RECENT PROGRESS IN TOPOLOGICAL GROUPS:
SELECTED TOPICS

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Some historical background on topological groups

**Theorem** (Pontryagin?): If the space of a topological group is a $T_0$-space, then it is automatically Tychonoff.

**Theorem** (Markov [1941]): There exists a topological group the space of which is not normal.

**Theorem** (Birkhoff-Kakutani [1930]): A topological group is metrizable if and only if it is first countable.

**Theorem**: Every locally compact group has a Haar measure. (This allows for integration on it.)

**Theorem**: Let $G$ be a locally compact abelian group, $g \in G$ and $g \neq 0$. Then there exists a continuous group homomorphism $\pi : G \to \mathbb{T}$ from $G$ into the torus group $\mathbb{T}$ such that $\pi(g) \neq 0$.

**Theorem** (Peter-Weyl-van Kampen): Let $G$ be a locally compact group, $g \in G$ and $g \neq 1_G$ where $1_G$ is the identity element of $G$. Then there exist a natural number $n$ and a continuous group homomorphism $\pi : G \to \mathbb{U}(n)$ from $G$ into the group $\mathbb{U}(n)$ of unitary $n \times n$ matrices over the complex number field such that $\pi(g) \neq I$. (Here $I$ is the identity matrix of $\mathbb{U}(n)$.) A cardinal $\tau$ is *Ulam nonmeasurable* provided that for every ultrafilter $\mathcal{F}$ on $\tau$ with the countable intersection property there exists $\alpha \in \tau$ such that $\mathcal{F} = \{A \subseteq \tau : \alpha \in A\}$.

**Theorem** (Varopolous [1964]): Let $G$ and $H$ be locally compact groups, and let $\pi : G \to H$ be a group homomorphism. Assume that:

(i) $|G|$ is an Ulam nonmeasurable cardinal, and

(ii) $\pi$ is *sequentially continuous*, i.e. for every sequence $S \subseteq G$ the image $\pi(S)$ is also a convergent sequence.

Then $\pi$ is continuous.

**Theorem** (Comfort-Remus [1994]): Let $G$ be a compact group that is either abelian or connected. Suppose also that every sequentially continuous group homomorphism
\[ \pi : G \to H \] from \( G \) into any compact group \( H \) is continuous. Then \( |G| \) is an Ulam measurable cardinal.

**Theorem** (Pasynkov [1961]): \( \text{ind } G = \text{Ind } G = \dim G \) for a locally compact group \( G \).

Note: Locally compact groups are paracompact (Pasynkov).

A continuous image of a Cantor cube \( \{0,1\}^\kappa \) is called a *dyadic* space.

**Theorem** (Kuz'minov [1959]): Compact groups are dyadic.

A compact space \( X \) is said to be *Dugundji* if any continuous function \( f : A \to X \) defined on a closed subset \( A \) of a Cantor cube \( \{0,1\}^\kappa \) has a continuous extension \( F : \{0,1\}^\kappa \to X \).

Since we can choose the above \( f \) to be onto, Dugundji spaces are dyadic.

**Theorem** (Čoban [1970s]): Let \( X \) be a compact \( G_\delta \)-subset of some topological group. Then \( X \) is a Dugundji space.

**Theorem** (Hagler, Gerlits and Efimov [1976/77]): An infinite compact group \( G \) contains a homeomorphic copy of the Cantor cube \( \{0,1\}^{w(G)} \).

As a corollary, one gets a particular version of Shapirovskii’s theorem about mappings onto Tychonoff cubes:

**Theorem:** Every infinite compact group \( G \) admits a continuous map onto a Tychonoff cube \( [0,1]^{w(G)} \).

Recall that a space \( X \) is *\( \sigma \)-compact* if it is a union of countable family of its compact subspaces.

A space \( X \) is *ccc* provided that \( X \) does not have an uncountable family of non-empty pairwise disjoint open subsets.

**Theorem** (Tkachenko [1981]): A *\( \sigma \)-compact* group is ccc.

A space is *pseudocompact* if every real-valued continuous function defined on it is bounded.

**Theorem** (Comfort and Ross [1966]): Let \( G \) be a dense subgroup of a compact group \( K \). Then the following conditions are equivalent:

(i) \( G \) is pseudocompact,

(ii) \( G \cap B \neq \emptyset \) for every non-empty \( G_\delta \)-subset \( B \) of \( K \).

**Corollary** (Comfort and Ross [1966]): The product of any family of pseudocompact groups is pseudocompact.

A (Hausdorff) topological group \( (G, \mathcal{T}) \) is called *minimal* provided that for every Hausdorff group topology \( \mathcal{T}' \) on \( G \) with \( \mathcal{T}' \subseteq \mathcal{T} \) one has \( \mathcal{T}' = \mathcal{T} \).
Clearly, compact groups are minimal.

**Theorem** (Prodanov, Stoyanov [1984]): A minimal abelian group $G$ is totally bounded, i.e. $G$ is (isomorphic to) a subgroup of some compact topological group.

**Generating dense subgroups of topological groups:**

**Suitable sets**

If $X$ is a subset of a group $G$, then $\langle X \rangle$ denotes the smallest subgroup of $G$ that contains $X$.

Let $X$ be a subspace $X$ of a topological group $G$.

We say that $X$ *algebraically generates* $G$ provided that $\langle X \rangle = G$.

We say that $X$ *topologically generates* $G$ if $\langle X \rangle$ is dense in $G$.

A compact connected abelian group $G$ has weight less than or equal to the continuum if and only if it is monothetic; that is, there exists an element $g \in G$ such that $G$ is topologically generated by the subset $\{g\}$.

This result was improved by Hofmann and Morris [1990] by showing that a compact connected group $G$ can be topologically generated by two elements if and only if the weight of $G$ is less than or equal to the continuum.

