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COXETER GROUPS の境界と VIRTUAL COHOMOLOGICAL DIMENSION について

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The purpose of this note is to introduce our recent paper [Ho-Y] about Coxeter groups and their boundaries. Let V be a finite set and $m\colon V \times V \to \mathbb{N} \cup \{\infty\}$ a function satisfying the following conditions:

(1) $m(v, w) = m(w, v)$ for all $v, w \in V$,
(2) $m(v, v) = 1$ for all $v \in V$, and
(3) $m(v, w) \geq 2$ for all $v \neq w \in V$.

A *Coxeter group* is a group Γ having the presentation

$$\langle V \mid (vw)^{m(v,w)} = 1 \text{ for } v, w \in V \rangle,$$

where if $m(v, w) = \infty$, then the corresponding relation is omitted, and the pair (Γ, V) is called a *Coxeter system.* If $m(v, w) = 2$ or ∞ for all $v \neq w \in V$, then (Γ, V) is said to be *right-angled.* For a Coxeter system (Γ, V) and a subset $W \subset V$, Γ_W is defined as the subgroup of Γ generated by W. The pair (Γ_W, W) is also a Coxeter system. Γ_W is called a *parabolic subgroup.*

For a Coxeter system (Γ, V), the simplicial complex $K(\Gamma, V)$ is defined by the following conditions:

(1) the vertex set of $K(\Gamma, V)$ is V, and
(2) for $W = \{v_0, \ldots, v_k\} \subset V$, $\{v_0, \ldots, v_k\}$ spans a k-simplex of $K(\Gamma, V)$ if and only if Γ_W is finite.

A simplicial complex K is called a *flag complex* if any finite set of vertices, which are pairwise joined by edges, spans a simplex of K. For example, the barycentric subdivision of a simplicial complex is a flag complex.

For any finite flag complex K, there exists a right-angled Coxeter system (Γ, V) with $K(\Gamma, V) = K$. Namely, let V be the vertex set of K and define $m\colon V \times V \to \mathbb{N} \cup \{\infty\}$ by

$$m(v, w) = \begin{cases} 1 & \text{if } v = w, \\ 2 & \text{if } \{v, w\} \text{ spans an edge in } K, \\ \infty & \text{otherwise.} \end{cases}$$

The associated right-angled Coxeter system (Γ, V) satisfies $K(\Gamma, V) = K$. Conversely, if (Γ, V) is a right-angled Coxeter system, then $K(\Gamma, V)$ is a finite flag complex ([D2, Corollary 9.4]).

For a group Γ and a ring R with identity, the *cohomological dimension of* Γ *over* R is defined as

$$\mathrm{cd}_R \Gamma = \sup\{i \mid H^i(\Gamma; M) \neq 0 \text{ for some } R\Gamma\text{-module } M\}.$$

If $R = \mathbb{Z}$ then $\mathrm{cd}_{\mathbb{Z}} \Gamma$ is simply called the cohomological dimension of Γ, and denoted $\mathrm{cd}\,\Gamma$. It is obvious that $\mathrm{cd}_R \Gamma \leq \mathrm{cd}\,\Gamma$ for a ring R with identity. It is known that $\mathrm{cd}\,\Gamma = \infty$ if Γ is not torsion-free ([Br, Corollary VIII.2.5]). A group Γ is said to be *virtually torsion-free* if Γ has a torsion-free subgroup of finite index. For a virtually torsion-free group Γ the *virtual cohomological dimension of* Γ *over a ring* R is defined as $\mathrm{cd}_R \Gamma'$, where Γ' is a torsion-free subgroup of Γ of finite index, and denoted $\mathrm{vcd}_R \Gamma$. It is a well-defined invariant by Serre's Theorem: if G is a torsion-free group and G' is a subgroup of finite index, then $\mathrm{cd}_R G' = \mathrm{cd}_R G$ ([Br, Theorem VIII.3.1]). If $R = \mathbb{Z}$ then $\mathrm{vcd}_{\mathbb{Z}} \Gamma$ is simply called the virtual cohomological dimension of Γ, and denoted $\mathrm{vcd}\,\Gamma$. It is known that every Coxeter group is virtually torsion-free and the virtual cohomological dimension of each Coxeter group is finite (cf. [D1, Corollary 5.2, Proposition 14.1]).

