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Related Topics to Transferable Utility Games — An Infinite Market Game —

Hidetoshi Komiya (小宮 英敏)
Faculty of Business and Commerce (商学部)
Keio University (慶應義塾大学)
e-mail:hkomiya@fbc.keio.ac.jp

1 Introduction

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space throughout this paper. A *game* v is a nonnegative real valued function, defined on the σ-field $\mathscr{F}$, which maps the empty set to zero. An *outcome* of a game v is a finitely additive real valued function α on $\mathscr{F}$ scuh that $\alpha(\Omega) = v(\Omega)$. For an outcome α of v, an integrable function f satisfying $\int_S f\,d\mu = \alpha(S)$ for all $S \in \mathscr{F}$ is said to be an *outcome density* of α with respect to μ. An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The *core* of v is the set of outcomes α satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathscr{F}$.

To every game v we associate an extended real number $|v|$ defined by

$$|v| = \sup\left\{\sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_\Omega\right\},$$

where $n = 1, 2, \ldots,$ $S_i \in \mathscr{F}$, λ_i is a real number. The notation χ_A denotes the characteristic function of a subset A of Ω. For a game v with $|v| < \infty$, we define two games $\overline{v}$ and $\hat{v}$ by

$$\overline{v}(S) = \sup\left\{\sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S\right\}, \quad S \in \mathscr{F},$$

$$\hat{v}(S) = \min\left\{\alpha(S) : \alpha \text{ is additive}, \alpha \geq v,\ \alpha(\Omega) = |v|\right\}, \quad S \in \mathscr{F},$$

following [3]. A game v is said to be *balanced* if $v(\Omega) = |v|$, *totally balanced* if $v = \overline{v}$ and *exact* if $v = \hat{v}$, respectively. It is proved in [3] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game v is said to be *monotone* if $S \subset T$ implies $v(S) \leq v(T)$ for any S and T in $\mathscr{F}$. A game v is said to be *continuous* at Ω if it follows that $\lim_{n\to\infty} v(S_n) = v(\Omega)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = \Omega$.

2 Market Games

We denote utilities of players by a Carathéodory type function u defined on $\Omega \times R^l_+$ to R_+, where R^l_+ denotes the nonnegative orthant of the l-dimensional Euclidean space R^l, and R_+ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player ω getting goods x. We always use the ordinary coordinatewise order when having concern with an order in R^l_+. We suppose that the function $u : \Omega \times R^l_+ \to R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R^l_+$;
2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all ω in Ω;
3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_+\} < \infty$, where $B_+ = \{x \in R^l_+ : \|x\| \leq 1\}$, and $\|x\|$ denotes the Euclidean norm of $x \in R^l_+$.

For any set S in $\mathscr{F}$, the set of integrable functions on S to R^l_+ is denoted by $L_1(S, R^l_+)$. We take an element e of $L_1(\Omega, R^l_+)$ as the density of initial endowments for the players. For any S in $\mathscr{F}$, define

$$v(S) \equiv \sup\left\{\int_S u(\omega, x(\omega))\, d\mu(w) : x \in L_1(S, R^l_+),\ \int_S x\, d\mu = \int_S e\, d\mu\right\}.$$

The set function v defined above is called a *market game* derived from the market $(\Omega, \mathscr{F}, \mu, u, e)$.

It is well known that the function $\omega \mapsto u(\omega, x(\omega))$ is measurable for any $x \in L_1(S, R^l_+)$. Moreover we need to show that the function $\omega \mapsto u(\omega, x(\omega))$ is integrable in order to define $v(S)$ as a real number.

Lemma 1 If $x \in L_1(S, R^l_+)$, then $u(\cdot, x(\cdot)) \in L_1(S, R_+)$ for any $S \in \mathscr{F}$ and the map $x \mapsto u(\cdot, x(\cdot))$ is continuous with respect to the norm topologies of $L_1(S, R^l_+)$ and $L_1(S, R_+)$.

