The purpose of this note is to introduce our recent paper [Ho-Y] about Coxeter groups and their boundaries. Let $V$ be a finite set and $m: V \times V \to \mathbb{N} \cup \{\infty\}$ a function satisfying the following conditions:

1. $m(v, w) = m(w, v)$ for all $v, w \in V$,
2. $m(v, v) = 1$ for all $v \in V$, and
3. $m(v, w) \geq 2$ for all $v \neq w \in V$.

A Coxeter group is a group $\Gamma$ having the presentation

$$\langle V \mid (vw)^{m(v, w)} = 1 \text{ for } v, w \in V \rangle,$$

where if $m(v, w) = \infty$, then the corresponding relation is omitted, and the pair $(\Gamma, V)$ is called a Coxeter system. If $m(v, w) = 2$ or $\infty$ for all $v \neq w \in V$, then $(\Gamma, V)$ is said to be right-angled. For a Coxeter system $(\Gamma, V)$ and a subset $W \subset V$, $\Gamma_W$ is defined as the subgroup of $\Gamma$ generated by $W$. The pair $(\Gamma_W, W)$ is also a Coxeter system. $\Gamma_W$ is called a parabolic subgroup.

For a Coxeter system $(\Gamma, V)$, the simplicial complex $K(\Gamma, V)$ is defined by the following conditions:

1. the vertex set of $K(\Gamma, V)$ is $V$, and
2. for $W = \{v_0, \ldots, v_k\} \subset V$, $\{v_0, \ldots, v_k\}$ spans a $k$-simplex of $K(\Gamma, V)$ if and only if $\Gamma_W$ is finite.

A simplicial complex $K$ is called a flag complex if any finite set of vertices, which are pairwise joined by edges, spans a simplex of $K$. For example, the barycentric subdivision of a simplicial complex is a flag complex.

For any finite flag complex $K$, there exists a right-angled Coxeter system $(\Gamma, V)$ with $K(\Gamma, V) = K$. Namely, let $V$ be the vertex set of $K$ and define $m: V \times V \to \mathbb{N} \cup \{\infty\}$ by

$$m(v, w) = \begin{cases} 1 & \text{if } v = w, \\ 2 & \text{if } \{v, w\} \text{ spans an edge in } K, \\ \infty & \text{otherwise.} \end{cases}$$

The associated right-angled Coxeter system $(\Gamma, V)$ satisfies $K(\Gamma, V) = K$. Conversely, if $(\Gamma, V)$ is a right-angled Coxeter system, then $K(\Gamma, V)$ is a finite flag complex ([D2, Corollary 9.4]).
For a group $\Gamma$ and a ring $R$ with identity, the cohomological dimension of $\Gamma$ over $R$ is defined as

$$\text{cd}_R \Gamma = \sup \{i \mid H^i(\Gamma; M) \neq 0 \text{ for some } R\Gamma\text{-module } M\}.$$ 

If $R = \mathbb{Z}$ then $\text{cd}_R \Gamma$ is simply called the cohomological dimension of $\Gamma$, and denoted $\text{cd} \Gamma$. It is obvious that $\text{cd}_R \Gamma \leq \text{cd} \Gamma$ for a ring $R$ with identity. It is known that $\text{cd} \Gamma = \infty$ if $\Gamma$ is not torsion-free ([Br, Corollary VIII.2.5]). A group $\Gamma$ is said to be virtually torsion-free if $\Gamma$ has a torsion-free subgroup of finite index. For a virtually torsion-free group $\Gamma$ the virtual cohomological dimension of $\Gamma$ over a ring $R$ is defined as $\text{cd}_R \Gamma'$, where $\Gamma'$ is a torsion-free subgroup of $\Gamma$ of finite index, and denoted $\text{vcd}_R \Gamma$. It is a well-defined invariant by Serre’s Theorem: if $G$ is a torsion-free group and $G'$ is a subgroup of finite index, then $\text{cd}_R G' = \text{cd}_R G$ ([Br, Theorem VIII.3.1]). If $R = \mathbb{Z}$ then $\text{vcd}_\mathbb{Z} \Gamma$ is simply called the virtual cohomological dimension of $\Gamma$, and denoted $\text{vcd} \Gamma$. It is known that every Coxeter group is virtually torsion-free and the virtual cohomological dimension of each Coxeter group is finite (cf. [D1, Corollary 5.2, Proposition 14.1]).

