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WEAK* FIXED POINT PROPERTY FOR DUAL BANACH SPACES ASSOCIATED TO LOCALLY COMPACT GROUPS

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1. Introduction

Let E be a Banach space and K be a bounded closed convex subset of E. We say that K has the fpp (= *fixed point property*) if every nonexpansive mapping $T : K \to K$ (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in K$) has a fixed point. The Banach space E has the *weak* fpp if every weakly compact convex subset $K \subseteq E$ has the fpp.

It is well known (Schauder's Theorem) that compact convex subsets of a Banach space has the fpp. In particular, any Banach space E having the Shur property (i.e. weakly compact subsets of E are norm compact) has the weak fpp. It is well-known (Browder's Theorem [2]) that uniformly convex Banach spaces have the weak fpp.

A closed bounded convex subset K of E is said to have *normal structure* if every non-trivial convex subset H of K contains a point x_0 such that

$$\sup\{\|x_0 - y\| : y \in H\} < \operatorname{diam}(H).$$

Here $\operatorname{diam}(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H. In [7], W. Kirk established the following fundamental existence theorem for nonexpansive mappings:

Theorem (W. Kirk [7]). *If K is a nonempty, weakly compact, convex subset of a Banach space and suppose K has normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

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A dual Banach space is said to have *weak*-normal structure* if every bounded closed convex subset of E has normal structure. In [12] T.C. Lim introduced the notion of weak*-normal structure and proved that the dual Banach space $E = \ell_1 = c_0^*$ has this property, and hence the weak* $-$ fpp [12, Theorem 1] (i.e. every weak*-compact convex subset of E has the fpp). In [11], Chris Lennard proved that $\mathcal{T}(H)$, the trace class operator on a Hilbert space H, has the weak*-normal structure when $\mathcal{T}(H)$ is identified as the dual of $\mathcal{C}(H)$, the space of compact operators on H.

It is the purpose of this note to report on some open problems and progress concerning the weak* $-$ fpp and other related geometries properties for the Fourier Stieltjes algebra of the locally compact group. It contains part of our talk given in the Symposium on Nonlinear and Convex Analysis held in August, 2000 held in Kyoto University. We would like to thank Professor Wataru Takahshi for kindly inviting me to the symposium and his warm hospitality, and for providing us with the most stimulating and friendly mathematical environment during our stay in Kyoto.

2. Fixed Point Property and Kadec-Klec Type Properties

A dual Banach space E is said to have the *weak* Kadec-Klee property* (KK^*) if whenever (x_n) is a sequence in the unit ball of E that converges to the weak*-topology on x, and $\operatorname{sep}(x_n) > 0$, where

$$\sup\big((x_n)\big) \equiv \sup\{\|x_n - x_m\|;\, n \neq m\}$$

then $\|x_n\| < 1$. We say that E has the *strong weak*-Kadec property* (SKK^*) if the weak*-topology and the norm topology agree on the unit sphere of E. It is known that a dual Banach space which is locally uniformly convex has property SKK^*, and that a space with property SKK^* has the Radon-Nikodym property.

A dual Banach space E is said to have *quasi-weak* normal structure* if each weak*-compact convex subset K of E there exists $x \in K$ such that

$$\|x - y\| < \operatorname{diam}(K)$$

for all $y \in K$, (see [15]) the following relationship between (SKK^*) and quasi-weak* normal structure was established in [8]:

Theorem 2.1 [8]. *Let E be a dual Banach space. If E has the property SKK^*, then E has quasi-weak* normal structure.*

Theorem 2.1 was used to show [8, Theorem 2] that if H is a Hilbert space, then $\mathcal{T}(H)$, the trace class operators on H, regarded as the dual Banach space of $\mathcal{C}(H)$, the space of compact linear operators on H, has the quasi-weak* normal structure.

