ON \( \mathcal{L} \)-STARCOMPACT SPACES

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ABSTRACT. A space \( X \) is \( \mathcal{L} \)-starcompact if for every open cover \( \mathcal{U} \) of \( X \), there exists a Lindelöf subset \( L \) of \( X \) such that \( St(L, \mathcal{U}) = X \). We clarify the relations between \( \mathcal{L} \)-starcompact spaces and other related spaces and investigate topological properties of \( \mathcal{L} \)-starcompact spaces. A question of Hiremath [3] is answered.

1. INTRODUCTION

By a space, we mean a topological space. Let us recall [6] that a space \( X \) is star-Lindelöf if for every open cover \( \mathcal{U} \) of \( X \), there exists a countable subset \( B \) of \( X \) such that \( St(B, \mathcal{U}) = X \), where \( St(B, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap B \neq \emptyset \} \). It is clear that every separable space is star-Lindelöf. Also, it is not difficult to see that every \( T_1 \)-space with countable extent is star-Lindelöf. Therefore, every countably compact \( T_1 \)-space is star-Lindelöf as well as every Lindelöf space. As generalities of star-Lindelöfness, the following classes of spaces are given (see [6]):

Definition 1.1. A space \( X \) is \( \mathcal{L} \)-starcompact if for every open cover \( \mathcal{U} \) of \( X \), there exists a Lindelöf subset \( L \) of \( X \) such that \( St(L, \mathcal{U}) = X \).

Definition 1.2. A space \( X \) is \( 1\frac{1}{2} \)-starLindelöf if for every open cover \( \mathcal{U} \) of \( X \), there exists a countable subset \( \mathcal{V} \) of \( \mathcal{U} \) such that \( St(\bigcup \mathcal{V}, \mathcal{U}) = X \).

In [3], \( \mathcal{L} \)-starcompactness is called sLc property, and in [1], a \( 1\frac{1}{2} \)-starLinde-löf space is called a star-Lindelöf space and a star-Lindelöf space is called a stronglystar-Lindelöf space.

From the above definitions, we have the following diagram:

\[
\text{star-Lindelöf} \longrightarrow \mathcal{L}\text{-starcompact} \longrightarrow 1\frac{1}{2}\text{-starLindelöf}.
\]

In the following section, we give examples showing that the converses in the above Diagram do not hold.

The cardinality of a set \( A \) is denoted by \( |A| \). Let \( \omega \) be the first infinite cardinal, \( \omega_1 \) the first uncountable cardinal and \( c \) the cardinality of the set of all real numbers. As usual,

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a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals $\alpha, \beta$ with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [2].

2. $\mathcal{L}$-STARCOMPACT SPACES AND RELATED SPACES

In [3], Hiremath asked if the product of two countably compact spaces is $\mathcal{L}$-starcompact. However, it is not difficult to see that the following well-known example gives a negative answer to the above question (see [8, Theorem 2.7]), we shall give the proof roughly for the sake of completeness. The symbol $\beta(X)$ means the Čech-Stone compactification of a Tychonoff space $X$.

**Example 2.1.** There exist two countably compact spaces $X$ and $Y$ such that $X \times Y$ is not $\mathcal{L}$-starcompact.

**Proof.** Let $D$ be a discrete space of the cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega} E_\alpha$, $Y = \bigcup_{\alpha<\omega} F_\alpha$, where $E_\alpha$ and $F_\alpha$ are the subsets of $\beta(D)$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

1. $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
2. $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
3. every infinite subset of $E_\alpha$ (resp. $F_\alpha$) has an accumulation point in $E_{\alpha+1}$ (resp. $F_{\alpha+1}$).

Those sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta(D)$ has the cardinality $2^\mathfrak{c}$ (see [5]). Then, $X \times Y$ is not $\mathcal{L}$-starcompact, because the diagonal $\{(d,d) : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$ and $\mathcal{L}$-starcompactness is preserved by open and closed subsets. $\square$

We end this section by giving examples which show the converses in the above diagram in §1 do not hold.

**Example 2.2.** There exists an $\mathcal{L}$-starcompact Tychonoff space which is not star-Lindelöf.

**Proof.** Let $D$ be a discrete space of the cardinality $\mathfrak{c}$. Define

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

Then, $X$ is $\mathcal{L}$-starcompact, since $\beta(D) \times \omega$ is a Lindelöf dense subset of $X$.