For a simplicial complex K and a simplex σ of K, the *closed star* $\mathrm{St}(\sigma, K)$ of σ in K is the union of all simplexes of K having σ *as a face*, and the *link* $\mathrm{Lk}(\sigma, K)$ of σ in K is the union of all simplexes of K lying in $\mathrm{St}(\sigma, K)$ that are disjoint from σ.

In [Dr2], Dranishnikov gave the following formula.

Theorem 1 (Dranishnikov [Dr2]**).** *Let* (Γ, V) *be a Coxeter system and* R *a principal ideal domain. Then there exists the formula*

$$\mathrm{vcd}_R \Gamma = \mathrm{lcd}_R CK = \max\{\mathrm{lcd}_R K,\ \mathrm{cd}_R K + 1\},$$

where $K = K(\Gamma, V)$ *and* CK *is the simplicial cone of* K.

Here, for a finite simplicial complex K and an abelian group G, the *local cohomological dimension of* K *over* G is defined as

$$\mathrm{lcd}_G K = \max_{\sigma \in K}\{i \mid H^i(\mathrm{St}(\sigma, K), \mathrm{Lk}(\sigma, K); G) \neq 0\},$$

and the *global cohomological dimension of* K *over* G is

$$\mathrm{cd}_G K = \max\{i \mid \tilde{H}^i(K; G) \neq 0\}.$$

When $\tilde{H}^i(K; G) = 0$ for each i, then we consider $\mathrm{cd}_G K = -1$. We note that $H^i(\mathrm{St}(\sigma, K), \mathrm{Lk}(\sigma, K); G)$ is isomorphic to $\tilde{H}^{i-1}(\mathrm{Lk}(\sigma, K); G)$. Hence, we have

$$\mathrm{lcd}_G K = \max_{\sigma \in K}\{\mathrm{cd}_G \mathrm{Lk}(\sigma, K) + 1\}.$$

Remark. We recall Dranishnikov's remark in [Dr3]. The definition of the local cohomological dimension in [Dr2] is given by the terminology of the normal star and link. Since $\mathrm{Lk}(\sigma, K)$ is homeomorphic to the normal link of σ in K, their definitions are equivalent by the formula above.

Dranishnikov also proved the following theorem as an application of Theorem 1.

Theorem 2 (Dranishnikov [Dr2]). *A Coxeter group* Γ *has the following properties:*

(a) $\operatorname{vcd}_{\mathbb{Q}}\Gamma \leq \operatorname{vcd}_R\Gamma$ *for any principal ideal domain* R.
(b) $\operatorname{vcd}_{\mathbb{Z}_p}\Gamma = \operatorname{vcd}_{\mathbb{Q}}\Gamma$ *for almost all primes* p.
(c) *There exists a prime* p *such that* $\operatorname{vcd}_{\mathbb{Z}_p}\Gamma = \operatorname{vcd}\Gamma$.
(d) $\operatorname{vcd}\Gamma\times\Gamma = 2\operatorname{vcd}\Gamma$.

We extend this theorem to one over principal ideal domain coefficients.

Theorem A. *Let* Γ *be a Coxeter group and* R *a principal ideal domain. Then* Γ *has the following properties:*

(a) $\operatorname{vcd}_{\mathbb{Q}}\Gamma \leq \operatorname{vcd}_{R/I}\Gamma \leq \operatorname{vcd}_R\Gamma \leq \operatorname{vcd}\Gamma$ *for any prime ideal* I *in* R.
(b) $\operatorname{vcd}_{R/I}\Gamma = \operatorname{vcd}_{\mathbb{Q}}\Gamma$ *for almost all prime ideals* I *in* R, *if* R *is not a field.*
(c) *There exists a non-trivial prime ideal* I *in* R *such that* $\operatorname{vcd}_{R/I}\Gamma = \operatorname{vcd}_R\Gamma$, *if* R *is not a field.*
(d) $\operatorname{vcd}_R\Gamma\times\Gamma = 2\operatorname{vcd}_R\Gamma$.

