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RECENT PROGRESS IN TOPOLOGICAL GROUPS: SELECTED TOPICS

愛媛大学理学部 Dmitri Shakhmatov

Department of Mathematics, Faculty of Science, Ehime University

Some historical background on topological groups

Theorem (Pontryagin?): If the space of a topological group is a T_0-space, then it is automatically Tychonoff.

Theorem (Markov [1941]): There exists a topological group the space of which is not normal.

Theorem (Birkhoff-Kakutani [1930s]): A topological group is metrizable if and only if it is first countable.

Theorem: Every locally compact group has a Haar measure. (This allows for integration on it.)

Theorem: Let G be a locally compact abelian group, $g \in G$ and $g \neq 0$. Then there exists a continuous group homomorphism $\pi : G \to \mathbf{T}$ from G into the torus group $\mathbf{T}$ such that $\pi(g) \neq 0$.

Theorem (Peter-Weyl-van Kampen): Let G be a locally compact group, $g \in G$ and $g \neq 1_G$ where 1_G is the identity element of G. Then there exist a natural number n and a continuous group homomorphism $\pi : G \to \mathbf{U}(n)$ from G into the group $\mathbf{U}(n)$ of unitary $n \times n$ matrices over the complex number field such that $\pi(g) \neq I$. (Here I is the identity matrix of $\mathbf{U}(n)$.) A cardinal τ is *Ulam nonmeasurable* provided that for every ultrafilter $\mathcal{F}$ on τ with the countable intersection property there exists $\alpha \in \tau$ such that $\mathcal{F} = \{A \subseteq \tau : \alpha \in A\}$.

Theorem (Varopolous [1964]): Let G and H be locally compact groups, and let $\pi : G \to H$ be a group homomorphism. Assume that:

(i) $|G|$ is an Ulam nonmeasurable cardinal, and

(ii) π is *sequentially continuous*, i.e. for every sequence $S \subseteq G$ the image $\pi(S)$ is also a convergent sequence.

Then π is continuous.

Theorem (Comfort-Remus [1994]): Let G be a compact group that is either abelian or connected. Suppose also that every sequentially continuous group homomorphism

$\pi : G \to H$ from G into any compact group H is continuous. Then $|G|$ is an Ulam measurable cardinal.

Theorem (Pasynkov [1961]): ind $G =$ Ind $G =$ dim G for a locally compact group G.

Note: Locally compact groups are paracompact (Pasynkov).

A continuous image of a Cantor cube $\{0,1\}^\kappa$ is called a *dyadic* space.

Theorem (Kuz'minov [1959]): Compact groups are dyadic.

A compact space X is said to be *Dugundji* if any continuous function $f: A \to X$ defined on a closed subset A of a Cantor cube $\{0,1\}^\kappa$ has a continuous extension $F: \{0,1\}^\kappa \to X$.

Since we can choose the above f to be onto, Dugundji spaces are dyadic.

Theorem (Čoban [1970s]): Let X be a compact G_δ-subset of some topological group. Then X is a Dugundji space.

Theorem (Hagler, Gerlits and Efimov [1976/77]): An infinite compact group G contains a homeomorphic copy of the Cantor cube $\{0,1\}^{w(G)}$.

As a corollary, one gets a particular version of Shapirovskii's theorem about mappings onto Tychonoff cubes:

Theorem: Every infinite compact group G admits a continuous map onto a Tychonoff cube $[0,1]^{w(G)}$.

Recall that a space X is σ*-compact* if it is a union of countable family of its compact subspaces.

A space X is *ccc* provided that X does not have an uncountable family of non-empty pairwise disjoint open subsets.

Theorem (Tkachenko [1981]): A σ-compact group is ccc.

A space is *pseudocompact* if every real-valued continuous function defined on it is bounded.

Theorem (Comfort and Ross [1966]): Let G be a dense subggroup of a compact group K. Then the following conditions are equivalent:

(i) G is pseudocompact,

(ii) $G \cap B \neq \emptyset$ for every non-empty G_δ-subset B of K.

Corollary (Comfort and Ross [1966]): The product of any family of pseudocompact groups is pseudocompact.

A (Hausdorff) topological group $(G, \mathcal{T})$ is called *minimal* provided that for every Hausdorff group topology $\mathcal{T}'$ on G with $\mathcal{T}' \subseteq \mathcal{T}$ one has $\mathcal{T}' = \mathcal{T}$.

Clearly, compact groups are minimal.

Theorem (Prodanov, Stoyanov [1984]): A minimal abelian group G is totally bounded, i.e. G is (isomorphic to) a subgroup of some compact topological group.

Generating dense subgroups of topological groups: Suitable sets

If X is a subset of a group G, then $\langle X \rangle$ denotes the smallest subgroup of G that contains X.

Let X be a subspace X of a topological group G.

We say that X *algebraically generates* G provided that $\langle X \rangle = G$.

We say that X *topologically generates* G if $\langle X \rangle$ is dense in G.

A compact connected abelian group G has weight less than or equal to the continuum if and only if it is monothetic; that is, there exists an element $g \in G$ such that G is topologically generated by the subset $\{g\}$.

This result was improved by Hofmann and Morris [1990] by showing that a compact connected group G can be topologically generated by two elements if and only if the weight of G is less than or equal to the continuum.