Clearly, neither finite nor countable subsets of a topological group $G$ with weight greater than the continuum can generate a dense subgroup of $G$. This fact led Hofmann and Morris to introduce the concept of suitable set as a way to define the notion of topological generating sets which are in some sense "close" to finite sets:

**Definition** (Hofmann and Morris [1990]): A subset $S$ of a topological group $G$ is said to be *suitable* for $G$ if $S$ is discrete in itself, generates a dense subgroup of $G$ and $S \cup \{1_G\}$ is closed in $G$, where $1_G$ is the identity of $G$.

**Theorem** (Hofmann and Morris [1990]): Every locally compact group has a suitable set.

**Theorem** (Comfort, Morris, Robbie, Svetlichny, and Tkačenko [1998]): Each metric group has a suitable set. A topological group $G$ is *almost metrizable* if there exists a compact subgroup $K$ of $G$ such that the space of left cosets $G/K$ is metrizable.

**Theorem** (Okunev and Tkachenko [1998]): An almost metrizable group has a suitable set.

**Theorem** (Dikranjan, Tkachenko, Tkachuk [1999]): A topological group representable as a countable union of closed metrizable subspaces has a suitable set.
Corollary (Dikranjan, Tkachenko, Tkachuk [1999]): A free (abelian) topological group over a metric space has a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Suppose that a topological group $G$ is a countable union of its metrizable subspaces. Does $G$ have a suitable set?

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): Every topological group with a $\sigma$-discrete network has a suitable set.

Corollary (Dikranjan, Tkachenko, Tkachuk [1999]): Every topological group with a countable network (i.e. a cosmic group) has a suitable set.

Corollary (Dikranjan, Tkachenko, Tkachuk [1999]): Stratifiable groups have suitable sets.

From the above results it follows that all countable groups have suitable sets. In fact, even more can be said for countable groups:

Theorem (Comfort, Morris, Robbie, Svetlichny, and Tkačenko [1998]):
Every countable topological group $G$ has a closed discrete subspace $S$ that algebraically generates $G$.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): A separable $\sigma$-compact group has a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Does every $\sigma$-compact group of size $< c$ have a suitable set?

Theorem (Comfort, Morris, Robbie, Svetlichny, and Tkačenko [1998]):
Let $G$ be the free (abelian) topological group of $\beta\mathbb{N} \setminus \mathbb{N}$. Then $G$ does not have a suitable set. In particular, a $\sigma$-compact group need not have a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Does every $\sigma$-compact group has a dense subgroup with a suitable set?

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): If $G$ is a topological group with a suitable set, then $d(G) \leq l(G) \cdot \psi(G)$. In particular, a non-separable Lindelöf group of countable pseudocharacter does not have a suitable set.

A space is submetrizable if it has a weaker metric topology.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): There exists a submetrizable Lindelöf non-separable linear topological space $L$ of countable tightness. Thus, $L$ does not have a suitable set.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): Under some additional set-theoretic assumptions (diamond) there exists a hereditarily Lindelöf non-separable linear topo-
logical space $L$ of countable tightness. Thus no dense additive subgroup of $L$ has a suitable set.

**Question** (Dikranjan, Tkachenko, Tkachuk [1999]): Can one construct in ZFC a topological group which does not contain a dense subgroup with a suitable set?

A space $X$ is $\omega$-bounded if the closure of each countable subset of $X$ is compact.

**Theorem** (Dikranjan, Tkachenko, Tkachuk [1999]): There exists an $\omega$-bounded group $G$ without a suitable set. Moreover, each power $G^\kappa$ of $G$ does not have a suitable set.

**Question:** In ZFC, does there exists a separable (pseudocompact) group without a suitable set?

**Theorem** (Dikranjan, Tkachenko, Tkachuk [1999]): A locally separable non-pseudocomapct group has a suitable set.

**Question** (Dikranjan, Tkachenko, Tkachuk [1999]): Does there exists an $\omega$-bounded topological group of size $c$ without a suitable set?

**Generating dense subgroups of topological groups:**

**Topologically generating weight**

We use $w(X)$ to denote the weight of a topological space $X$, i.e. the smallest size of a base for the topology of $X$ if such a base is infinite, or $\omega$ otherwise.

Define

$$agw(G) = \min\{w(X) : X \text{ is closed in } G \text{ and algebraically generates } G\}$$

and

$$tgw(G) = \min\{w(F) : F \text{ is closed in } G \text{ and topologically generates } G\}.$$ 

We will call $agw(G)$ an algebraically generating weight of $G$ and $tgw(G)$ a topologically generating weight of $G$.

Clearly $tgw(G) \leq agw(G) \leq w(G)$. While the definition of algebraically generating weight appears to be more natural than that of topologically generating weight, it does not lead to anything new for compact groups:

**Theorem** (Arhangel'skii): $agw(G) = w(G)$ holds for every compact group $G$.

For an infinite cardinal $\tau$ define $\sqrt{\tau}$ to be the smallest infinite cardinal $\kappa$ with $\tau \leq \kappa^\omega$. Clearly, $\sqrt{\tau} \leq \tau$.

**Theorem** (Dikranjan and Shakhmatov [1998]): $tgw(G) = \sqrt{w(c(G))} \cdot w(G/c(G))$ for every compact group $G$, where $c(G)$ is the connected component of $G$. 
Corollary (Dikranjan and Shakhmatov [1998]): \( tgw(G) = w(G) \) for a totally disconnected compact group \( G \).

Corollary (Dikranjan and Shakhmatov [1998]): \( tgw(G) = \sqrt{w(G)} \) for every connected compact group \( G \). A super-sequence is a compact space with at most one non-isolated point.

Suitable sets in compact groups are precisely super-sequences, so Hofmann-Morris' theorem justifies an introduction of the following cardinal number for a compact group \( G \):

\[ \text{seq}(G) = \omega \cdot \min \{|S| : S \subseteq G \text{ is a super-sequence topologically generating } G \} \]

Clearly \( tgw(G) \leq \text{seq}(G) \leq w(G) \).