Next, we shall show that $X$ is not star-Lindelöf. Let us consider the open cover

$$U = \{\{d\} \times (\omega + 1) : d \in D\} \cup \{\beta(D) \times \{n\} : n \in \omega\}$$

of $X$. Let $B$ be a countable subset of $X$. Then, there exists a $d^* \in D$ such that $B \cap (\{d^*\} \times (\omega + 1)) = \emptyset$. This means that $U = \{d^*\} \times (\omega + 1)$ is the only element of $U$ containing the point $\langle d^*, \omega \rangle$, and hence $\langle d^*, \omega \rangle \notin St(B, \mathcal{V})$. $\square$
Example 2.3. There exists a $1\frac{1}{2}$-starLindelöf Tychonoff space which is not $\mathcal{L}$-starcompact.

Proof. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of $\omega$ with $|\mathcal{R}| = c$. Define

$$X = \mathcal{R} \cup (c \times \omega).$$

We topologize $X$ as follows: $c \times \omega$ has the usual product topology and is an open subspace of $X$. On the other hand a basic neighbourhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < c\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < c$ and a finite subset $K$ of $\omega$. To show that $X$ is $1\frac{1}{2}$-starLindelöf, let $\mathcal{U}$ be an open cover of $X$. Let

$$M = \{n \in \omega : (\exists U \in \mathcal{U})(\exists \beta < c)((\beta, c) \times \{n\} \subseteq U)\}.$$ 

For each $n \in M$, there exist $U_n \in \mathcal{U}$ and $\beta_n < c$ such that $(\beta_n, c) \times \{n\} \subseteq U_n$. If we put $\mathcal{V}' = \{U_n : n \in M\}$, then

$$\mathcal{R} \subseteq St(\cup \mathcal{V}', \mathcal{U}).$$

On the other hand, for each $n < \omega$, since $c \times \{n\}$ is countably compact, we can find a finite subfamily $\mathcal{V}_n$ of $\mathcal{U}$ such that

$$c \times \{n\} \subseteq St(\cup \mathcal{V}_n, \mathcal{U}).$$

Consequently, if we put $\mathcal{V} = \mathcal{V}' \cup \{\mathcal{V}_n : n < \omega\}$, then, $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$ and $X = St(\cup \mathcal{V}, \mathcal{U})$. Hence, $X$ is $1\frac{1}{2}$-starLindelöf.

Next, we shall show that $X$ is not $\mathcal{L}$-starcompact. Since $|\mathcal{R}| = c$, enumerate $\mathcal{R}$ as $\{r_{\alpha} : \alpha < c\}$. For each $\alpha < c$, let $U_\alpha = \{r_{\alpha}\} \cup ((\alpha, c) \times r_{\alpha})$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < c\} \cup \{c \times \omega\}$$

of $X$ and let $L$ be a Lindelöf subset of $X$. Since $\mathcal{R}$ is discrete closed in $X$, $L \cap \mathcal{R}$ is countable. Hence, there exists $\beta' < c$ such that

$$L \cap \{r_\alpha : \alpha > \beta'\} = \emptyset. \quad (1)$$

On the other hand, $L \cap (c \times \{n\})$ is bounded in $c \times \{n\}$ for each $n < \omega$. Thus, there exists $\beta_n < c$ such that $\beta_n > \sup\{\alpha < c : (\alpha, n) \in L\}$. Pick $\beta'' < c$ such that $\beta'' > \beta_n$ for each $n \in \omega$. Then,

$$((\beta'', c) \times \omega) \cap L = \emptyset. \quad (2)$$

Choose $\gamma < c$ such that $\gamma > \max\{\beta', \beta''\}$. Then, $U_\gamma$ is the only element of $\mathcal{U}$ containing the point $r_\gamma$ and $U_\gamma \cap L = \emptyset$ by (1) and (2). It follows that $r_\gamma \notin St(L, \mathcal{U})$, and which shows that $X$ is not $\mathcal{L}$-starcompact. \hfill $\square$

Remark 1. The author does not know if each arrow in the above diagram can be reversed in the realm of normal spaces.
3. Properties of $\mathcal{L}$-starcompact spaces

Topological behavior of $\mathcal{L}$-starcompact spaces are extensively studied by Hiremath [3] and Ikenaga [4]. The purpose of this section is to prove some results which supply their investigation. In [3, Example 3.6], Hiremath proved that a closed subspace of an $\mathcal{L}$-starcompact space need not be $\mathcal{L}$-starcompact. The following example shows that a regular closed subspace of an $\mathcal{L}$-starcompact space need not be $\mathcal{L}$-starcompact.

