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{\operatorname{co}\{T(s)x : s \in S\}}$ for every $x \in X$.

**Sketch of proof.** Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on $D$ and let $\{\mu_\beta : \beta \in B\}$ be a net of finite means on $S$ such that

$$\lim_\beta ||\mu_\beta - r_t^*\mu_\beta|| = 0 \quad \text{for every } t \in S. \quad (\ast)$$

From Lemma 4.5, we have that $\int T(h+t)xd\mu_\beta(t)$ converges strongly to a common fixed point $y_0$ of $T(t), t \in S$ uniformly in $h \in S$. Let $\varepsilon > 0$ and let $\mu$ be an invariant mean on $D$. From Lemma 4.5, we also know $y_0 = T_{\mu}x$. Further, there exists $\beta_1$ such that

$$\left\| \int T(h+t)xd\mu_\beta(t) - T_{\mu}x \right\| < \frac{\varepsilon}{\sup_\alpha ||\lambda_\alpha||}$$

for all $\beta \geq \beta_1$ and $h \in S$. Suppose

$$\mu_{\beta_1} = \sum_{i=1}^n b_i\delta_{t_i} \quad (b_i \geq 0, \sum_{i=1}^n b_i = 1) \quad (3)$$

and put $\mu_1 = \mu_{\beta_1}$. Then, we have

$$\left\| \int T(h+t)xd\mu_1(t) - T_{\mu}x \right\| < \frac{\varepsilon}{\sup_\alpha ||\lambda_\alpha||} \quad (4)$$
for every $h \in S$. Since $\{\lambda_{\alpha}\}$ is strongly regular, there exists $\alpha_{0}$ such that

$$|1 - \lambda_{\alpha}(1)| < \frac{\varepsilon}{\max\{1, \|T_{\mu}x\|\}}$$

and

$$\|\lambda_{\alpha} - r_{t_i}^*\lambda_{\alpha}\| < \frac{\varepsilon}{\max\{1, M\}}$$

(5)

for every $i \in \{1, 2, \cdots, n\}$ and $\alpha \geq \alpha_{0}$, where $M = \sup_{g \in S} \|T(g)x\|$. Then, we have

$$\|T_{\mu}x - \int T_{\mu}xd\lambda_{\alpha}(s)\| \leq \sup_{x^* \in S_{1}(E^*)} \langle T_{\mu}x, x^* \rangle \cdot |1 - \lambda_{\alpha}(1)| < \varepsilon$$

for every $\alpha \geq \alpha_{0}$ and from (4),

$$\left\| \int T(h + s + t)xd\mu_{1}(t)d\lambda_{\alpha}(s) - \int T_{\mu}xd\lambda_{\alpha}(s) \right\| < \varepsilon$$

for every $h \in S$ and $\alpha \in A$. Thus, we obtain

$$\left\| \int T(h + s + t)xd\mu_{1}(t)d\lambda_{\alpha}(s) - T_{\mu}x \right\| < \varepsilon + \varepsilon = 2\varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_{0}$. On the other hand, from (3) and (5), we have

$$\left\| \int T(h + s)x  d\mu_{1}(t)d\lambda_{\alpha}(s) - \int T(h + s + t)x  d\mu_{1}(t)d\lambda_{\alpha}(s) \right\| \leq \sum_{i=1}^{n} b_i \|\lambda_{\alpha} - r_{t_i}^*\lambda_{\alpha}\| \cdot M < \varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_{0}$. Therefore, we obtain

$$\left\| \int T(h + s)x  d\lambda_{\alpha}(s) - T_{\mu}x \right\| < \varepsilon + 2\varepsilon = 3\varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_{0}$. Then, $\int T(h + t)x  d\lambda_{\alpha}(t)$ converges strongly to a common fixed point $y_0$ of $T(t), t \in S$ uniformly in $h$. Further, such an element $y_0$ is independent of $\{\lambda_{\alpha}\}$ and $y_0 = T_{\mu}x$ for any invariant mean $\mu$ on $D$. If $Qx = \lim_{\alpha} \int T(t)x  d\lambda_{\alpha}(t)$ for each $x \in X$, then $Q$ is a nonexpansive mapping of $X$ onto $F(T)$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{\text{co}}\{T(s)x : s \in S\}$ for every $x \in X$.