Theorem (Dikranjan and Shakhmatov [1998]): \( tgw(G) = \text{seq}(G) \) for every compact group \( G \).

For topological spaces \( X \) and \( Y \) we use \( C(X, Y) \) to denote the family of all continuous maps from \( X \) to \( Y \). No topology is assumed on \( C(X, Y) \).

For topological groups \( G \) and \( H \) we will use \( \text{Hom}(G, H) \) to denote the family of all continuous homomorphisms from \( G \) to \( H \). No topology is assumed on \( \text{Hom}(G, H) \).

Lemma 1: Let \( X \) be a subset of a topological group \( G \). Assume that \( X \) topologically generates \( G \). Then \( |\text{Hom}(G, H)| \leq |C(X, H)| \) for every topological group \( H \).

Proof: Define a map \( f : \text{Hom}(G, H) \to C(X, H) \) by \( f(\pi) = \pi|_X \) for \( \pi \in \text{Hom}(G, H) \). We claim that \( f \) is an injection. Indeed, assume that \( \pi, \omega \in \text{Hom}(G, H) \) and \( f(\pi) = f(\omega) \). Then \( \pi|_X = \omega|_X \). Since both \( \pi \) and \( \omega \) are group homomorphisms from \( G \) to \( H \), one has \( \pi|_{\langle X \rangle} = \omega|_{\langle X \rangle} \). Since \( \langle X \rangle \) is dense in \( G \), continuity of \( \pi \) and \( \omega \) implies now that \( \pi = \omega \).

PROOF OF THE TOTALLY DISCONNECTED CASE

Lemma 2: Let \( X \) be a totally disconnected compact space and \( H \) be a discrete space. Then \( |C(X, H)| \leq w(X) \).

Let \( X \) be a closed subset of \( G \) that topologically generates \( G \). Since \( G \) is compact and totally disconnected, it is profinite, i.e. its topology is determined by the family of all continuous homomorphisms into finite discrete groups. Let \( H \) be one of these discrete groups.

Since \( G \) is totally disconnected, so is \( X \). Therefore \( |C(X, H)| \leq w(X) \) by Lemma 2.

We also have \( |\text{Hom}(G, H)| \leq |C(X, H)| \) since \( X \) topologically generates \( G \) (Lemma 1).

Since there are only countably many pairwise non-isomorphic finite discrete groups \( H \), it now follows that \( w(G) \leq \omega \cdot w(X) = w(X) \).

PROOF OF THE INEQUALITY \( \sqrt{w(G)} \leq tgw(G) \)
Lemma 3: Let $X$ be a compact space and $H$ be a separable metric space. Then $|C(X, H)| \leq w(X)^\omega$.

Theorem: $\sqrt{w(G)} \leq tgw(G)$ for every compact group $G$.

Proof: Let $G$ be a compact group. By Peter-Weyl-van Kampen theorem the topology of every compact group is determined by the set of its homomorphisms into the compact metric group $H = \prod_n U(n)$, where $U(n)$ is the group of unitary $n \times n$ matrices over the complex number field.

Therefore $w(G) \leq |\text{Hom}(G, H)|$.

Let $X$ be a closed subspace of $G$ that topologically generates $G$ and satisfies the equality $w(X) = tgw(G)$. From Lemmas 1 and 3 we have the following:

$$|\text{Hom}(G, H)| \leq |C(X, H)| \leq w(X)^\omega = tgw(G)^\omega.$$ 

Therefore $\sqrt{w(G)} \leq \sqrt{tgw(G)^\omega} \leq tgw(G)$.

STRONGLY TOPOLOGICALLY FINITELY GENERATED GROUPS

Recall that a topological group $G$ is topologically finitely generated provided that there exists a finite subset of $G$ topologically generating $G$.

Definition (Dikranjan and Shakhmatov): We say that a topological group $G$ is strongly topologically finitely generated provided that for every open set $U$ containing the identity element of $G$ one can find a finite set $F \subseteq U$ such that $F$ topologically generates $G$.

Lemma 4: Let $G$ be a topologically finitely generated group that has no proper open subgroups. Then $G$ is strongly topologically finitely generated. Proof: Let $D = \langle g_1, \ldots, g_n \rangle$ be a dense finitely generated subgroup of $G$.

Let $U$ be an open neighbourhood of $e$ in $G$. Then the subgroup $H = \langle D \cap U \rangle$ of $D$ is obviously open in $D$, hence also closed in $D$. On the other hand, its closure $\overline{H}$ in $G$ contains $D \cap U \supseteq \overline{U}$ since $U$ is open and $D$ is dense in $G$. Therefore $\overline{H}$ is an open subgroup of $G$. Our hypothesis gives $\overline{H} = G$.

Now closedness of $H$ in $D$ yields $H = \overline{H} \cap D = G \cap D = D$. We have proved in this way that $D = H$.

Let $i = 1, \ldots, n$. Since $g_i \in D = H = \langle D \cap U \rangle$, there exists a finite subset $F_i \subseteq D \cap U$ such that $g_i \in \langle F_i \rangle$. Clearly the finite set set $F = \bigcup_{i=1}^n F_i$ generates the whole group $D$ and $F \subseteq U$. Since $D$ is dense in $G$, $F$ topologically generates $G$.

Lemma 5: Let $G$ be a metric (not necessarily compact!) group that is strongly topologically finitely generated. Then for every infinite cardinal $\tau$ one has seq($G^{\tau^+}$) $\leq \tau$. 

Proof: Fix an infinite cardinal $\tau$, and let $\{U_n : n \in \omega\}$ be a decreasing open base at the identity element $e$ of $G$. For each $n \in \omega$ use the hypothesis of our lemma to fix a finite set $F_n = \{g^n_i : i < m_n\} \subseteq U_n$ such that $\langle F_n \rangle$ is dense in $G$.