For a simplicial complex $K$ and a simplex $\sigma$ of $K$, the closed star $\text{St}(\sigma, K)$ of $\sigma$ in $K$ is the union of all simplexes of $K$ having $\sigma$ as a face, and the link $\text{Lk}(\sigma, K)$ of $\sigma$ in $K$ is the union of all simplexes of $K$ lying in $\text{St}(\sigma, K)$ that are disjoint from $\sigma$.

In [Dr2], Dranishnikov gave the following formula.

**Theorem 1 (Dranishnikov [Dr2]).** Let $(\Gamma, V)$ be a Coxeter system and $R$ a principal ideal domain. Then there exists the formula

$$\text{vcd}_R \Gamma = \text{lcd}_R CK = \max \{\text{lcd}_R K, \text{cd}_R K + 1\},$$

where $K = K(\Gamma, V)$ and $CK$ is the simplicial cone of $K$.

Here, for a finite simplicial complex $K$ and an abelian group $G$, the local cohomological dimension of $K$ over $G$ is defined as

$$\text{lcd}_G K = \max_{\sigma \in K} \{i \mid H^i(\text{St}(\sigma, K), \text{Lk}(\sigma, K); G) \neq 0\},$$

and the global cohomological dimension of $K$ over $G$ is

$$\text{cd}_G K = \max \{i \mid \tilde{H}^i(K; G) \neq 0\}.$$

When $\tilde{H}^i(K; G) = 0$ for each $i$, then we consider $\text{cd}_G K = -1$. We note that $H^i(\text{St}(\sigma, K), \text{Lk}(\sigma, K); G)$ is isomorphic to $\tilde{H}^{i-1}(\text{Lk}(\sigma, K); G)$. Hence, we have

$$\text{lcd}_G K = \max_{\sigma \in K} \{\text{cd}_G \text{Lk}(\sigma, K) + 1\}.$$ 

**Remark.** We recall Dranishnikov’s remark in [Dr3]. The definition of the local cohomological dimension in [Dr2] is given by the terminology of the normal star and link. Since $\text{Lk}(\sigma, K)$ is homeomorphic to the normal link of $\sigma$ in $K$, their definitions are equivalent by the formula above.

Dranishnikov also proved the following theorem as an application of Theorem 1.
Theorem 2 (Dranishnikov [Dr2]). A Coxeter group $\Gamma$ has the following properties:

(a) $\text{vcd}_Q \Gamma \leq \text{vcd}_R \Gamma$ for any principal ideal domain $R$.
(b) $\text{vcd}_{\mathbb{Z}_p} \Gamma = \text{vcd}_Q \Gamma$ for almost all primes $p$.
(c) There exists a prime $p$ such that $\text{vcd}_{\mathbb{Z}_p} \Gamma = \text{vcd} \Gamma$.
(d) $\text{vcd} \Gamma \times \Gamma = 2 \text{vcd} \Gamma$.

We extend this theorem to one over principal ideal domain coefficients.

Theorem A. Let $\Gamma$ be a Coxeter group and $R$ a principal ideal domain. Then $\Gamma$ has the following properties:

(a) $\text{vcd}_Q \Gamma \leq \text{vcd}_{R/I} \Gamma \leq \text{vcd}_R \Gamma \leq \text{vcd} \Gamma$ for any prime ideal $I$ in $R$.
(b) $\text{vcd}_{R/I} \Gamma = \text{vcd}_Q \Gamma$ for almost all prime ideals $I$ in $R$, if $R$ is not a field.
(c) There exists a non-trivial prime ideal $I$ in $R$ such that $\text{vcd}_{R/I} \Gamma = \text{vcd}_R \Gamma$, if $R$ is not a field.
(d) $\text{vcd}_R \Gamma \times \Gamma = 2 \text{vcd}_R \Gamma$.

Let $(\Gamma, V)$ be a Coxeter system and $K = K(\Gamma, V)$. Consider the product space $\Gamma \times |CK|$ of $\Gamma$ with the discrete topology and the underlying space $|CK|$ of the cone of $K$. Define an equivalence relation $\sim$ on the space as follows: for $(\gamma_1, x_1), (\gamma_2, x_2) \in \Gamma \times |CK|$, $(\gamma_1, x_1) \sim (\gamma_2, x_2)$ if and only if $x_1 = x_2$ and $\gamma_1^{-1}\gamma_2 \in \Gamma_{V(x_1)}$, where $V(x) = \{v \in V | x \in \text{St}(v, \beta^1K)\}$. Here we consider that $|K|$ is naturally embedded in $|CK|$ as the base of the cone and $\beta^1K$ denotes the barycentric subdivision of $K$. The natural left $\Gamma$-action on $\Gamma \times |CK|$ is compatible with the equivalence relation; hence, it passes to a left action on the quotient space $\Gamma \times |CK|/\sim$. Denote this quotient space by $A(\Gamma, V)$. The space $A(\Gamma, V)$ is contractible and $\Gamma$ acts cocompactly and properly discontinuously on the space ([D1, Theorem 13.5]).