Clearly, neither finite nor countable subsets of a topological group G with weight greater than the continuum can generate a dense subgroup of G. This fact led Hofmann and Morris to introduce the concept of suitable set as a way to define the notion of topological generating sets which are in some sense "close" to finite sets:

Definition (Hofmann and Morris [1990]): A subset S of a topological group G is said to be *suitable* for G if S is discrete in itself, generates a dense subgroup of G and $S \cup \{1_G\}$ is closed in G, where 1_G is the identity of G.

Theorem (Hofmann and Morris [1990]): Every locally compact group has a suitable set.

Theorem (Comfort, Morris, Robbie, Svetlichny, and Tkačenko [1998]):

Each metric group has a suitable set. A topological group G is *almost metrizable* if there exists a compact subgroup K of G such that the space of left cosets G/K is metrizable.

Theorem (Okunev and Tkachenko [1998]): An almost metrizable group has a suitable set.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): A topological group representable as a countable union of closed metrizable subspaces has a suitable set.

Corollary (Dikranjan, Tkachenko, Tkachuk [1999]): A free (abelian) topological group over a metric space has a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Suppose that a topological group G is a countable union of its metrizable subspaces. Does G have a suitable set?

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): Every topological group with a σ-discrete network has a suitable set.

Corollary (Dikranjan, Tkachenko, Tkachuk [1999]): Every topological group with a countable network (i.e. a cosmic group) has a suitable set.

Corollary (Dikranjan, Tkachenko, Tkachuk [1999]): Stratifiable groups have suitable sets.

From the above results it follows that all countable groups have suitable sets. In fact, even more can be said for countable groups:

Theorem (Comfort, Morris, Robbie, Svetlichny, and Tkačenko [1998]):

Every countable topological group G has a closed discrete subspace S that algrebraically generates G.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): A separable σ-compact group has a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Does every σ-compact group of size $< c$ have a suitable set?

Theorem (Comfort, Morris, Robbie, Svetlichny, and Tkačenko [1998]):

Let G be the free (abelian) topological group of $\beta\mathbf{N} \setminus \mathbf{N}$. Then G does not have a suitable set. In particular, a σ-compact group need not have a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Does every σ-compact group has a dense subgroup with a suitable set?

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): If G is a topological group with a suitable set, then $d(G) \leq l(G) \cdot \psi(G)$. In particular, a non-separable Lindelöf group of countable pseudocharacter does not have a suitable set.

A space is *submetrizable* if it has a weaker metric topology.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): There exists a submetrizable Lindelöf non-separable linear topological space L of countable tightness. Thus, L does not have a suitable set.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): Under some additional set-theoretic assumptions (diamond) there exists a hereditarily Lindelöf non-separable linear topological space L of countable tightness. Thus no dense additive subgroup of L has a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Can one construct in ZFC a topological group which does not contain a dense subgroup with a suitable set?

A space X is *ω-bounded* if the closure of each countable subset of X is compact.

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): There exists an ω-bounded group G without a suitable set. Moreover, each power G^κ of G does not have a suitable set.

Question: In ZFC, does there exists a separable (pseudocompact) group without a suitable set?

Theorem (Dikranjan, Tkachenko, Tkachuk [1999]): A locally separable non-pseudocomapct group has a suitable set.

Question (Dikranjan, Tkachenko, Tkachuk [1999]): Does there exists an ω-bounded topological group of size c without a suitable set?

Generating dense subgroups of topological groups: Topologically generating weight

We use $w(X)$ to denote the *weight* of a topological space X, i.e. the smallest size of a base for the topology of X if such a base is infinite, or ω otherwise.

Define

$agw(G) = \min\{w(X) : X$ is closed in G and algebraically generates $G\}$

and

$tgw(G) = \min\{w(F) : F$ is closed in G and topologically generates $G\}$.

We will call $agw(G)$ an *algebraically generating weight of* G and $tgw(G)$ a *topologically generating weight of* G.

Clearly $tgw(G) \leq agw(G) \leq w(G)$. While the definition of algebraically generating weight appears to be more natural than that of topologically generating weight, it does not lead to anything new for compact groups:

Theorem (Arhangel'skii): $agw(G) = w(G)$ holds for every compact group G.

For an infinite cardinal τ define $\sqrt{\tau}$ to be the smallest infinite cardinal κ with $\tau \leq \kappa^\omega$. Clearly, $\sqrt{\tau} \leq \tau$.

Theorem (Dikranjan and Shakhmatov [1998]): $tgw(G) = \sqrt{w(c(G))} \cdot w(G/c(G))$ for every compact group G, where $c(G)$ is the connected component of G.

Corollary (Dikranjan and Shakhmatov [1998]): $tgw(G) = w(G)$ for a totally disconnected compact group G.

Corollary (Dikranjan and Shakhmatov [1998]): $tgw(G) = \sqrt{w(G)}$ for every connected compact group G. A *super-sequence* is a compact space with at most one non-isolated point.

Suitable sets in compact groups are precisely super-sequences, so Hofmann-Morris' theorem justifies an introduction of the following cardinal number for a compact group G:

$seq(G) = \omega \cdot \min\{|S| : S \subseteq G \text{ is a super-sequence topologically generating } G\}$.

Clearly $tgw(G) \leq seq(G) \leq w(G)$.

Theorem (Dikranjan and Shakhmatov [1998]): $tgw(G) = seq(G)$ for every compact group G.