For $f \in \tau^\omega$ and $n \in \omega$ let $f|n \in \tau^n$ be the restriction of the function $f$ to $n = \{0, 1, \ldots, n-1\}$.

For $n \in \omega$, $i < m_n$ and $\phi \in \tau^n$ we define a point $x_{n,i,\phi} \in G^{\tau^\omega}$ as follows:

$$x_{n,i,\phi}(f) = g^n_i$$

if $f|n = \phi$ and $x_{n,i,\phi}(f) = e$ otherwise. Then

$$X = \{x_{n,i,\phi} : n \in \omega, i < m_n, \phi \in \tau^n\}$$

is a subset of $G^{\tau^\omega}$ of size at most $\tau$.

**CLAIM 1.** For every open set $W$ which contains the identity element $e$ of $G^{\tau^\omega}$ the set $X \setminus W$ is at most finite.

Claim 1 implies that $X \cup \{e\}$ is a super-sequence.

**Proof of Claim 1.** Since $W$ contains a finite intersection of sets of the form

$$V_{f,n} = \{x \in G^{\tau^\omega} : x(f) \in U_n\},$$

it suffices to prove that, for each $f \in \tau^\omega$ and for every $n \in \omega$, $x(f) \in U_n$ for all but finitely many $x \in X$, i.e., the set $\{x \in X : x(f) \notin U_n\}$ is finite.

So let $f \in \tau^\omega$ and $n \in \omega$. Our construction implies that if $k \in \omega$, $j < m_k$, $\phi \in \tau^k$ and $x_{k,j,\phi}(f) \notin U_n$, then:

(i) $k < n$ (because $n \leq k$ implies $U_k \subseteq U_n$), and

(ii) $f|k = \phi$ (because $f|k \neq \phi$ implies $x_{k,j,\phi}(f) = e \in U_n$).

There are only finitely many of such $x_{k,j,\phi}$, and the result follows.

**CLAIM 2.** For every finite subset $F$ of $\tau^\omega$ there exists $n \in \omega$ (depending on $F$) such that, for each $f \in F$, the finite set

$$\{x_{n,i,f|n} : i < m_n\} \subseteq X$$

satisfies the following two properties:

(i) $\langle \{x_{n,i,f|n}(f) : i < m_n\} \rangle$ is dense in $G$,

(ii) $x_{n,i,f|n}(f') = e$ whenever $f' \in F \setminus \{f\}$.

From Claim 2 it immediately follows that, for every finite set $F \subseteq \tau^\omega$, the projection of

$$\langle \{x_{n,i,f|n} : i \in F, i < m_n\} \rangle$$

(where $n$ is as in Claim 2) onto the subproduct $G^F$ is dense in $G^F$. Since

$$\{x_{n,i,f|n} : i \in F, i < m_n\} \subseteq X,$$
this implies that $\langle X \cup \{e\} \rangle$ is dense in $G^{\tau^\omega}$. Proof of Claim 2. There exists $n \in \omega$ such that $f'|n \neq f''|n$ whenever $f', f'' \in F$ and $f' \neq f''$. We will show that this $n$ works.

Indeed, let $f \in F$. By our construction, one has $x_{n,i,f|n}(f) = g_i^n$ for all $i < m_n$, so

$$\{x_{n,i,f|n}(f) : i < m_n\} = \{g_i^n : i < m_n\},$$

and the latter set generates a dense subgroup of $G$. This implies (i).

Again by our construction, $f' \in F \setminus \{f\}$ implies $f'|n \neq f|n$ and so $x_{n,i,f|n}(f') = e$. This gives (ii).

PROOF OF THE CONNECTED CASE

Theorem: (Universal compact connected group of a given weight)

There exists a sequence $\{L_n : n \in \omega\}$ of compact connected simple Lie groups $L_n$ such that every compact connected group of weight $\leq \tau$ is a quotient group of the group

$$G_\tau = (\hat{\mathbb{Q}})^\tau \times \prod_n L_n^\tau,$$

where $\hat{\mathbb{Q}}$ is the Pontryagin dual of the discrete group $\mathbb{Q}$ of rational numbers. (Note that $G_\tau$ is a connected group of weight $\tau$.)

Theorem: $\text{seq}(G) \leq \sqrt{w(G)}$ for a compact connected group $G$.

Proof: Let $\tau = \sqrt{w(G)}$. By the above theorem, $G$ is a quotient group of the group

$$H = (\hat{\mathbb{Q}})^{w(G)} \times \prod_n L_n^{w(G)}$$

for a suitable sequence $\{L_n : n \in \omega\}$ of compact connected simple Lie groups $L_n$. Since $w(G) \leq \tau^\omega$, $H$ is a natural quotient group (under projection map) of the group $K^{\tau^\omega}$, where

$$K = (\hat{\mathbb{Q}}) \times \prod_n L_n.$$

Therefore $\text{seq}(G) \leq \text{seq}(H) \leq \text{seq}(K^{\tau^\omega})$.

Since $K$ is connected, it has no proper open subgroups. Since $K$ is also topologically finitely generated, $K$ is strongly topologically finitely generated (Lemma 4).

Therefore $\text{seq}(K^{\tau^\omega}) \leq \tau$ by Lemma 5.

Finally, $\text{seq}(G) \leq \text{seq}(K^{\tau^\omega}) \leq \tau = \sqrt{w(G)}$.

Applications of Michael's selection theorem to proving results about (mostly compact) topological groups
Uspenskii [1988] was the first to notice how Michael’s selection theorem can be applied to get a simple topological proof of the classical result of Kuzminov that compact groups are dyadic. Recall that a set-valued map $F : Y \to Z$ is a map which assigns a non-empty closed set $F(y) \subseteq Z$ to every point $y \in Y$.