We can also give the space $A(\Gamma, V)$ a structure of a piecewise Euclidean cell complex with the vertex set $\Gamma \times \{v_0\}$ ([D2, §9]). $\Sigma(\Gamma, V)$ denotes this piecewise Euclidean cell complex. Refer to [D2, Definition 2.2] for the definition of a piecewise Euclidean cell complex. In particular, if $(\Gamma, V)$ is right-angled, then each cell of $\Sigma(\Gamma, V)$ is a cube, hence, $\Sigma(\Gamma, V)$ is a cubical complex. More precisely, for a right-angled Coxeter system $(\Gamma, V)$, we can define the cubical complex $\Sigma(\Gamma, V)$ by the following conditions:

1. the vertex set of $\Sigma(\Gamma, V)$ is $\Gamma$,
2. for $\gamma, \gamma' \in \Gamma$, $\{\gamma, \gamma'\}$ spans an edge in $\Sigma(\Gamma, V)$ if and only if the length $l_V(\gamma^{-1}\gamma') = 1$, and
3. for $\gamma \in \Gamma$ and $v_0, \ldots, v_k \in V$, the edges $|\gamma, \gamma v_0|, \ldots, |\gamma, \gamma v_k|$ form a $(k+1)$-cube in $\Sigma(\Gamma, V)$ if and only if $\{v_0, \ldots, v_k\}$ spans a $k$-simplex in $K(\Gamma, V)$.

We note the 1-skeleton of this cell complex is isomorphic to the Cayley graph of $\Gamma$ with respect to $V$. For $\gamma \in \Gamma$ and a $k$-simplex $\sigma = [v_0, \ldots, v_k]$ of $K(\Gamma, V)$, let $C_{\gamma, \sigma}$ be the $(k+1)$-cube in $\Sigma(\Gamma, V)$ formed by $|\gamma, \gamma v_0|, \ldots, |\gamma, \gamma v_k|$. Then the vertex set of $C_{\gamma, \sigma}$ is $\gamma \Gamma \{v_0, \ldots, v_k\}$. We note that

\[ \gamma \Gamma \{v_0, \ldots, v_k\} = \{\gamma v_0^{e_0} \cdots v_k^{e_k} | e_i \in \{0, 1\}, i = 0, \ldots, k\}. \]

For every Coxeter system $(\Gamma, V)$, $\Sigma(\Gamma, V)$ is a CAT(0) geodesic space by a piecewise Euclidean metric (cf. [D2, Theorem 7.8]). We define the boundary $\partial \Gamma$ as the set of
geodesic rays in \( \Sigma(\Gamma, V) \) emanating from the unit element \( e \in \Gamma \subset \Sigma(\Gamma, V) \) with the topology of the uniform convergence on compact sets, i.e., \( \partial \Gamma \) is the visual sphere of \( \Sigma(\Gamma, V) \) at the point \( e \in \Sigma(\Gamma, V) \). In general, for all points \( x, y \) in a CAT(0) space \( X \), the visual spheres of \( X \) at points \( x \) and \( y \) are homeomorphic (cf. [Dr1, Assertion 1]). This boundary is known to be a finite-dimensional compactum (i.e., metrizable compact space). Details of the boundaries of CAT(0) spaces can be found in [D2] and [D-J].

It is still unknown whether the following conjecture holds.

**Rigidity Conjecture (Dranishnikov [Dr4]).** Isomorphic Coxeter groups have homeomorphic boundaries.

We note that there exists a Coxeter group \( \Gamma \) with different Coxeter systems \((\Gamma, V_1)\) and \((\Gamma, V_2)\).

Let \( X \) be a compact metric space and \( G \) an abelian group. The **cohomological dimension of \( X \) over \( G \)** is defined as

\[
c-\dim_G X = \sup \{ i \mid \check{H}^i(X, A; G) \neq 0 \text{ for some closed set } A \subset X \},
\]

where \( \check{H}^i(X, A; G) \) is the Čech cohomology of \((X, A)\) over \( G \).