For topological spaces X and Y we use $C(X, Y)$ to denote the family of all continuous maps from X to Y. No topology is assumed on $C(X, Y)$.

For topological groups G and H we will use $\mathrm{Hom}(G, H)$ to denote the family of all continuous homomorphisms from G to H. No topology is assumed on $\mathrm{Hom}(G, H)$.

Lemma 1: Let X be a subset of a topological group G. Assume that X topologically generates G. Then $|\mathrm{Hom}(G, H)| \leq |C(X, H)|$ for every topological group H.

Proof: Define a map $f : \mathrm{Hom}(G, H) \to C(X, H)$ by $f(\pi) = \pi|_X$ for $\pi \in \mathrm{Hom}(G, H)$. We claim that f is an injection. Indeed, assume that $\pi, \varpi \in \mathrm{Hom}(G, H)$ and $f(\pi) = f(\varpi)$. Then $\pi|_X = \varpi|_X$. Since both π and ϖ are group homomorphisms from G to H, one has $\pi|_{\langle X \rangle} = \varpi|_{\langle X \rangle}$. Since $\langle X \rangle$ is dense in G, continuity of π and ϖ implies now that $\pi = \varpi$.

PROOF OF THE TOTALLY DISCONNECTED CASE

Lemma 2: Let X be a totally disconnected compact space and H be a discrete space. Then $|C(X, H)| \leq w(X)$.

Let X be a closed subset of G that topologically generates G. Since G is compact and totally disconnected, it is profinite, i.e. its topology is determined by the family of all continuous homomorphisms into finite discrete groups. Let H be one of these discrete groups.

Since G is totally disconnected, so is X. Therefore $|C(X, H)| \leq w(X)$ by Lemma 2.

We also have $|\mathrm{Hom}(G, H)| \leq |C(X, H)|$ since X topologically generates G (Lemma 1).

Since there are only countably many pairwise non-isomorphic finite discrete groups H, it now follows that $w(G) \leq \omega \cdot w(X) = w(X)$.

PROOF OF THE INEQUALITY $\sqrt{w(G)} \leq tgw(G)$

Lemma 3: Let X be a compact space and H be a separable metric space. Then $|C(X,H)| \leq w(X)^{\omega}$.

Theorem: $\sqrt{w(G)} \leq tgw(G)$ for every compact group G.

Proof: Let G be a compact group. By Peter-Weyl-van Kampen theorem the topology of every compact group is determined by the set of its homomorphisms into the compact metric group $H = \prod_n \mathbf{U}(n)$, where $\mathbf{U}(n)$ is the group of unitary $n \times n$ matrices over the complex number field.

Therefore $w(G) \leq |\mathrm{Hom}(G,H)|$.

Let X be a closed subspace of G that topologically generates G and satisfies the equality $w(X) = tgw(G)$. From Lemmas 1 and 3 we have the following:

$$|\mathrm{Hom}(G,H)| \leq |C(X,H)| \leq w(X)^{\omega} = tgw(G)^{\omega}.$$

Therefore $\sqrt{w(G)} \leq \sqrt{tgw(G)^{\omega}} \leq tgw(G)$.

STRONGLY TOPOLOGICALLY FINITELY GENERATED GROUPS

Recall that a topological group G is *topologically finitely generated* provided that there exists a finite subset of G topologically generating G.

Definition (Dikranjan and Shakhmatov): We say that a topological group G is *strongly topologically finitely generated* provided that for every open set U containing the identity element of G one can find a finite set $F \subseteq U$ such that F topologically generates G.

Lemma 4: Let G be a topologically finitely generated group that has no proper open subgroups. Then G is strongly topologically finitely generated. *Proof:* Let $D = \langle g_1, \dots, g_n \rangle$ be a dense finitely generated subgroup of G.

Let U be an open neighbourhood of e in G. Then the subgroup $H = \langle D \cap U \rangle$ of D is obviosuly open in D, hence also closed in D. On the other hand, its closure $\overline{H}$ in G contains $\overline{D \cap U} \supseteq \overline{U}$ since U is open and D is dense in G. Therefore $\overline{H}$ is an open subgroup of G. Our hypothesis gives $\overline{H} = G$.

Now closedness of H in D yields $H = \overline{H} \cap D = G \cap D = D$. We have proved in this way that $D = H$.

Let $i = 1, \dots, n$. Since

$$g_i \in D = H = \langle D \cap U \rangle,$$

there exists a finite subset $F_i \subseteq D \cap U$ such that $g_i \in \langle F_i \rangle$. Clearly the finite set set $F = \bigcup_{i=1}^{n} F_i$ generates the whole group D and $F \subseteq U$. Since D is dense in G, F topologically generates G.

Lemma 5: Let G be a metric (not necessarily compact!) group that is strongly topologically finitely generated. Then for every infinite cardinal τ one has $seq(G^{\tau^{\omega}}) \leq \tau$.

Proof: Fix an infinite cardinal τ, and let $\{U_n : n \in \omega\}$ be a decreasing open base at the identity element e of G. For each $n \in \omega$ use the hypothesis of our lemma to fix a finite set $F_n = \{g_i^n : i < m_n\} \subseteq U_n$ such that $\langle F_n \rangle$ is dense in G.

For $f \in \tau^\omega$ and $n \in \omega$ let $f|n \in \tau^n$ be the restriction of the function f to $n = \{0, 1, \ldots, n-1\}$.