This set-valued map is lower semicontinuous if

$$V = \{ y \in Y : F(y) \cap U \neq \emptyset \}$$

is open in $Y$ for every set $U$ open in $Z$.

A selection for a set-valued map $F : Y \to Z$ is a (single-valued) continuous map $f : Y \to Z$ such that $f(y) \in F(y)$ for all $y \in Y$.

**Theorem** (Michael [1956]): Every lower semicontinuous set-valued map $F : Y \to Z$ from a zero-dimensional compact space $Y$ into a complete metric space (in particular, compact metric space) $Z$ has a selection.

**Lemma**: Suppose that $H$ and $H'$ are topological groups, $G$ is a subgroup of the product $H \times H'$, $\varphi : H \times H' \to H$ and $\pi : H \times H' \to H'$ are projections onto the first and second coordinates respectively. Assume also that:

(i) the restriction $\varphi|_G : G \to \varphi(G)$ of $\varphi$ to $G$ is an open map,

(ii) the restriction $\pi|_G : G \to \pi(G)$ of $\pi$ to $G$ is a closed map, and

(iii) the subgroup $\pi(G)$ of $H'$ is a complete metric group.

Then for every compact zero-dimensional space $Y \subseteq \varphi(G)$ there exists a homeomorphic embedding $f : Y \to G$ such that $(\varphi \circ f)(y) = y$ for every $y \in Y$. Proof: Define $Z = \pi(G)$ and note that $G \subseteq H \times Z$.

For $y \in Y$ define $F(y) = \{ z \in Z : (y, z) \in G \}$.

The set $G \cap (\{ y \} \times H')$ is closed in $G$, so from (ii) it follows that

$$F(y) = \pi(G \cap (\{ y \} \times H'))$$

is closed in $Z = \pi(G)$.

For $y \in Y$, since $y \in Y \subseteq \varphi(G)$, we have $F(y) \neq \emptyset$. Therefore $F : Y \to Z$ is a set-valued map.

We claim that $F$ is lower semicontinuous. Indeed, let $U$ be an open subset of $Z$. We have to check that the set

$$V = \{ y \in Y : F(y) \cap U \neq \emptyset \}$$

is open in $Y$. To see this note that the set $G \cap (H \times U)$ is open in $G$, so $\varphi(G \cap (H \times U))$ is open in $\varphi(G)$ by (i). Since $Y \subseteq \varphi(G)$,

$$V = Y \cap \varphi(G \cap (H \times U))$$
is open in $Y$.

Since $\pi(G) = \mathbb{Z}$ is a complete metric group, we can use Michael's selection theorem to pick a (single-valued) continuous selection $f : Y \to \mathbb{Z}$ of $F$.

From the definition of $F$ it follows that $(\varphi \circ f)(y) = y$ for all $y \in Y$. In particular, $f$ is one-to-one. Since $Y$ is compact, $f$ is a homeomorphism.

**Corollary:** Suppose that $H$ is a topological group, $H'$ is a metric group, $G$ is a compact subgroup of the product $H \times H'$, and $\varphi : H \times H' \to H$ is the projection onto the first coordinate.

Then for every compact zero-dimensional space $Y \subseteq \varphi(G)$ there exists a homeomorphic embedding $f : Y \to G$ such that $(\varphi \circ f)(y) = y$ for every $y \in Y$.

**Proof:** Let $\pi : H \times H' \to H'$ be the projection onto the second coordinate.

Since $G$ is compact, the restriction $\varphi|_G : G \to \varphi(G)$ of $\varphi$ to $G$ is a closed continuous map, so a quotient map, and so an open map. This gives (i).

Since $G$ is compact, the restriction $\pi|_G : G \to \pi(G)$ of $\pi$ to $G$ is a closed map. This gives (ii).

The subgroup $\pi(G)$ of $H'$ is compact, being a continuous image of the compact group $G$. Since $H'$ is metric, so is $\pi(G)$. In particular, $\pi(G)$ is a complete metric group. This gives (iii).

A subset $X$ of an abelian group $G$ is *independent* provided that $\langle A \rangle \cap \langle X \setminus A \rangle = \{0\}$ for every $A \subseteq X$.

For a prime number $p \geq 2$, a subset $X$ of an abelian group $G$ is called *$p$-independent* provided that $X$ is independent and

$$\min\{1 \leq n \leq p : nx = 0\} = p$$

for every $x \in X$. For an abelian group $G$ and a prime number $p$, cardinal numbers

$$r_0(G) = \sup\{|X| : X \subseteq G \text{ is independent}\}$$

and

$$r_p(G) = \sup\{|X| : X \subseteq G \text{ is } p\text{-independent}\}$$

are called *rank* and *$p$-rank* of $G$ respectively.

For a cardinal number $\tau$ we define $\log(\tau)$ to be the smallest infinite cardinal $\sigma$ such that $2^\sigma \geq \tau$.

**Theorem** (Shakhmatov): Let $G$ be an infinite compact abelian group. Then:

(i) $G$ contains an independent subset $X$ homeomorphic to the Cantor cube $\{0, 1\}^{\log r_0(G)}$ of weight $\log r_0(G)$, and
(ii) for every prime number $p \geq 2$ the group $G$ contains a $p$-independent subset $X$ homeomorphic to the Cantor cube $\{0,1\}^{|\log r_p(G)|}$ of weight $\log r_p(G)$.

Even the following corollary to the above general theorem is new:

**Corollary** (Shakhmatov): Let $G$ be an infinite compact abelian group. Then:

(i) $G$ contains a closed independent subset $X$ with $|X| = r_0(G)$, and

(ii) for every prime number $p \geq 2$ the group $G$ contains a closed $p$-independent subset $X$ with $|X| = r_p(G)$.

Wallace's problem and continuity of separately continuous multiplication in semigroups

A *semigroup* is a pair $(S, \cdot)$ consisting of a set $S$ and a binary associative operation $\cdot$ on $S$.