A dual Banach space E has the *weak* uniform Kadec-Klee property* (UKK^*) if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever A is a subset of the closed unit ball of E containing a sequence (x_n) with $\operatorname{sep}\left((x_n)\right) > \varepsilon$ then there is an x in the weak*-closure of A such that $\|x\| \leq \delta$.

In [3], van Dust and Sims define the notion of UKK^* and show that if a dual Banach space has the property UKK^*, then E has the weak* normal structure (i.e. every w^*-compact convex subset has normal structure). In particular E has the weak* fpp.

We summarize the relationships among the various concepts in the following diagram

$$\begin{tikzcd}
UKK^* \arrow[r, Rightarrow] \arrow[d, Rightarrow] & KK^* & SKK^* \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\text{weak* normal structure} \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \text{quasi-weak* normal structure} \\
\text{weak* fpp} & &
\end{tikzcd}$$

In general, $SKK^* \not\Longrightarrow UKK^*$, $SKK^* \not\Longrightarrow$ weak*normal structure, and quasi-weak* normal structure $\not\Longrightarrow$ weak* normal structure (see [8] and [9]).

Let G be a locally compact group and $M(G)$ be the space bounded regular Borel measures on G with the total variation norm. Let $C_0(G)$ be the Banach space of all continuous functions $f : G \to \mathbb{C}$ vanishing at infinity with the supremum norm. Then as well known $M(G)$ may be identified the continuous dual of $C_0(G)$.

Theorem 2.2 ([9]). *Let G be a locally compact group, the following are equivalent:*

(a) *G is discrete*

(b) *$M(G)$ has property UKK^**

(c) *$M(G)$ has property SKK^**

(d) *$M(G)$ has property KK^**

(e) *$M(G)$ has weak* normal structure*

(f) *$M(G)$ has weak** fpp.

Problem 1. When does $M(G)$ have quasi-weak* normal structure?

In a remarkable paper of C. Lennard [11], he showed that $\mathcal{T}(H)$ has the property UKK^*. Consequently $\mathcal{T}(H)$ has weak*-normal structure. This answers affirmatively a question raised by Lau and Mah in [9].

Let G be a locally compact group, and let $B(G)$ denote the Fourier Stieltjes algebra of G, i.e. $B(G)$ is the subalgebra of $CB(G)$ (bounded complex-valued continuous functions on G) consisting of all functions ϕ of the form

$$\phi(x) = \langle \pi(x)h, k \rangle \quad h, k \in H_\pi$$

where $\{\pi, H_\pi\}$ is a continuous unitary representation on G. Then $B(G)$ is a commutative Banach algebra with pointwise multiplication and norm

$$\|\phi\| = \sup\left\{ \left| \int f(t)\phi(t)d\lambda(t) \right|,\ f \in L^1(G),\ |||f||| \leq 1 \right\}$$

where λ is a fixed left Haar measure on G, $|||f||| = \sup\{\|\pi(f)\|;\ \pi$ is a representation of $L^1(G)\}$. Then $B(G)$ is the continuous dual of $C^*(G)$, the completion of $\left(L^1(G), |||\cdot|||\right)$.

In the case that G is abelian, then $B(G) \cong M(\widehat{G})$, and $C^*(G) \cong C_0(\widehat{G})$, where $\widehat{G}$ is the dual group of G (see [4] for details).

Theorem 2.3 ([9]). *If G is a compact, then $B(G)$ has UKK^*.*

The following theorem was proved for the case when G is amenable in [9, Theorem 5], and more recently for all G:

Theorem 2.4 ([1]). *Let G be a locally compact group. Then G is compact if and only if $B(G)$ has SKK^*.*

The following problem still remains open:

Problem 2. Does any of the following properties on $B(G)$ imply G is compact?

(i) UKK^*

(ii) weak* normal structure

(iii) weak* fpp.

The following follows from Lemma 3.1 in [10]:

Proposition 2.5 ([10]). *If $B(G)$ has the Radon Nikodym Property, then $B(G)$ has the weak* fpp.