For $n \in \omega$, $i < m_n$ and $\phi \in \tau^n$ we define a point $x_{n,i,\phi} \in G^{\tau^\omega}$ as follows:

for each $f \in \tau^\omega$ let $x_{n,i,\phi}(f) = g_i^n$ if $f|n = \phi$ and $x_{n,i,\phi}(f) = e$ otherwise. Then

$$X = \{x_{n,i,\phi} : n \in \omega, i < m_n, \phi \in \tau^n\}$$

is a subset of G^{τ^ω} of size at most τ.

CLAIM 1. For every open set W which contains the identity element e of G^{τ^ω} the set $X \setminus W$ is at most finite.

Claim 1 implies that $X \cup \{e\}$ is a super-sequence.

Proof of Claim 1. Since W contains a finite intersection of sets of the form

$$V_{f,n} = \{x \in G^{\tau^\omega} : x(f) \in U_n\},$$

it suffices to prove that, for each $f \in \tau^\omega$ and for every $n \in \omega$, $x(f) \in U_n$ for all but finitely many $x \in X$, i.e., the set $\{x \in X : x(f) \notin U_n\}$ is finite.

So let $f \in \tau^\omega$ and $n \in \omega$. Our construction implies that if $k \in \omega$, $j < m_k$, $\phi \in \tau^k$ and $x_{k,j,\phi}(f) \notin U_n$, then:
(i) $k < n$ (because $n \leq k$ implies $U_k \subseteq U_n$), and
(ii) $f|k = \phi$ (because $f|k \neq \phi$ implies $x_{k,j,\phi}(f) = e \in U_n$).

There are only finitely many of such $x_{k,j,\phi}$, and the result follows.

CLAIM 2. For every finite subset F of τ^ω there exists $n \in \omega$ (depending on F) such that, for each $f \in F$, the finite set

$$\{x_{n,i,f|n} : i < m_n\} \subseteq X$$

satisfies the following two properties:

(i) $\langle \{x_{n,i,f|n}(f) : i < m_n\} \rangle$ is dense in G,

(ii) $x_{n,i,f|n}(f') = e$ whenever $f' \in F \setminus \{f\}$.

From Claim 2 it immediately follows that, for every finite set $F \subseteq \tau^\omega$, the projection of

$$\langle \{x_{n,i,f|n} : f \in F, i < m_n\} \rangle$$

(where n is as in Claim 2) onto the subproduct G^F is dense in G^F. Since

$$\{x_{n,i,f|n} : f \in F, i < m_n\} \subseteq X,$$

this implies that $\langle X \cup \{e\} \rangle$ is dense in G^{τ^ω}. *Proof of Claim 2.* There exists $n \in \omega$ such that $f'|n \neq f''|n$ whenever $f', f'' \in F$ and $f' \neq f''$. We will show that this n works.

Indeed, let $f \in F$. By our construction, one has $x_{n,i,f|n}(f) = g_i^n$ for all $i < m_n$, so

$$\{x_{n,i,f|n}(f) : i < m_n\} = \{g_i^n : i < m_n\},$$

and the latter set generates a dense subgroup of G. This implies (i).

Again by our construction, $f' \in F \setminus \{f\}$ implies $f'|n \neq f|n$ and so $x_{n,i,f|n}(f') = e$. This gives (ii).

PROOF OF THE CONNECTED CASE

Theorem: (Universal compact connected group of a given weight)

There exists a sequence $\{L_n : n \in \omega\}$ of compact connected simple Lie groups L_n such that every compact connected group of weight $\leq \tau$ is a quotient group of the group

$$G_\tau = (\hat{\mathbf{Q}})^\tau \times \prod_n L_n^\tau,$$

where $\hat{\mathbf{Q}}$ is the Pontryagin dual of the discrete group $\mathbf{Q}$ of rational numbers. (Note that G_τ is a connected group of weight τ.)

Theorem: $seq(G) \leq \sqrt{w(G)}$ for a compact connected group G.

Proof: Let $\tau = \sqrt{w(G)}$. By the above theorem, G is a quotient group of the group

$$H = (\hat{\mathbf{Q}})^{w(G)} \times \prod_n L_n^{w(G)}$$

for a suitable sequence $\{L_n : n \in \omega\}$ of compact connected simple Lie groups L_n. Since $w(G) \leq \tau^\omega$, H is a natural quotient group (under projection map) of the group K^{τ^ω}, where

$$K = (\hat{\mathbf{Q}}) \times \prod_n L_n.$$

Therefore $seq(G) \leq seq(H) \leq seq(K^{\tau^\omega})$.

Since K is connected, it has no proper open subgroups. Since K is also topologically finitely generated, K is strongly topologically finitely generated (Lemma 4).

Therefore $seq(K^{\tau^\omega}) \leq \tau$ by Lemma 5.

Finally, $seq(G) \leq seq(K^{\tau^\omega}) \leq \tau = \sqrt{w(G)}$.

Applications of Michael's selection theorem to proving results about (mostly compact) topological groups

Uspenskii [1988] was the first to notice how Michael's selection theorem can be applied to get a simple topological proof of the classical result of Kuzminov that *compact groups are dyadic.* Recall that a *set-valued map* $F : Y \to Z$ is a map which assigns a non-empty closed set $F(y) \subseteq Z$ to every point $y \in Y$.