A semigroup $S$ has the *cancellation property* provided that either of $sx = sy$ and $xs = ys$ implies $x = y$ whenever $x, y, s \in S$.

A *topological semigroup* is a semigroup equipped with a topology which makes its binary operation continuous.

Clearly, every topological group is a topological semigroup with the cancellation property.

**Theorem** (Gelbaum, Kalish and Olmsted [1951]): A compact semigroup with the cancellation property is a topological group.

**Problem** (Wallace [1955]): Is a countably compact Hausdorff semigroup with the cancellation property a topological group?

A series of positive results by Mukhurjea-Tserpes, Grant, Korovin, Reznichenko, Yur'eva culminated in the following most general result:

**Theorem** (Bokalo-Guran [1996]): A sequentially compact Hausdorff semigroup with the cancellation property is a topological group.

**Theorem** (Robbie, Svetlichny [1996]): Suppose that there exists an abelian topological group $G$ with the following properties:

(i) $G$ is countably compact,

(ii) every infinite closed subset of $G$ has cardinality greater or equal than the continuum,

(iii) $G$ is torsion-free, i.e. for every $x \in G$ and each $n \geq 1$ one has $ng \neq 1_G$.

Then, (inside of $G$) one can find a Tychonoff counterexample to the Wallace problem, i.e. there exists a commutative Tychonoff countably compact semigroup with the
cancellation property that is not a topological group.

**Theorem** (Tkačenko [1990]): Assume CH. Then there exists a topological group $G$ with the following properties:

(i) $G$ is countably compact,

(ii) every infinite closed subset of $G$ has cardinality greater or equal than the continuum,

(iii) $G$ is a free abelian group (in particular, $G$ is torsion-free).

Tomita [1997] constructed similar group under Martin’s Axiom for Countable Sets.

**Question:** Is there such a group in ZFC?

**Theorem** (Ellis [1957]): A group equipped with a locally compact topology such that multiplication is separately continuous is a topological group.

**Theorem** (Korovin [1992]): A group equipped with a countably compact topology such that multiplication is separately continuous is a topological group.

**Theorem** (Reznichenko [1994]): Let $G$ be group equipped with a pseudocompact topology such that multiplication is separately continuous. Then $G$ is a topological group provided that one of the following conditions holds:

(i) $G$ has countable tightness,

(ii) $G$ is separable,

(iii) $G$ is a $k$-space.

**Theorem** (Korovin [1992]): There exists an abelian group (of period 2) equipped with a pseudocompact group topology such that multiplication is separately continuous but is not jointly continuous.

Since the group is of period 2, i.e. $x + x = 0$ and so $x = -x$ for all $x \in G$, the inverse operation is just the identity map, and so the inverse operation is automatically continuous.

Thus a pseudocompact group with a separately continuous multiplication (and even continuous inverse) need not be a topological group.

**Convergence properties in topological groups and function spaces**

Let $X$ be a topological space. For $A \subseteq X$ we use $\overline{A}$ to denote the closure of $A$ in $X$.

A *sequence converging to* $x \in X$ is a countable infinite set $S$ such that $S \setminus U$ is finite for every open neighbourhood $U$ of $x$. 
A space $X$ is Fréchet-Urysohn provided that for each set $A \subseteq X$ if $x \in \overline{A}$, then there exists a sequence $S \subseteq A$ converging to $x$.

**Definition** (Arhangel'skii [1970]): The tightness $t(X)$ of a topological space $X$ is defined as the smallest cardinal $\tau$ such that

$$\overline{A} = \bigcup\{\overline{B} : B \in [A]^\leq \tau\}$$

for every $A \subseteq X$.

**Definition** (Arhangel'skii [1972]): Let $X$ be a topological space. For $i = 1, 2, 3$ and 4 we say that $X$ is an $\alpha_i$-space if for every countable family $\{S_n : n \in \omega\}$ of sequences converging to some point $x \in X$ there exists a (kind of diagonal) sequence $S$ converging to $x$ such that:

- $(\alpha_1) S_n \setminus S$ is finite for all $n \in \omega$,
- $(\alpha_2) S_n \cap S$ is infinite for all $n \in \omega$,
- $(\alpha_3) S_n \cap S$ is infinite for infinitely many $n \in \omega$,
- $(\alpha_4) S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

**Definition** (Nyikos [1990]): We say that a space $X$ is an $\alpha_{3/2}$-space if for every countable family $\{S_n : n \in \omega\}$ of sequences converging to some point $x \in X$ such that $S_n \cap S_m = \emptyset$ for $n \neq m$, there exists a sequence $S$ converging to $x$ such that $S_n \setminus S$ is finite for infinitely many $n \in \omega$.

The only nontrivial implication $\alpha_{3/2} \rightarrow \alpha_2$ is due to Nyikos [1992].

**GENERAL TOPOLOGICAL SPACES**

**Theorem** (Simon [1980]): There exists a compact Fréchet-Urysohn $\alpha_4$-space that is not $\alpha_3$.

**Theorem** (Reznichenko [1986], Gerlits, Nagy [1988] and Nyikos [1989]): There exists a compact Fréchet-Urysohn $\alpha_3$-space that is not $\alpha_2$.

**Theorem** (Dow [1990]): $\alpha_2$ implies $\alpha_1$ in the Laver model for the Borel conjecture.

For $f, g \in \omega^\omega$ we write $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \omega$.

A family $\mathcal{F} \subseteq \omega^\omega$ is unbounded if for every function $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $g <^* f$. 

A family $\mathcal{F} \subseteq \omega^\omega$ is unbounded if for every function $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $g <^* f$. 


We define $b$ to be the smallest cardinality of an unbounded family in $(\omega^\omega, <^*)$.