**Example 3.1.** There exists a star-Lindelöf (hence, an $\mathcal{L}$-starcompact) Tychonoff space having a regular-closed subset which is not $\mathcal{L}$-starcompact.

*Proof.* Let $S_1 = (c \times \omega) \cup \mathcal{R}$ be the same space as the space $X$ in Example 2.3. As we prove above, $S_1$ is not $\mathcal{L}$-starcompact. Let $S_2 = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [7], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of $\omega$ with $|\mathcal{R}| = \mathfrak{c}$. Then, $S_2$ is $\mathcal{L}$-starcompact because it is separable.

Assume $S_1 \cap S_2 = \emptyset$ and let $X$ be the quotient image of the disjoint sum $S_1 \oplus S_2$ identifying the subspace $\mathcal{R}$ of $S_1$ with the subspace $\mathcal{R}$ of $S_2$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then, $\varphi[S_1]$ is a regular-closed subspace of $X$ which is not $\mathcal{L}$-starcompact.

We shall show that $X$ is star-Lindelöf. Let $\mathcal{U}$ be an open cover of $X$. For each $n \in \omega$, since $\varphi[c \times \{n\}]$ is countably compact, there exists a finite subset $F_n \subseteq \varphi[c \times \{n\}]$ such that $\varphi[c \times \{n\}] \subseteq \text{St}(F_n, \mathcal{U})$. Thus, if we put $B' = \cup\{F_n : n \in \omega\}$, then

$$\varphi[c \times \omega] \subseteq \text{St}(B', \mathcal{U}).$$

On the other hand, since $\varphi[S_2]$ is separable, there exists a countable subset $B''$ of $\varphi[S_2]$ such that $\varphi[S_2] \subseteq \text{St}(B'', \mathcal{U})$. Consequently, we can show that $\text{St}(B' \cup B'', \mathcal{U}) = X$, and which shows that $X$ is star-Lindelöf. $\square$

**Theorem 3.2.** An open $F_\delta$-subset of an $\mathcal{L}$-starcompact space is $\mathcal{L}$-starcompact.

*Proof.* Let $X$ be an $\mathcal{L}$-starcompact space and let $Y = \cup\{H_n : n \in \omega\}$ be an open $F_\delta$-subset of $X$, where the set $H_n$ is closed in $X$ for each $n \in \omega$. To show that $Y$ is $\mathcal{L}$-starcompact, let $\mathcal{U}$ be an open cover of $Y$. We have to find a Lindelöf subset $L$ of $Y$ such that $\text{St}(L, \mathcal{U}) = Y$. For each $n \in \omega$, consider the open cover

$$\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$$

of $X$. Since $X$ is $\mathcal{L}$-starcompact, there exists a Lindelöf subset $L_n$ of $X$ such that $\text{St}(L_n, \mathcal{U}_n) = X$. Let $M_n = L_n \cap Y$. Since $Y$ is a $F_\delta$-set, $M_n$ is Lindelöf, and clearly $H_n \subseteq \text{St}(M_n, \mathcal{U})$. Thus, if we put $L = \cup\{M_n : n \in \omega\}$, then $L$ is a Lindelöf subset of $Y$ and $\text{St}(L, \mathcal{U}) = Y$. Hence, $Y$ is $\mathcal{L}$-starcompact. $\square$

A cozero-set in a space $X$ is a set of the form $f^{-1}(R \setminus \{0\})$ for some real-valued continuous function $f$ on $X$. Since a cozero-set is an open $F_\sigma$-set, we have the following corollary:

**Corollary 3.3.** A cozero-set of an $\mathcal{L}$-starcompact space is $\mathcal{L}$-starcompact.

Let $\tau$ be an infinite cardinal. Recall that a space $X$ is *Lindelöf-$\tau$-bounded* if every subset of $X$ of cardinality $\leq \tau$ is contained in a Lindelöf subset of $X$ ([6]).
Theorem 3.4. Every Lindelöf-\(\omega_1\)-bounded space is star-Lindelöf.