Let (Γ, V) be a Coxeter system and $K = K(\Gamma, V)$. Consider the product space $\Gamma\times|CK|$ of Γ with the discrete topology and the underlying space $|CK|$ of the cone of K. Define an equivalence relation $\sim$ on the space as follows: for $(\gamma_1, x_1), (\gamma_2, x_2) \in \Gamma\times|CK|$, $(\gamma_1, x_1)\sim(\gamma_2, x_2)$ if and only if $x_1 = x_2$ and $\gamma_1^{-1}\gamma_2 \in \Gamma_{V(x_1)}$, where $V(x) = \{v\in V | x \in \operatorname{St}(v, \beta^1K)\}$. Here we consider that $|K|$ is naturally embedded in $|CK|$ as the base of the cone and β^1K denotes the barycentric subdivision of K. The natural left Γ-action on $\Gamma\times|CK|$ is compatible with the equivalence relation; hence, it passes to a left action on the quotient space $\Gamma\times|CK|/\sim$. Denote this quotient space by $A(\Gamma, V)$. The space $A(\Gamma, V)$ is contractible and Γ acts cocompactly and properly discontinuously on the space ([D1, Theorem 13.5]).

We can also give the space $A(\Gamma, V)$ a structure of a piecewise Euclidean cell complex with the vertex set $\Gamma\times\{v_0\}$ ([D2, §9]). $\Sigma(\Gamma, V)$ denotes this piecewise Euclidean cell complex. Refer to [D2, Definition 2.2] for the definition of a piecewise Euclidean cell complex. In particular, if (Γ, V) is right-angled, then each cell of $\Sigma(\Gamma, V)$ is a cube, hence, $\Sigma(\Gamma, V)$ is a cubical complex. More precisely, for a *right-angled* Coxeter system (Γ, V), we can define the cubical complex $\Sigma(\Gamma, V)$ by the following conditions:

(1) the vertex set of $\Sigma(\Gamma, V)$ is Γ,
(2) for $\gamma, \gamma' \in \Gamma$, $\{\gamma, \gamma'\}$ spans an edge in $\Sigma(\Gamma, V)$ if and only if the length $l_V(\gamma^{-1}\gamma') = 1$, and
(3) for $\gamma\in\Gamma$ and $v_0, \dots, v_k \in V$, the edges $|\gamma, \gamma v_0|, \dots, |\gamma, \gamma v_k|$ form a $(k+1)$-cube in $\Sigma(\Gamma, V)$ if and only if $\{v_0, \dots, v_k\}$ spans a k-simplex in $K(\Gamma, V)$.

We note the 1-skeleton of this cell complex is isomorphic to the Cayley graph of Γ with respect to V. For $\gamma\in\Gamma$ and a k-simplex $\sigma = |v_0, \dots, v_k|$ of $K(\Gamma, V)$, let $C_{\gamma,\sigma}$ be the $(k+1)$-cube in $\Sigma(\Gamma, V)$ formed by $|\gamma, \gamma v_0|, \dots, |\gamma, \gamma v_k|$. Then the vertex set of $C_{\gamma,\sigma}$ is $\gamma\Gamma_{\{v_0,\dots,v_k\}}$. We note that

$$\gamma\Gamma_{\{v_0,\dots,v_k\}} = \{\gamma v_0^{\epsilon_0}\cdots v_k^{\epsilon_k} \mid \epsilon_i \in \{0,1\},\ i = 0, \dots, k\}.$$

For every Coxeter system (Γ, V), $\Sigma(\Gamma, V)$ is a CAT(0) geodesic space by a piecewise Euclidean metric (cf. [D2, Theorem 7.8]). We define the boundary $\partial\Gamma$ as the set of geodesic rays in $\Sigma(\Gamma, V)$ emanating from the unit element $e \in \Gamma \subset \Sigma(\Gamma, V)$ with the topology of the uniform convergence on compact sets, i.e., $\partial\Gamma$ is the visual sphere of $\Sigma(\Gamma, V)$ at the point $e \in \Sigma(\Gamma, V)$. In general, for all points x, y in a CAT(0) space X, the visual spheres of X at points x and y are homeomorphic (cf. [Dr1, Assertion 1]). This boundary is known to be a finite-dimensional compactum (i.e., metrizable compact space). Details of the boundaries of CAT(0) spaces can be found in [D2] and [D-J].

It is still unknown whether the following conjecture holds.

Rigidity Conjecture (Dranishnikov [Dr4]**).** *Isomorphic Coxeter groups have homeomorphic boundaries.*

We note that there exists a Coxeter group Γ with different Coxeter systems (Γ, V_1) and (Γ, V_2).

Let X be a compact metric space and G an abelian group. The *cohomological dimension of X over G* is defined as

$$\text{c-dim}_G X = \sup\{i \mid \check{H}^i(X, A; G) \neq 0 \text{ for some closed set } A \subset X\},$$

where $\check{H}^i(X, A; G)$ is the Čech cohomology of (X, A) over G.