**Theorem** (Nyikos [1992]): If $b = \omega_1$ holds, then there exists a countable Fréchet-Urysohn $\alpha_2$-space that is not $\alpha_1$.

**Corollary**: The existence of a (Fréchet-Urysohn) $\alpha_2$-space that is not $\alpha_1$ is both consistent with and independent of ZFC.

**Theorem** (Gerlits, Nagy [1988] and Nyikos [1989]): There exists a countable Fréchet-Urysohn $\alpha_2$-space that is not first countable.

**Theorem** (Gerlits, Nagy [1982]): There exists a (uncountable) Fréchet-Urysohn $\alpha_1$-space that is not first countable.

**Theorem** (Nyikos [1989]): Every space of character $< b$ is $\alpha_1$.

$c$ is the cardinality of the continuum.

**Theorem** (Malyhin, Shapirovskiĭ [1974]): If $MA + \neg CH$ holds, then every countable space of character $< c$ is Fréchet-Urysohn.

**Corollary**: $MA + \neg CH$ implies the existence of a countable Fréchet-Urysohn $\alpha_1$-space that is not first countable.

**Theorem** (Dow, Steprans [1990]): There is a model of ZFC in which all countable Fréchet-Urysohn $\alpha_1$-spaces are first countable.

**Corollary**: The existence of a countable Fréchet-Urysohn $\alpha_1$ space that is not first countable is both consistent with and independent of ZFC.

**Theorem** (folklore): Let

$$G = \{f \in 2^{\omega_1} : |\{\beta \in \omega_1 : f(\beta) = 1\}| \leq \omega\}.$$ 

Then $G$ is a Fréchet-Urysohn topological group that is $\alpha_1$ but is not first countable.

**TOPOLOGICAL GROUPS**

**Theorem** (Nyikos [1981]): Every Fréchet-Urysohn topological group is $\alpha_4$.

**Theorem** (Shakhmatov [1990]): Let $M$ be a model of ZFC obtained by adding $\omega_1$ many Cohen reals to an arbitrary model of ZFC. Then $M$ contains a countable Fréchet-Urysohn topological group $G$ that is not $\alpha_3$. (Note that $G$ is $\alpha_4$ by Nyikos' theorem.)

**Theorem** (Shibakov [1999]): CH implies the existence of a countable Fréchet-Urysohn topological group that is $\alpha_3$ but is not $\alpha_2$.

**Theorem** (Shakhmatov [1990]): Let $M$ be a model of ZFC obtained by adding $\omega_1$ many Cohen reals to an arbitrary model of ZFC. Then $M$ contains a countable Fréchet-Urysohn topological group $G$ that is $\alpha_2$ but is not $\alpha_{3/2}$.

**Theorem** (Shibakov [1999]): A Fréchet-Urysohn topological group that is an $\alpha_{3/2}$-space is $\alpha_1$. Thus $\alpha_{3/2}$ and $\alpha_1$ are equivalent for Fréchet-Urysohn topological groups.
Theorem (Birkhoff, Kakutani [1936]): A topological group is metrizable if and only if it is first countable.

Question (Shakhmatov [1990]): Is it consistent with ZFC that every Fréchet-Urysohn topological group is $\alpha_2$? What about countable Fréchet-Urysohn topological groups?

Question: Is it consistent with ZFC that every Fréchet-Urysohn topological group that is an $\alpha_2$-space is automatically $\alpha_2$? What about countable Fréchet-Urysohn topological groups?

Question (Shakhmatov [1990]): Is it consistent with ZFC that every countable Fréchet-Urysohn topological group that is an $\alpha_2$-space is first countable?

Question (Malyhin [197?]): Without any additional set-theoretic assumptions beyond ZFC, does there exist a countable Fréchet-Urysohn topological group that is not first countable?

Theorem (Malyhin [197?): $MA + \neg CH$ implies the existence of such a group.

Definition (Sipacheva [1998]): Let $\mathcal{F}$ be a filter on $\omega$. We say that $\mathcal{F}$ is a FUF-filter provided that the following property holds:

If $K \subseteq [\omega]<\omega$ is a family of finite subsets of $\omega$ such that for every $F \in \mathcal{F}$ there exists $K \in K$ with $K \subseteq F$, then there exists a sequence $\{K_n : n \in \omega\} \subseteq K$ so that for every $F \in \mathcal{F}$ one can find $n \in \omega$ with $K_m \subseteq F$ for all $m \geq n$.

For a filter $\mathcal{F}$ on $\omega$ let $\omega_\mathcal{F}$ be the space obtained by adding to the discrete copy of $\omega$ a single point $*$ whose filter of open neighbourhoods is $\{F \cup \{*\} : F \in \mathcal{F}\}$.

Theorem (Sipacheva [1998]): If $\mathcal{F}$ is a FUF-filter on $\omega$, then the space $\omega_\mathcal{F}$ is $\alpha_2$. For $A, B \in [\omega]<\omega$ define

$$A \cdot B = (A \setminus B) \cup (B \setminus A) \in [\omega]<\omega.$$

This operation makes $[\omega]<\omega$ into an Abelian group with $\emptyset$ as the identity element such that $A \cdot A = \emptyset$ (thus $A$ coincides with its own inverse, and all elements of $[\omega]<\omega$ have order 2).

For a filter $\mathcal{F}$ on $\omega$ let $G(\mathcal{F})$ be the group $([\omega]<\omega, \cdot, \emptyset)$ equipped with the topology whose base of open neighbourhoods of $\emptyset$ is given by the family $\{[F]<\omega : F \in \mathcal{F}\}$.

Theorem (folklore): Let $\mathcal{F}$ be a filter on $\omega$. Then:

(i) $G(\mathcal{F})$ is Hausdorff if and only if $\mathcal{F}$ is free (i.e. $\bigcap \mathcal{F} = \emptyset$),

(ii) $G(\mathcal{F})$ is Fréchet-Urysohn if and only if $\mathcal{F}$ is an FUF-filter,

(iii) $G(\mathcal{F})$ is first countable if and only if $\mathcal{F}$ is countably generated.