This set-valued map is *lower semicontinuous* if

$$V = \{y \in Y : F(y) \cap U \neq \emptyset\}$$

is open in Y for every set U open in Z.

A *selection* for a set-valued map $F : Y \to Z$ is a a (single-valued) continuous map $f : Y \to Z$ such that $f(y) \in F(y)$ for all $y \in Y$.

Theorem (Michael [1956]): Every lower semicontinuous set-valued map $F : Y \to Z$ from a zero-dimensional compact space Y into a complete metric space (in particular, compact metric space) Z has a selection.

Lemma: Suppose that H and H' are topologcal groups, G is a subgroup of the product $H \times H'$, $\varphi : H \times H' \to H$ and $\pi : H \times H' \to H'$ are projections onto the first and second coordinates respectively. Assume also that:

(i) the restriction $\varphi|_G : G \to \varphi(G)$ of φ to G is an open map,

(ii) the restriction $\pi|_G : G \to \pi(G)$ of π to G is a closed map, and

(iii) the subgroup $\pi(G)$ of H' is a complete metric group.

Then for every compact zero-dimensional space $Y \subseteq \varphi(G)$ there exists a homeomorphic embedding $f : Y \to G$ such that $(\varphi \circ f)(y) = y$ for every $y \in Y$. Proof: Define $Z = \pi(G)$ and note that $G \subseteq H \times Z$.

For $y \in Y$ define $F(y) = \{z \in Z : (y, z) \in G\}$.

The set $G \cap (\{y\} \times H')$ is closed in G, so from (ii) it follows that

$$F(y) = \pi(G \cap (\{y\} \times H'))$$

is closed in $Z = \pi(G)$.

For $y \in Y$, since $y \in Y \subseteq \varphi(G)$, we have $F(y) \neq \emptyset$. Therefore $F : Y \to Z$ is a set-valued map.

We claim that F is lower semicontinuous. Indeed, let U be an open subset of Z. We have to check that the set

$$V = \{y \in Y : F(y) \cap U \neq \emptyset\}$$

is open in Y. To see this note that the set $G \cap (H \times U)$ is open in G, so $\varphi(G \cap (H \times U))$ is open in $\varphi(G)$ by (i). Since $Y \subseteq \varphi(G)$,

$$V = Y \cap \varphi(G \cap (H \times U))$$

is open in Y.

Since $\pi(G) = Z$ is a complete metric group, we can use Michael's selection theorem to pick a (single-valued) continuous selection $f : Y \to Z$ of F.

From the definition of F it follows that $(\varphi \circ f)(y) = y$ for all $y \in Y$. In particular, f is one-to-one. Since Y is compact, f is a homeomorphism.

Corollary: Suppose that H is a topologcal group, H' is a metric group, G is a compact subgroup of the product $H \times H'$, and $\varphi : H \times H' \to H$ is the projection onto the first coordinate.

Then for every compact zero-dimensional space $Y \subseteq \varphi(G)$ there exists a homeomorphic embedding $f : Y \to G$ such that $(\varphi \circ f)(y) = y$ for every $y \in Y$.

Proof: Let $\pi : H \times H' \to H'$ be the projection onto the second coordinate.

Since G is compact, the restriction $\varphi|_G : G \to \varphi(G)$ of φ to G is a closed continuous map, soa quotient map, and so an open map. This gives (i).

Since G is compact, the restriction $\pi|_G : G \to \pi(G)$ of π to G is a closed map. This gives (ii).

The subgroup $\pi(G)$ of H' is compact, being a continuous image of the compact group G. Since H' is metric, so is $\pi(G)$. In particular, $\pi(G)$ is a complete metric group. This gives (iii).

A subset X of an abelian group G is *independent* provided that $\langle A \rangle \cap \langle X \setminus A \rangle = \{0\}$ for every $A \subseteq X$.

For a prime number $p \geq 2$, a subset X of an abelian group G is called *p-independent* provided that X is independent and

$$\min\{1 \leq n \leq p : nx = 0\} = p$$

for every $x \in X$. For an abelian group G and a prime number p, cardinal numbers

$$r_0(G) = \sup\{|X| : X \subseteq G \text{ is independent}\}$$

and

$$r_p(G) = \sup\{|X| : X \subseteq G \text{ is } p\text{-independent}\}$$

are called *rank* and *p-rank* of G respectively.

For a cardinal number τ we define $\log(\tau)$ to be the smallest infinite cardinal σ such that $2^\sigma \geq \tau$.

Theorem (Shakhmatov): Let G be an infinite compact abelian group. Then:

(i) G contains an independent subset X homeomorphic to the Cantor cube $\{0,1\}^{\log r_0(G)}$ of weight $\log r_0(G)$, and

(ii) for every prime number $p \geq 2$ the group G contains a p-independent subset X homeomorphic to the Cantor cube $\{0,1\}^{\log r_p(G)}$ of weight $\log r_p(G)$.

Even the following corollary to the above general theorem is new:

Corollary (Shakhmatov): Let G be an infinite compact abelian group. Then:

(i) G contains a closed independent subset X with $|X| = r_0(G)$, and

(ii) for every prime number $p \geq 2$ the group G contains a closed p-independent subset X with $|X| = r_p(G)$.

Wallace's problem and continuity of separately continuous multiplication in semigroups

A *semigroup* is a pair $(S, \cdot)$ consisting of a set S and a binary associative operation $\cdot$ on S.