5. Applications of the Main Theorem

We now apply Theorem 4.6 to obtain other nonlinear strong ergodic theorems with compact domains.

Theorem 5.1 ([4]). Let $X$ be a nonempty closed convex subset of $E$. Let $T$ be a nonexpansive mapping of $X$ into itself such that $T(X)$ is relatively compact. Then, for each $x \in X$, $(1 - s) \sum_{i=0}^{\infty} s^i T^{i+k}x$ converges strongly to some $y \in F(T)$, as $s \uparrow 1$, uniformly in $k \in \mathbb{Z}^+$. Let $Q = \{q_{n,m}\}_{n,m \in \mathbb{Z}^+}$ be a matrix satisfying the following conditions:
(a) $\sup_{n \in \mathbb{Z}^+} \sum_{m=0}^{\infty} |q_{n,m}| < \infty$;
(b) $\lim_{n \to \infty} \sum_{m=0}^{\infty} q_{n,m} = 1$;
(c) $\lim_{n \to \infty} \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0$.

Then, according to Lorentz [14], $Q$ is called a strongly regular matrix. If $Q$ is a strongly regular matrix, then for each $m \in \mathbb{Z}^+$, we have that $|q_{n,m}| \to 0$, as $n \to \infty$ (see also [11]).

**Theorem 5.2** ([4]). Let $E, X$ and $T$ be as in Theorem 5.1. Let $Q = \{q_{n,m}\}_{n,m \in \mathbb{Z}^+}$ be a strongly regular matrix. Then, for any $x \in X$, $\sum_{m=0}^{\infty} q_{n,m} T^{m+k} x$ converges strongly to some $y \in F(T)$, as $n \to \infty$, uniformly in $k \in \mathbb{Z}^+$.

**Theorem 5.3** ([4]). Let $X$ be a nonempty closed convex subset of $E$. Let $U$ and $T$ be nonexpansive mappings of $X$ into itself with $UT = TU$. Assume $(U(X) \cup T(X)) \subset K$ for some compact subset $K$ of $X$. Then, for each $x \in X$, $(1/n^2) \sum_{i,j=0}^{n-1} U^{i+k} T^{j+h} x$ converges strongly to some $y \in F(U) \cap F(T)$, as $n \to \infty$, uniformly in $k, h \in \mathbb{Z}^+$.

**Theorem 5.4** ([4]). Let $X$ be a nonempty compact convex subset of $E$ and let $S = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on $X$. Then, for any $x \in X$, $r \int_0^\infty e^{-rt} T(t+k)x dt$ converges strongly to some $y \in F(S)$, as $r \downarrow 0$, uniformly in $k \in \mathbb{R}^+$.

Let $Q = \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions:

(a) $\sup_{s \in \mathbb{R}^+} \int_0^\infty |Q(s,t)| dt < \infty$;
(b) $\lim_{s \to \infty} \int_0^\infty Q(s,t) dt = 1$;
(c) $\lim_{s \to \infty} \int_0^\infty |Q(s, t+h) - Q(s, t)| dt = 0$ for every $h \in \mathbb{R}^+$.

Then, $Q$ is called a strongly regular kernel.

**Theorem 5.5** ([4]). Let $E, X, S = \{T(t) : t \in \mathbb{R}^+\}$ be as in Theorem 5.4. Let $Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a strongly regular kernel. Then, for any $x \in X$, $\int_0^\infty Q(s,t) T(t+h)x dt$ converges strongly to some $y \in F(S)$, as $s \to \infty$, uniformly in $h \in \mathbb{R}^+$.

**References**


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