Remark 2.6. If G is the Fell's group (which is the natural semi-direct product of the p-adic numbers with the compact group of p-adic units for a fixed prime p) then G is non-compact, totally disconnected and has countable dual; $B(G)$ has the Radon-Nikodym property [14, Remark 4.6]. So $B(G)$ has the weak fpp. However we do not know if $B(G)$ has the weak* fpp for this G.

Let G be a locally compact group and $1 < p < \infty$. For $f \in L^1(G)$, let $\rho(f)$ be the operator on $L^p(G)$ defined by $\rho(f)(h) = f * h$, $h \in L^p(G)$. Let $PF_p(G)$ be the norm closure of $\{\rho(f); f \in L^1(G)\}$ in $\mathcal{B}(L^p(G))$. Then $PF_p(G) \subseteq A_p(G)^*$, where

$A_p(G)$ is the Figa-Talamanca-Herz algebra (see [6] or [13]) consisting of all functions f on G which can be represented as $f = \sum_{n=1}^{\infty} v_n * \check{u}_n$ as absolutely and uniformly convergent sums such that $\sum_{n=1}^{\infty} \|v_n\|_{p'}\|u_n\|_p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, where $\check{u}_n(x) = u_n(x^{-1})$, $x \in G$, $u_n \in L^p(G)$, $v_n \in L^{p'}(G)$. We define norm $\|f\|_{A_p} = \inf\{\sum \|v_n\|_{p'}\|u_n\|_p\}$ over all such representations. Let $B_p(G)$ be all complex-valued functions u on G such that $uv \in A_p(G)$ for each $v \in A_p(G)$. It then follows by the closed graph theorem that $\|u\|_M = \sup\{\|uv\|_{A_p}; \|v\|_{A_p} \leq 1\}$ is finite. We equip $B_p(G)$ with this multiplier norm. Then $B_p(G) \subseteq CB(G)$, the space of bounded complex-valued continuous functions on G, and $B_p(G)$ becomes in this way a translation invariant Banach algebra with pointwise multiplication. If G is amenable, $B_p(G)$ is isometrically isomorphic to the dual space of $PF_p(G)$, and in this case, $B_2(G) = B(G)$, $PG_2(G) = C^*(G)$ defined earlier (see [6] for details).

The following was proved in [9, Theorem 5] for the case $p = 2$:

Proposition 2.7. *Let $1 < p < \infty$. If G is amenable and $B_p(G)$ has property SKK^*, then G is compact.*

Proof. Suppose $B_p(G)$ has property SKK^*. Since G is amenable, $A_p(G)$ has a bounded approximate unit (ϕ_α), $\|\phi_\alpha\| \leq 1$ ([6, Theorem 6]. Let θ be a weak*-cluster point of $\{\phi_\alpha\}$ in $B_p(G)$. By passing to a subnet if necessary, we may assume that ϕ_α converges to θ in the weak*-toplogy. If $x \in G$, let $\psi \in A_p(G)$ such that $\psi(x) = 1$ (see [6, Proposition 3]). Since multiplication in $B_p(G)$ is separately continuous in the weak* topology, and $\|\phi_\alpha\psi - \psi\| \to 0$, it follows that $\psi = \psi\theta$. Hence $\theta(x) = 1$. Consequently $\theta \equiv 1$. Now since $\|\phi_\alpha\| \to \|\theta\| = 1$, the net $\widetilde{\phi}_\alpha = \phi_\alpha/\|\phi_\alpha\|$ has norm 1, and $\widetilde{\phi}_\alpha \to \theta$ in the weak* topology. Now if $B_p(G)$ has SKK^*, $\|\widetilde{\phi}_\alpha - \theta\| \to 0$. Hence $1 \in A_p(G) \subseteq C_0(G)$. Consequently G is compact.

$\square$

Problem 3. If G is compact, does $B_p(G)$ have property UKK^* or weak*-normal structure, or weak* fpp?

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