A semigroup S has the *cancellation property* provided that either of $sx = sy$ and $xs = ys$ implies $x = y$ whenever $x, y, s \in S$.

A *topological semigroup* is a semigroup equipped with a topology which makes its binary operation continuous.

Clearly, every topological group is a topological semigroup with the cancellation property.

Theorem (Gelbaum, Kalish and Olmsted [1951]): A compact semigroup with the cancellation property is a topological group.

Problem (Wallace [1955]): Is a countably compact Hausdorff semigroup with the cancellation property a topological group?

A series of positive results by Mukhurjea-Tserpes, Grant, Korovin, Reznichenko, Yur'eva culminated in the following most general result:

Theorem (Bokalo-Guran [1996]): A sequentially compact Hausdorff semigroup with the cancellation property is a topological group.

Theorem (Robbie, Svetlichny [1996]): Suppose that there exists an abelian topological group G with the following properties:

(i) G is countably compact,

(ii) every infinite closed subset of G has cardinality greater or equal than the continuum,

(iii) G is torsion-free, i.e. for every $x \in G$ and each $n \geq 1$ one has $ng \neq 1_G$.

Then, (inside of G) one can find a Tychonoff counterexample to the Wallace problem, i.e. there exists a commutative Tychonoff countably compact semigroup with the cancellation property that is not a topological group.

Theorem (Tkačenko [1990]): Assume CH. Than there exists a topological group G with the following properties:

(i) G is countably compact,

(ii) every infinite closed subset of G has cardinality greater or equal than the continuum,

(iii) G is a free abelian group (in particular, G is torsion-free).

Tomita [1997] constructed similar group under Martin's Axiom for Countable Sets.

Question: Is there such a group in ZFC?

Theorem (Ellis [1957]): A group equipped with a locally compact topology such that multiplication is separately continuous is a topological group.

Theorem (Korovin [1992]): A group equipped with a countably compact topology such that multiplication is separately continuous is a topological group.

Theorem (Reznichenko [1994]): Let G be group equipped with a pseudocompact topology such that multiplication is separately continuous. Then G is a topological group provided that one of the following conditions holds:

(i) G has countable tightness,

(ii) G is separable,

(iii) G is a k-space.

Theorem (Korovin [1992]): There exists an abelian group (of period 2) equipped with a pseudocompact group topology such that multiplication is separately continuous but is not jointly continuous.

Since the group is of period 2, i.e. $x + x = 0$ and so $x = -x$ for all $x \in G$, the inverse operation is just the identity map, and so the inverse operation is automatically continuous.

Thus a pseudocompact group with a separately continuous multiplcation (and even continuous inverse) need not be a topological group.

Convergence properties in topological groups and function spaces

Let X be a topological space. For $A \subseteq X$ we use $\overline{A}$ to denote the closure of A in X.

A *sequence converging to* $x \in X$ is a countable infinite set S such that $S \setminus U$ is finite for every open neighbourhood U of x.

A space X is *Fréchet-Urysohn* provided that for each set $A \subseteq X$ if $x \in \overline{A}$, then there exists a sequence $S \subseteq A$ converging to x.

Definition (Arhangel'skii [1970]): The *tightness* $t(X)$ of a topological space X is defined as the smallest cardinal τ such that

$$\overline{A} = \bigcup\{\overline{B} : B \in [A]^{\leq\tau}\} \text{ for every } A \subseteq X.$$

$$\text{metric} \to \text{first countable} \to$$

$$\to \text{Fréchet-Urysohn} \to t(X) = \omega$$

Definition (Arhangel'skii [1972]): Let X be a topological space. For $i = 1, 2, 3$ and 4 we say that X is an α_i*-space* if for every countable family $\{S_n : n \in \omega\}$ of sequences converging to some point $x \in X$ there exists a (kind of diagonal) sequence S converging to x such that:

(α_1) $S_n \setminus S$ is finite for all $n \in \omega$,

(α_2) $S_n \cap S$ is infinite for all $n \in \omega$,

(α_3) $S_n \cap S$ is infinite for infinitely many $n \in \omega$,

(α_4) $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

Definition (Nyikos [1990]): We say that a space X is an $\alpha_{3/2}$*-space* if for every countable family $\{S_n : n \in \omega\}$ of sequences converging to some point $x \in X$ such that $S_n \cap S_m = \emptyset$ for $n \neq m$, there exists a sequence S converging to x such that $S_n \setminus S$ is finite for infinitely many $n \in \omega$.

$$\text{metric} \to \text{first countable} \to$$

$$\to \alpha_1 \to \alpha_{3/2} \to \alpha_2 \to \alpha_3 \to \alpha_4$$

The only nontrivial implication $\alpha_{3/2} \to \alpha_2$ is due to Nyikos [1992].

GENERAL TOPOLOGICAL SPACES

Theorem (Simon [1980]): There exists a compact Fréchet-Urysohn α_4-space that is not α_3.

Theorem (Reznichenko [1986], Gerlits, Nagy [1988] and Nyikos [1989]): There exists a compact Fréchet-Urysohn α_3-space that is not α_2.

Theorem (Dow [1990]): α_2 implies α_1 in the Laver model for the Borel conjecture.

For $f, g \in \omega^\omega$ we write $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \omega$.

A family $\mathcal{F} \subseteq \omega^\omega$ is *unbounded* if for every function $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $g <^* f$.