Proof. Let \(X\) be a Lindelöf-\(\omega_1\)-bounded space . Suppose that \(X\) is not star-Lindelöf. Then, there exists an open cover \(\mathcal{U}\) of \(X\) such that \(\text{St}(B,\mathcal{U}) \neq X\) for every countable subset \(B\) of \(X\). By induction, we can define a sequence \(\{x_\alpha : \alpha < \omega_1\}\) of points of \(X\) such that
\[
x_\alpha \notin \text{St}(\{x_\beta : \beta < \alpha\},\mathcal{U}) \quad \text{for each } \alpha < \omega_1.
\]
Since \(X\) be Lindelöf-\(\omega_1\)-bounded, the set \(\{x_\alpha : \alpha < \omega_1\}\) is contained in a Lindelöf subspace \(L \subseteq X\). Thus, there exists a countable subfamily \(\mathcal{V} \subseteq \mathcal{U}\) which covers \(L\). Then at least one element of \(\mathcal{V}\) contains uncountably many points \(x_\alpha\), which is a contradiction. Hence, \(X\) is star-Lindelöf. \(\square\)

For a space \(X\), let \(l(X)\) be the Lindelöf number of \(X\), i.e., the smallest cardinal \(\lambda\) such that every open cover of \(X\) has an open refinement \(\mathcal{V}\) with \(|\mathcal{V}| \leq \lambda\).

Theorem 3.5. Let \(\tau \geq \omega_1\). Let \(X = Y \cup Z\), where \(Y\) is dense in \(X\), \(Y\) is Lindelöf-\(\tau\)-bounded and \(l(Z) \leq \tau\). Then, \(X\) is \(\mathcal{L}\)-starcompact.

Proof. Let \(\mathcal{U}\) be an open cover of \(X\). Since \(Y\) is Lindelöf-\(\tau\)-bounded, from Theorem 3.4, there exists a countable subset \(B\) of \(Y\) such that \(Y \subseteq \text{St}(B,\mathcal{U})\). So it remains to find a Lindelöf subset \(L' \subseteq Y\) such that \(Z \subseteq \text{St}(L',\mathcal{U})\). Since \(l(Z) \leq \tau\), there is a subfamily \(\mathcal{V} \subseteq \mathcal{U}\) such that \(|\mathcal{V}| \leq \tau\) and \(Z \subseteq \cup \mathcal{V}\). Pick \(x_\mathcal{V} \in V \cap Y\) for each \(V \in \mathcal{V}\). Since \(Y\) is Lindelöf-\(\tau\)-bounded, the subset \(\{x_\mathcal{V} : V \in \mathcal{V}\}\) of \(Y\) is included in some Lindelöf subspace \(L' \subseteq Y\). Hence, \(Z \subseteq \text{St}(L',\mathcal{U})\). Let \(L = L' \cup B\). Then, \(L\) is a Lindelöf subspace of \(X\) and \(X = \text{St}(L,\mathcal{U})\), which completes the proof. \(\square\)

In [3], Hiremath proved that a continuous image of an \(\mathcal{L}\)-starcompact space is \(\mathcal{L}\)-starcompact. By contrast, he also showed a perfect preimage of an \(\mathcal{L}\)-starcompact space need not be \(\mathcal{L}\)-starcompact. Now we give a positive result:

Theorem 3.6. Let \(f\) be an open perfect map from a space \(X\) to an \(\mathcal{L}\)-starcompact space \(Y\). Then, \(X\) is \(\mathcal{L}\)-starcompact.