Proof Let $x \in L_1(S, R^l_+)$. Since $u(\omega, \cdot)$ is concave, for any $x \in R^l_+$ with $\|x\| > 1$, we have the inequality

$$\frac{u(\omega, x) - u(\omega, x/\|x\|)}{\| x - x/\|x\| \|} \leq \frac{u(\omega, x/\|x\|) - u(\omega, 0)}{\| x/\|x\| - 0 \|},$$

and hence we have $u(\omega, x) \leq \|x\|\sigma$ for any $\omega \in \Omega$ and $x \in R^l_+$ with $\|x\| > 1$. It is obvious from the definition of σ that $u(\omega, x) \leq \sigma$ for any $\omega \in \Omega$ and $x \in R^l_+$ with $\|x\| \leq 1$. Thus we have $u(\omega, x) \leq \sigma(1 + \|x\|)$ for any $(\omega, x) \in \Omega \times R^l_+$ and this leads to the inequalities

$$\begin{aligned}\int_S u(\omega, x(\omega))\, d\mu(\omega) &\leq \int_S \sigma(1 + \|x(\omega)\|)\, d\mu(\omega) \\ &= \sigma\left(\mu(S) + \int_S \|x(\omega)\|\, d\mu(\omega)\right) \\ &< \infty.\end{aligned}$$

Thus it follows that $u(\cdot, x(\cdot)) \in L_1(S, R_+)$. The second part of the assertion is verified in Theorem 2.1 of [2]. Although Theorem 2.1 of [2] is proved under the hypotheses that S is a measurable set in R^l and the second argument x of the function u runs over R, the proof of Theorem 2.1 of [2] is valid even in our setting. Thus the map $x \mapsto u(\cdot, x(\cdot))$ is norm continuous. Q.E.D.

Lemma 2 A market game v is actually a game and is monotone.

Proof It is obvious $v(\emptyset) = 0$. The finiteness of $v(S)$ follows since the inequalities

$$\begin{gathered}\int_S u(\omega, x(\omega))\, d\mu(\omega) \leq \sigma \int_S (1 + \|x\|)\, d\mu \\ \leq \sigma\left(\mu(S) + \sum_{i=1}^{l} \int_S x^i\, d\mu\right) = \sigma\left(\mu(S) + \sum_{i=1}^{l} \int_S e^i\, d\mu\right)\end{gathered}$$

hold if

$$\int_S x\, d\mu = \int_S e\, d\mu,$$

where x^i and e^i are the i-th coordinate functions of x and e, respectively. Moreover v is monotone because u has nonnegative values. Q.E.D.

3 Cores of Market Games

We start with a lemma on concave functions.

Lemma 3 If $f : R^l_+ \to R$ is concave and $f(0) = 0$, then for any $x_1, \ldots, x_n \in R^l_+$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i \leq 1$, it follows that

$$\sum_{i=1}^n \lambda_i f(x_i) \leq f(\sum_{i=1}^n \lambda_i x_i).$$

Proof We can assume that $\lambda = \sum_{i=1}^n \lambda_i > 0$ without loss of generality. It follows that

$$\begin{aligned}
\sum_{i=1}^n \lambda_i f(x_i) &= \lambda \sum_{i=1}^n \frac{\lambda_i}{\lambda} f(x_i) \\
&\leq \lambda f(\sum_{i=1}^n \frac{\lambda_i}{\lambda} x_i) \\
&= (1-\lambda) f(0) + \lambda f(\frac{1}{\lambda} \sum_{i=1}^n \lambda_i x_i) \\
&\leq f(\sum_{i=1}^n \lambda_i x_i).
\end{aligned}$$

Q.E.D.

Let S be a measurable set. For any $x \in L_1(S, R^l_+)$, define $\overline{x} \in L_1(\Omega, R^l_+)$ by

$$\overline{x}(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in S; \\ 0, & \text{if } \omega \in S^c. \end{cases}$$

Proposition 1 A market game v is totally balanced.

Proof Take any $S \in \mathscr{F}$ and $S_i \in \mathscr{F}$ and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i \chi_{S_i} \leq \chi_S$. We can assume that $\mu(S) > 0$ without loss of generality.

Let ϵ be an arbitrary positive number. Take $x_i \in L_1(S_i, R^l_+)$ such that

$$\int_{S_i} x_i \, d\mu = \int_{S_i} e \, d\mu \quad \text{and} \quad v(S_i) - \frac{\epsilon}{n} < \int_{S_i} u(\omega, x_i(\omega)) \, d\mu(\omega),$$

and define $y \in L_1(S, R^l_+)$ by

$$y = \sum_{i=1}^{n} \lambda_i \overline{x}_i.$$

Then we have the following:

$$\begin{aligned}
\int_S y \, d\mu &= \sum_{i=1}^{n} \lambda_i \int_S \overline{x}_i \, d\mu \\
&= \sum_{i=1}^{n} \lambda_i \int_{S_i} e \, d\mu \\
&= \int_S e \sum_{i=1}^{n} \lambda_i \chi_{S_i} \, d\mu \\
&\leq \int_S e \, d\mu.
\end{aligned}$$

Define $y' \in L_1(S, R^l_+)$ by

$$y' = y + \frac{1}{\mu(S)} \left(\int_S e \, d\mu - \int_S y \, d\mu \right).$$

Then it is easily seen that $\int_S y' \, d\mu = \int_S e \, d\mu$.