We define b to be the smallest cardinality of an unbounded family in $(\omega^\omega, <^*)$.

Theorem (Nyikos [1992]): If $b = \omega_1$ holds, then there exists a countable Fréchet-Urysohn α_2-space that is not α_1.

Corollary: The existence of a (Fréchet-Urysohn) α_2-space that is not α_1 is both consistent with and independent of ZFC.

Theorem (Gerlits, Nagy [1988] and Nyikos [1989]): There exists a countable Fréchet-Urysohn α_2-space that is not first countable.

Theorem (Gerlits, Nagy [1982]): There exists a (uncountable) Fréchet-Urysohn α_1-space that is not first countable.

Theorem (Nyikos [1989]): Every space of character $< b$ is α_1.

c is the cardinality of the continuum.

Theorem (Malyhin, Shapirovskiĭ [1974]): If $MA + \neg CH$ holds, then every countable space of character $< c$ is Fréchet-Urysohn.

Corollary: $MA + \neg CH$ implies the existense of a countable Fréchet-Urysohn α_1-space that is not first countable.

Theorem (Dow, Steprans [1990]): There is a model of ZFC in which all countable Fréchet-Urysohn α_1-spaces are first countable.

Corollary: The existence of a countable Fréchet-Urysohn α_1 space that is not first countable is both consistent with and independent of ZFC.

Theorem (folklore): Let

$$G = \{f \in 2^{\omega_1} : |\{\beta \in \omega_1 : f(\beta) = 1\}| \leq \omega\}.$$

Then G is a Fréchet-Urysohn topological group that is α_1 but is not first countable.

TOPOLOGICAL GROUPS

Theorem (Nyikos [1981]): Every Fréchet-Urysohn topological group is α_4.

Theorem (Shakhmatov [1990]): Let M be a model of ZFC obtained by adding ω_1 many Cohen reals to an arbitrary model of ZFC. Then M contains a countable Fréchet-Urysohn topological group G that is not α_3. (Note that G is α_4 by Nyikos' theorem.)

Theorem (Shibakov [1999]): CH implies the existence of a countable Fréchet-Urysohn topological group that is α_3 but is not α_2.

Theorem (Shakhmatov [1990]): Let M be a model of ZFC obtained by adding ω_1 many Cohen reals to an arbitrary model of ZFC. Then M contains a countable Fréchet-Urysohn topological group G that is α_2 but is not $\alpha_{3/2}$.

Theorem (Shibakov [1999]): A Fréchet-Urysohn topological group that is an $\alpha_{3/2}$-space is α_1. Thus $\alpha_{3/2}$ and α_1 are equivalent for Fréchet-Urysohn topological groups.

Theorem (Birkhoff, Kakutani [1936]): A topological group is metrizable if and only if it is first countable.

Question (Shakhmatov [1990]): Is it consistent with ZFC that every Fréchet-Urysohn topological group is α_3? What about countable Fréchet-Urysohn topological groups?

Question: Is it consistent with ZFC that every Fréchet-Urysohn topological group that is an α_3-space is automatically α_2? What about countable Fréchet-Urysohn topological groups?

Question (Shakhmatov [1990]): Is it consistent with ZFC that every *countable* Fréchet-Urysohn topological group that is an α_2-space is first countable?

Question (Malyhin [197?]): Without any additional set-theoretic assumptions beyond ZFC, does there exist a *countable* Fréchet-Urysohn topological group that is not first countable?

Theorem (Malyhin [197?]): $MA + \neg CH$ implies the existence of such a group.

Definition (Sipacheva [1998]): Let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{F}$ is a *FUF-filter* privided that the following property holds:

if $\mathcal{K} \subseteq [\omega]^{<\omega}$ is a family of finite subsets of ω such that for every $F \in \mathcal{F}$ there exists $K \in \mathcal{K}$ with $K \subseteq F$, then there exists a sequence $\{K_n : n \in \omega\} \subseteq \mathcal{K}$ so that for every $F \in \mathcal{F}$ one can find $n \in \omega$ with $K_m \subseteq F$ for all $m \geq n$.

For a filter $\mathcal{F}$ on ω let $\omega_{\mathcal{F}}$ be the space obtained by adding to the discrete copy of ω a single point $*$ whose filter of open neighbourhoods is $\{F \cup \{*\} : F \in \mathcal{F}\}$.

Theorem (Sipacheva [1998]): If $\mathcal{F}$ is a FUF-filter on ω, then the space $\omega_{\mathcal{F}}$ is α_2. For $A, B \in [\omega]^{<\omega}$ define

$$A \cdot B = (A \setminus B) \cup (B \setminus A) \in [\omega]^{<\omega}.$$

This operation makes $[\omega]^{<\omega}$ into an Abelian group with $\emptyset$ as the identity element such that $A \cdot A = \emptyset$ (thus A coincides with its own inverse, and all elements of $[\omega]^{<\omega}$ have order 2).

For a filter $\mathcal{F}$ on ω let $G(\mathcal{F})$ be the group $([\omega]^{<\omega}, \cdot, \emptyset)$ equipped with the topology whose base of open neighbourhoods of $\emptyset$ is given by the family $\{[F]^{<\omega} : F \in \mathcal{F}\}$.