Theorem (folklore): If there exists a free FUF-filter on $\omega$ that is not countably generated, then there exists a countable Fréchet-Urysohn topological group that is not first countable.

Question (folklore): Is there, in ZFC only, a free FUF-filter on $\omega$ that is not countably generated?
**Theorem** (Nogura, Shakhmatov [1995]): All $\alpha_i$ properties ($i = 1, 3/2, 2, 3, 4$) coincide for locally compact topological groups.

**Theorem** (Nogura, Shakhmatov [1995]): The following conditions are equivalent:

(i) every compact group that is an $\alpha_1$-space is metrizable,

(ii) every locally compact group that is an $\alpha_4$-space is metrizable,

(iii) $b = \omega_1$.

**Corollary** (Nogura, Shakhmatov [1995]): Under CH, a locally compact group is metrizable if and only if it is $\alpha_4$. 
FUNCTION SPACES $C_p(X)$

For a topological space $X$ let $C_p(X)$ be the set of all real-valued continuous functions on $X$ equipped with the topology of pointwise convergence, i.e., with the topology which the set $C_p(X)$ inherits from $R^X$, the latter space having the Tychonoff product topology.

For every space $X$, $C_p(X)$ is both a (locally convex) topological vector space and a topological ring.

**Theorem** (Scheepers [1998]): Let $X$ be a topological space. Then $C_p(X)$ is $\alpha_2$ if and only if $C_p(X)$ is $\alpha_4$. Therefore, all three properties $\alpha_4$, $\alpha_3$ and $\alpha_2$ coincide for spaces of the form $C_p(X)$.

**Corollary** (Scheepers [1998]): If $C_p(X)$ is Fréchet-Urysohn, then $C_p(X)$ is $\alpha_2$.

**Theorem** (Scheepers [1998]): It is consistent with ZFC that there exists a subset of real numbers $X \subseteq R$ such that $C_p(X)$ is Fréchet-Urysohn (and thus $\alpha_2$) but is not $\alpha_1$.

Note that the existence of the above space is not only consistent with ZFC but also independent of ZFC by Dow’s theorem.

**Theorem** (Scheepers [1998]): It is consistent with ZFC that there exists a subset of real numbers $X \subseteq R$ such that $C_p(X)$ is $\alpha_1$ but is not Fréchet-Urysohn.

**PRODUCTS OF GENERAL SPACES**

**Theorem** (Nogura [1985]):

(i) For $i = 1, 2, 3$, if $X$ and $Y$ are $\alpha_i$-spaces, then $X \times Y$ is also an $\alpha_i$-space.

(ii) There exist compact Fréchet-Urysohn $\alpha_4$-spaces $X$ and $Y$ such that $X \times Y$ is neither Fréchet-Urysohn nor $\alpha_4$.

**Theorem** (Arangel'skii [1971]): If $X$ is a Fréchet-Urysohn $\alpha_3$-space and $Y$ is a (countably) compact Fréchet-Urysohn space, then $X \times Y$ is Fréchet-Urysohn.

**Theorem** (Costantini, Simon [1999]): There exist two countable Fréchet-Urysohn $\alpha_4$-spaces $X$ and $Y$ such that $X \times Y$ is $\alpha_4$ but fails to be Fréchet-Urysohn.

**Theorem** (Simon [1999]): Under CH, there exist two countable Fréchet-Urysohn $\alpha_4$-spaces $X$ and $Y$ such that $X \times Y$ is Fréchet-Urysohn but is not $\alpha_4$.

**Question:** Is there such an example in ZFC?

**PRODUCTS OF TOPOLOGICAL GROUPS**

**Theorem** (Todorčević [1993]): There exist two (compactly generated) Fréchet-Urysohn groups $G$ and $H$ such that $t(G \times H) > \omega$ (in particular, $G \times H$ is not Fréchet-Urysohn). Moreover, every countable subset of $G$ and $H$ is metrizable, and so both $G$ and $H$ are $\alpha_1$.

**Theorem** (Malyhin, Shakhmatov [1992]):

Add a single Cohen real to a model of $MA + \neg CH$. Then, in the generic extension,
the exists a (hereditarily separable) Fréchet-Urysohn topological group $G$ such that $t(G \times G) > \omega$ (in particular, $G \times G$ is not Fréchet-Urysohn). Moreover, $G$ is an $\alpha_1$-space.

**Theorem** (Shibakov [1999]): Under CH, there exists a *countable* Fréchet-Urysohn topological group $G$ such that $G \times G$ is not Fréchet-Urysohn.

**Question:** Is there such an example in ZFC only?

**Question:** In ZFC only, does there exist two *countable* Fréchet-Urysohn topological groups $G$ and $H$ such that $G \times H$ is not Fréchet-Urysohn?

**Question:** In ZFC only, is there a Fréchet-Urysohn topological group $G$ such that $G$ is $\alpha_1$ but $G \times G$ is not Fréchet-Urysohn?

**PRODUCTS OF $C_p(X)$**

**Theorem** (Tkáčuk [1984]): If $C_p(X)$ is Fréchet-Urysohn, then even its countable power $C_p(X)^\omega$ is Fréchet-Urysohn.

**Theorem** (Todorčević [1993]): There exist two spaces $X$ and $Y$ such that both $C_p(X)$ and $C_p(Y)$ are Fréchet-Urysohn but

$$t(C_p(X) \times C_p(Y)) > \omega$$

(in particular, $C_p(X) \times C_p(Y)$ is not Fréchet-Urysohn). Moreover, every countable subset of $C_p(X)$ and $C_p(Y)$ is metrizable, and so both $C_p(X)$ and $C_p(Y)$ are $\alpha_1$. 