On the other hand, let $\mathcal{A}$ be the family of all nonempty subsets A of $\{1, \ldots, n\}$ such that $T_A \equiv \bigcap_{i \in A} S_i \cap \bigcap_{j \in A^c} (S \setminus S_j) \neq \emptyset$. Then it is easily seen that $S_i = \bigcup_{A \ni i} T_A$ for $i = 1, \ldots, n$ and $\{T_A : A \in \mathcal{A}\}$ is a partition of $\bigcup_{i=1}^{n} S_i$, and $\sum_{i \in A} \lambda_i \leq 1$ for all $A \in \mathcal{A}$. For any i and A with $i \in A \in \mathcal{A}$, define $x_i^A = x_i|_{T_A}$, the restriction of x_i to T_A. Then we have

$$\overline{x}_i = \sum_{A \ni i} \overline{x}_i^A \quad \text{and} \quad y = \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \overline{x}_i^A.$$

Thus we have

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i v(S_i) - \epsilon &< \sum_{i=1}^{n} \lambda_i \int_{S_i} u(\omega, x_i(\omega))\, d\mu(\omega) \\
&= \sum_{i=1}^{n} \sum_{A \ni i} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega))\, d\mu(\omega) \\
&= \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega))\, d\mu(\omega) \\
&= \sum_{A \in \mathcal{A}} \int_{T_A} \sum_{i \in A} \lambda_i u(\omega, x_i^A(\omega))\, d\mu(\omega) \\
&\leq \sum_{A \in \mathcal{A}} \int_{T_A} u(\omega, \sum_{i \in A} \lambda_i x_i^A(\omega))\, d\mu(\omega) \quad \text{by Lemma 3} \\
&= \int_{S} u(\omega, \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \overline{x}_i^A(\omega))\, d\mu(\omega) \quad \text{by } u(\omega, 0) = 0 \\
&= \int_{S} u(\omega, y(\omega))\, d\mu(\omega) \\
&\leq \int_{S} u(\omega, y'(\omega))\, d\mu(\omega) \quad \text{by monotonicity of } u(\omega, \cdot) \\
&\leq v(S).
\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{n} \lambda_i v(S_i) \leq v(S).$$

Thus $\overline{v}(S) \leq v(S)$ and the reverse inequality is obvious. Hence we have $\overline{v} = v$. Q.E.D.

Proposition 2 A market game v is continuous at Ω.

Proof Let $\{S_n\}$ be a nondecreasing sequence of measurable sets with $\Omega = \bigcup_{n=1}^{\infty} S_n$ and ϵ an arbitrary positive number. Then, there is $x \in L_1(S, R_+^l)$ such that

$$v(\Omega) - \epsilon < \int_{\Omega} u(\omega, x(\omega))\, d\mu(\omega) \quad \text{and} \quad \int_{\Omega} x\, d\mu = \int_{\Omega} e\, d\mu.$$

Let x_n be the restriction $x|_{S_n}$ and define a sequence $\{y_n\}$ of functions in $L_1(S_n, R^l_+)$ by

$$y^i_n = \begin{cases} \frac{\int_{S_n} e^i \, d\mu}{\int_{S_n} x^i_n \, d\mu} x^i_n, & \text{if } \int_{S_n} x^i_n \, d\mu > \int_{S_n} e^i \, d\mu; \\ x^i_n + \frac{1}{\mu(S_n)} \left(\int_{S_n} e^i \, d\mu - \int_{S_n} x^i_n \, d\mu \right), & \text{if } \int_{S_n} x^i_n \, d\mu \leq \int_{S_n} e^i \, d\mu, \end{cases}$$

for $i = 1, \ldots, l$. It is obvious that

$$\int_{S_n} y_n \, d\mu = \int_{S_n} e \, d\mu.$$

On the other hand, since

$$\lim_{n\to\infty} \int_{S_n} |y^i_n - x^i_n| \, d\mu = \lim_{n\to\infty} \left| \int_{S_n} e^i \, d\mu - \int_{S_n} x^i_n \, d\mu \right| = 0,$$