In [B-M], Bestvina-Mess proved the following theorem for hyperbolic groups. An analogous theorem for Coxeter groups is proved by the same argument (cf. [Dr1]).

**Theorem 3 (Bestvina-Mess [B-M]).** Let \( \Gamma \) be a Coxeter group and \( R \) a ring with identity. Then there exists the formula

\[
c-\dim_R \partial \Gamma = \text{vcd}_R \Gamma - 1.
\]

We have a dimension theoretic theorem in the study of Coxeter groups.

**Theorem B.** Let \((\Gamma, V)\) be a right-angled Coxeter system with \( \text{vcd}_R \Gamma = n \), where \( R \) is a principal ideal domain. Then there exists a sequence \( W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset V \) such that \( \text{vcd}_R \Gamma_{W_i} = i \) for \( i = 0, \ldots, n-1 \). In particular, we can obtain a sequence of simplexes \( \tau_0 \succ \tau_1 \succ \cdots \succ \tau_{n-1} \) such that \( W_i \) is the vertex set of \( \text{Lk}(\tau_i, K(\Gamma, V)) \) and \( K(\Gamma_{W_i}, W_i) = \text{Lk}(\tau_i, K(\Gamma, V)) \).

We note that Theorem B is not always true for general Coxeter groups.

**Example.** We consider the Coxeter system \((\Gamma, V)\) defined by \( V = \{v_1, v_2, v_3\} \) and

\[
m(v_i, v_j) = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}
\]

Then \( \Gamma \) is not right-angled, and \( K(\Gamma, V) \) is not a flag complex. Indeed, \( \Gamma_{\{v_i, v_j\}} \) is finite for each \( i, j \in \{1, 2, 3\} \), but \( \Gamma \) is infinite (cf. [Bo, p.98, Proposition 8]). Since \( \text{cd} K(\Gamma, V) = 1 \) and \( \text{lcd} K(\Gamma, V) = 1 \), we have \( \text{vcd} \Gamma = 2 \) by Theorem 1. For any proper subset \( W \subset V \), \( \text{vcd} \Gamma_{W} = 0 \), because \( \Gamma_{W} \) is a finite group. Hence there does not exist a subset \( W \subset V \) such that \( \text{vcd} \Gamma_{W} = 1 \). □

By Theorem 3, we can obtain the following corollary.
Corollary B'. For a right-angled Coxeter system $(\Gamma, V)$ with $c\dim R \partial \Gamma = n$, where $R$ is a principal ideal domain, there exists a sequence $\partial \Gamma_{W_0} \subset \partial \Gamma_{W_1} \subset \cdots \subset \partial \Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups of $(\Gamma, V)$ such that $c\dim R \partial \Gamma_{W_i} = i$ for each $i = 0, 1, \ldots, n - 1$.

In general, for a finite dimensional compactum $X$, the equality $c\dim Z X = \dim X$ holds ([K, §2, Remark 4]). Since the boundaries of Coxeter groups are always finite dimensional, we obtain the following corollary.

Corollary B''. For a right-angled Coxeter system $(\Gamma, V)$ with $\dim \partial \Gamma = n$, there exists a sequence $\partial \Gamma_{W_0} \subset \partial \Gamma_{W_1} \subset \cdots \subset \partial \Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups of $(\Gamma, V)$ such that $\dim \partial \Gamma_{W_i} = i$ for each $i = 0, 1, \ldots, n - 1$.

Finally, we see a relation between a subgroup of a Coxeter group which is of finite index and their boundaries.

If $X$ and $Y$ are topological spaces, let us define $X \ast Y$ to be the quotient space of $X \times Y \times [0, 1]$ obtained by identifying each set $x \times Y \times 0$ to a point and each set $X \times y \times 1$ to a point.

Theorem C. Let $(\Gamma, V)$ be a right-angled Coxeter system and $W$ a subset of $V$. Then the following conditions are equivalent:

1. The parabolic subgroup $\Gamma_W \subset \Gamma$ is of finite index.
2. $\{v, v'\}$ spans an edge of $K(\Gamma, V)$ for any $v \in V \setminus W$ and $v' \in V$.
3. $\Gamma = \Gamma_W \times \Gamma_{V \setminus W}$ and $\Gamma_{V \setminus W} \approx \mathbb{Z}_2^{V \setminus W}$.
4. $\partial \Gamma = \partial \Gamma_W$.

References


**Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan**

*E-mail address:* thosaka@math.tsukuba.ac.jp

**Department of Mathematics, Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue, 690-8504, Japan**

*E-mail address:* yokoi@math.shimane-u.ac.jp