In [B-M], Bestvina-Mess proved the following theorem for hyperbolic groups. An analogous theorem for Coxeter groups is proved by the same argument (cf. [Dr1]).

Theorem 3 (Bestvina-Mess [B-M]**).** *Let Γ be a Coxeter group and R a ring with identity. Then there exists the formula*

$$\text{c-dim}_R \partial\Gamma = \text{vcd}_R \Gamma - 1.$$

We have a dimension theoritic theorem in the study of Coxeter groups.

Theorem B. *Let (Γ, V) be a right-angled Coxeter system with $\text{vcd}_R \Gamma = n$, where R is a principal ideal domain. Then there exists a sequence $W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset V$ such that $\text{vcd}_R \Gamma_{W_i} = i$ for $i = 0, \ldots, n-1$. In particular, we can obtain a sequence of simplexes $\tau_0 \succ \tau_1 \succ \cdots \succ \tau_{n-1}$ such that W_i is the vertex set of $\text{Lk}(\tau_i, K(\Gamma, V))$ and $K(\Gamma_{W_i}, W_i) = \text{Lk}(\tau_i, K(\Gamma, V))$.*

We note that Theorem B is not always true for general Coxeter groups.

Example. We consider the Coxeter system (Γ, V) defined by $V = \{v_1, v_2, v_3\}$ and

$$m(v_i, v_j) = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}$$

Then Γ is not right-angled, and $K(\Gamma, V)$ is not a flag complex. Indeed, $\Gamma_{\{v_i, v_j\}}$ is finite for each $i, j \in \{1, 2, 3\}$, but Γ is infinite (cf. [Bo, p.98, Proposition 8]). Since $\text{cd}\, K(\Gamma, V) = 1$ and $\text{lcd}\, K(\Gamma, V) = 1$, we have $\text{vcd}\, \Gamma = 2$ by Theorem 1. For any proper subset $W \subset V$, $\text{vcd}\, \Gamma_W = 0$, because Γ_W is a finite group. Hence there does not exist a subset $W \subset V$ such that $\text{vcd}\, \Gamma_W = 1$. □

By Theorem 3, we can obtain the following corollary.

Corollary B'. *For a right-angled Coxeter system* (Γ, V) *with* $\text{c-dim}_R\, \partial\Gamma = n$, *where* R *is a principal ideal domain, there exists a sequence* $\partial\Gamma_{W_0} \subset \partial\Gamma_{W_1} \subset \cdots \subset \partial\Gamma_{W_{n-1}}$ *of the boundaries of parabolic subgroups of* (Γ, V) *such that* $\text{c-dim}_R\, \partial\Gamma_{W_i} = i$ *for each* $i = 0, 1, \ldots, n-1$.

In general, for a finite dimensional compactum X, the equality $\text{c-dim}_{\mathbb{Z}}\, X = \dim X$ holds ([K, §2, Remark 4]). Since the boundaries of Coxeter groups are always finite dimensional, we obtain the following corollary.

Corollary B". *For a right-angled Coxeter system* (Γ, V) *with* $\dim \partial\Gamma = n$, *there exists a sequence* $\partial\Gamma_{W_0} \subset \partial\Gamma_{W_1} \subset \cdots \subset \partial\Gamma_{W_{n-1}}$ *of the boundaries of parabolic subgroups of* (Γ, V) *such that* $\dim \partial\Gamma_{W_i} = i$ *for each* $i = 0, 1, \ldots, n-1$.

Finally, we see a relation between a subgroup of a Coxeter group which is of finte index and their boundaries.

If X and Y are topological spaces, let us define $X * Y$ to be the quotient space of $X \times Y \times [0,1]$ obtained by identifying each set $x \times Y \times 0$ to a point and each set $X \times y \times 1$ to a point.

Theorem C. *Let* (Γ, V) *be a right-angled Coxeter system and* W *a subset of* V. *Then the following conditions are equivalent:*

(1) *The parabolic subgroup* $\Gamma_W \subset \Gamma$ *is of finite index.*
(2) $\{v, v'\}$ *spans an edge of* $K(\Gamma, V)$ *for any* $v \in V \setminus W$ *and* $v' \in V$.
(3) $\Gamma = \Gamma_W \times \Gamma_{V \setminus W}$ *and* $\Gamma_{V\setminus W} \approx \mathbb{Z}_2^{|V \setminus W|}$.
(4) $\partial\Gamma = \partial\Gamma_W$.

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