for $i = 1, \ldots, l$, we have

$$\lim_{n\to\infty} \int_{\Omega} \|\overline{y}_n - x\| \, d\mu = \lim_{n\to\infty} \int_{S_n} \|y_n - x\| \, d\mu + \lim_{n\to\infty} \int_{S^c_n} \|x\| \, d\mu = 0,$$

and hence $\overline{y}_n$ converges to x with respect to the norm topology of $L_1(\Omega, R^l_+)$. Therefore, by Lemma 1, it follows that

$$\lim_{n\to\infty} \int_{S_n} u(\omega, y_n(\omega)) \, d\mu(\omega) = \lim_{n\to\infty} \int_{\Omega} u(\omega, \overline{y}_n(\omega)) \, d\mu(\omega) = \int_{\Omega} u(\omega, x(\omega)) \, d\mu(\omega)$$

and hence, for sufficiently large n,

$$v(\Omega) - \epsilon < \int_{S_n} u(\omega, y_n(\omega)) \, d\mu(\omega) \leq v(S_n).$$

Thus we have $\lim_{n\to\infty} v(S_n) = v(\Omega)$. Q.E.D.

Now we have reached our main theorem combining Proposition 1 and Proposition 2.

Theorem 1 A market game v has a nonempty core, and every element α of the core is countably additive and has a unique outcome density f in $L_1(\Omega, R_+)$ with respect to μ, and hence it follows that

$$\alpha(S) = \int_S f \, d\mu, \quad S \in \mathscr{F}.$$

Proof The core is nonempty by Proposition 1. Since v is continuous at Ω by Proposition 2, any element α of the core is continuous at Ω, which implies that α is countably additive. To prove existence of an outcome density for α, it is sufficient to show that α is absolutely continuous with respect to μ by virtue of the Radon-Nikodym theorem. If $\mu(S) = 0$, then $v(S^c) = v(\Omega)$ by the definition of the market game v, and hence we have $\alpha(S^c) \geq v(S^c) = v(\Omega) = \alpha(\Omega)$, that is, $\alpha(S) = 0$. Q.E.D.

Remark 1 Similar to the assertion of Theorem 1, an exact game which is continuous at Ω has a nonempty core and every element of the core is countably additive. Moreover, there is a measure λ on $\mathscr{F}$ such that every element of the core is absolutely continuous with respect to λ according to [3]. The following example shows that there is a market game which is not exact and Theorem 1 is independent of the results of [3].

Example 1 [[1], pp. 192] Let $l = 1$, $\Omega = [0,1]$ and μ be the Lebesgue measure. Define $u : [0,1] \times R_+ \to R_+$ by

$$u(\omega, x) = \sqrt{x+\omega} - \sqrt{\omega} \quad \text{and} \quad e(\omega) = \frac{1}{32} \text{ for all } \omega \in [0,1].$$

According to [1], the supremum is attained for every measurable sets in the definition of the market game, and the core has only one element α and the outcome density f of α is given by

$$f(\omega) = \begin{cases} (\frac{1}{2} - \sqrt{\omega})^2 + \frac{1}{32}, & \text{if } \omega \in [0, \frac{1}{4}]; \\ \frac{1}{32}, & \text{if } \omega \in [\frac{1}{4}, 1]. \end{cases}$$

Thus it follows $\alpha([\frac{1}{2}, 1]) = \frac{1}{64}$, and hence $\hat{v}([\frac{1}{2}, 1]) = \frac{1}{64}$. On the other hand, we have

$$\sqrt{x+\omega} - \sqrt{\omega} \leq \sqrt{x + \frac{1}{2}} - \sqrt{\frac{1}{2}} \leq \sqrt{\frac{1}{2}x}$$

for $1/2 \leq \omega \leq 1$ and $x \geq 0$. Thus, if $x \in L_1([0,1], R_+)$ satisfies

$$\int_{\frac{1}{2}}^{1} x \, d\mu = \int_{\frac{1}{2}}^{1} e \, d\mu = \frac{1}{64},$$

then

$$\int_{\frac{1}{2}}^{1} u(\omega, x(\omega)) \, d\mu(\omega) \leq \int_{\frac{1}{2}}^{1} \sqrt{\frac{1}{2}x} \, d\mu = \frac{1}{64\sqrt{2}} < \frac{1}{64}.$$

Therefore we have $v([\frac{1}{2}, 1]) < \hat{v}([\frac{1}{2}, 1])$ and v is not exact.

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