Theorem (folklore): Let $\mathcal{F}$ be a filter on ω. Then:

(i) $G(\mathcal{F})$ is Hausdorff if and only if $\mathcal{F}$ is free (i.e. $\bigcap \mathcal{F} = \emptyset$),

(ii) $G(\mathcal{F})$ is Fréchet-Urysohn if and only if $\mathcal{F}$ is an FUF-filter,

(iii) $G(\mathcal{F})$ is first countable if and only if $\mathcal{F}$ is countably generated.

Theorem (folklore): If there exists a free FUF-filter on ω that is not countably generated, then there exists a countable Fréchet-Urysohn topological group that is not first countable.

Question (folklore): Is there, in ZFC only, a free FUF-filter on ω that is not countably generated?

Theorem (Nogura, Shakhmatov [1995]): All α_i properties ($i = 1, 3/2, 2, 3, 4$) coincide for locally compact topological groups.

Theorem (Nogura, Shakhmatov [1995]): The following conditions are equivalent:
(i) every compact group that is an α_1-space is metrizable,
(ii) every locally compact group that is an α_4-space is metrizable,
(iii) $b = \omega_1$.

Corollary (Nogura, Shakhmatov [1995]): Under CH, a locally compact group is metrizable if and only if it is α_4.

FUNCTION SPACES $C_p(X)$

For a topological space X let $C_p(X)$ be the set of all real-valued continuous functions on X equipped with the topology of pointwise convergence, i.e with the topology which the set $C_p(X)$ inherits from R^X, the latter space having the Tychonoff product topology.

For every space X, $C_p(X)$ is both a (locally convex) topological vector space and a topological ring.

Theorem (Scheepers [1998]): Let X be a topological space. Then $C_p(X)$ is α_2 if and only if $C_p(X)$ is α_4. Therefore, all three properties α_4, α_3 and α_2 coincide for spaces of the form $C_p(X)$.

Corollary (Scheepers [1998]): If $C_p(X)$ is Fréchet-Urysohn, then $C_p(X)$ is α_2.

Theorem (Scheepers [1998]): It is consistent with ZFC that there exists a subset of real numbers $X \subseteq R$ such that $C_p(X)$ is Fréchet-Urysohn (and thus α_2) but is not α_1.

Note that the existence of the above space is not only consistent with ZFC but also independent of ZFC by Dow's theorem.

Theorem (Scheepers [1998]): It is consistent with ZFC that there exists a subset of real numbers $X \subseteq R$ such that $C_p(X)$ is α_1 but is not Fréchet-Urysohn.

PRODUCTS OF GENERAL SPACES

Theorem (Nogura [1985]):

(i) For $i = 1, 2, 3$, if X and Y are α_i-spaces, then $X \times Y$ is also an α_i-space.

(ii) There exist compact Fréchet-Urysohn α_4-spaces X and Y such that $X \times Y$ is neither Fréchet-Urysohn nor α_4.

Theorem (Arangel'skii [1971]): If X is a Fréchet-Urysohn α_3-space and Y is a (countably) compact Fréchet-Urysohn space, then $X \times Y$ is Fréchet-Urysohn.

Theorem (Costantini, Simon [1999]): There exist two countable Fréchet-Urysohn α_4-spaces X and Y such that $X \times Y$ is α_4 but fails to be Fréchet-Urysohn.

Theorem (Simon [1999]): Under CH, there exist two countable Fréchet-Urysohn α_4-spaces X and Y such that $X \times Y$ is Fréchet-Urysohn but is not α_4.

Question: Is there such an example in ZFC?

PRODUCTS OF TOPOLOGICAL GROUPS

Theorem (Todorčević [1993]): There exist two (compactly generated) Fréchet-Urysohn groups G and H such that $t(G \times H) > \omega$ (in particular, $G \times H$ is not Fréchet-Urysohn). Moreover, every countable subset of G and H is metrizable, and so both G and H are α_1.

Theorem (Malyhin, Shakhmatov [1992]):

Add a single Cohen real to a model of $MA + \neg CH$. Then, in the generic extension, the exists a (hereditarily separable) Fréchet-Urysohn topological group G such that $t(G \times G) > \omega$ (in particular, $G \times G$ is not Fréchet-Urysohn). Moreover, G is an α_1-space.

Theorem (Shibakov [1999]): Under CH, there exists a *countable* Fréchet-Urysohn topological group G such that $G \times G$ is not Fréchet-Urysohn.

Question: Is there such an example in ZFC only?

Question: In ZFC only, does there exist two *countable* Fréchet-Urysohn topological groups G and H such that $G \times H$ is not Fréchet-Urysohn?

Question: In ZFC only, is there a Fréchet-Urysohn topological group G such that G is α_1 but $G \times G$ is not Fréchet-Urysohn?

PRODUCTS OF $C_p(X)$

Theorem (Tkačuk [1984]): If $C_p(X)$ is Fréchet-Urysohn, then even its countable power $C_p(X)^\omega$ is Fréchet-Urysohn.

Theorem (Todorčević [1993]): There exist two spaces X and Y such that both $C_p(X)$ and $C_p(Y)$ are Fréchet-Urysohn but

$$t(C_p(X) \times C_p(Y)) > \omega$$

(in particular, $C_p(X) \times C_p(Y)$ is not Fréchet-Urysohn). Moreover, every countable subset of $C_p(X)$ and $C_p(Y)$ is metrizable, and so both $C_p(X)$ and $C_p(Y)$ are α_1.