Proof. Since \(f[X]\) is open and closed in \(Y\), we may assume that \(f[X] = Y\). Let \(\mathcal{U}\) be an open cover of \(X\) and let \(y \in Y\). Since \(f^{-1}(y)\) is compact, there exists a finite subcollection \(\mathcal{U}_y\) of \(\mathcal{U}\) such that \(f^{-1}(y) \subseteq \cup \mathcal{U}_y\) and \(U \cap f^{-1}(y) \neq \emptyset\) for each \(U \in \mathcal{U}_y\). Pick an open neighbourhood \(V_y\) of \(y\) in \(Y\) such that \(f^{-1}[V_y] \subseteq \cup \{U : U \in \mathcal{U}_y\}\), and we can assume that
\[
(1) \quad V_y \subseteq \cap \{f[U] : U \in \mathcal{U}_y\},
\]
because \(f\) is open. Taking such open set \(V_y\) for each \(y \in Y\), we have an open cover \(\mathcal{V} = \{V_y : y \in Y\}\) of \(Y\). Let \(L\) be a Lindelöf subset of the \(\mathcal{L}\)-starcompact space \(Y\) such that \(\text{St}(L,\mathcal{V}) = Y\). Since \(f\) is perfect, the set \(f^{-1}(L)\) is a Lindelöf subset of \(X\). To show that \(\text{St}(f^{-1}(L),\mathcal{V}) = X\), let \(x \in X\). Then, there exists \(y \in Y\) such that \(f(x) \in V_y\) and \(V_y \cap L \neq \emptyset\). Since
\[
x \in f^{-1}[V_y] \subseteq \cup \{U : U \in \mathcal{U}_y\},
\]
we can choose \(U \in \mathcal{U}_y\) with \(x \in U\). Then \(V_y \subseteq f[U]\) by (1), and hence \(U \cap f^{-1}[L] \neq \emptyset\). Therefore, \(x \in \text{St}(f^{-1}[L],\mathcal{U})\). Consequently, we have that \(\text{St}(f^{-1}(L),\mathcal{U}) = X\). \(\square\)
Corollary 3.7. (Hiremath [3]) Let $X$ be an $\mathcal{L}$-starcompact space and $Y$ a compact space. Then, $X \times Y$ is $\mathcal{L}$-starcompact.

The following theorem is a generalization of Corollary 3.7.

Theorem 3.8. Let $X$ be an $\mathcal{L}$-starcompact space and $Y$ a locally compact, Lindelöf space. Then, $X \times Y$ is $\mathcal{L}$-starcompact.

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $y \in Y$, there exists an open neighbourhood $V_y$ of $y$ in $Y$ such that $\text{cl}_Y V_y$ is compact. By the Corollary 3.7, the subspace $X \times \text{cl}_Y V_y$ is $\mathcal{L}$-starcompact. Thus, there exists a Lindelöf subset $L_y \subseteq X \times \text{cl}_Y V_y$ such that

$$X \times \text{cl}_Y V_y \subseteq St(L_y, \mathcal{U}).$$

Since $Y$ is Lindelöf, there exists a countable cover $\{V_{y_i} : i \in \omega\}$ of $Y$. Let $L = \cup\{L_{y_i} : i \in \omega\}$. Then, $L$ is a Lindelöf subset of $X \times Y$ such that $St(L, \mathcal{U}) = X \times Y$. Hence, $X \times Y$ is $\mathcal{L}$-starcompact. 

Hiremath [3] showed that the product of two Lindelöf spaces need not be $\mathcal{L}$-starcompact. In [1, Example 3.3.3], van Douwen-Reed-Roscoe-Tree also gave an example of a countably compact (and hence, starcompact) space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not star-Lindelöf. Now, we shall show that the product $X \times Y$ is not $\mathcal{L}$-starcompact:

Example 3.3.9. There exist a countably compact space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not $\mathcal{L}$-starcompact.

Proof. Let $X = \omega_1$ with the usual order topology. $Y = \omega_1 + 1$ with the following topology. Each point $\alpha$ with $\alpha < \omega_1$ is isolated and a set $U$ containing $\omega_1$ is open if and only if $Y \setminus U$ is countable. Then, $X$ is countably compact and $Y$ is Lindelöf. Now, we show that $X \times Y$ is not $\mathcal{L}$-starcompact. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$, and $V_\alpha = (\alpha, \omega_1) \times \{\alpha\}$. Consider the open cover

$$U = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$$

of $X \times Y$ and let $L$ be a Lindelöf subset of $X \times Y$. Then, $\pi_X[L]$ is a Lindelöf subset of $X$, where $\pi_X : X \times Y \to X$ is the projection. Thus, there exists $\beta < \omega_1$ such that $L \cap ((\beta, \omega_1) \times Y) = \emptyset$. Pick $\alpha$ with $\alpha > \beta$. Then, $\langle \alpha, \beta \rangle \notin St(L, \mathcal{U})$ since $V_\beta$ is the only element of $\mathcal{U}$ containing $\langle \alpha, \beta \rangle$. Hence, $X \times Y$ is not $\mathcal{L}$-starcompact. which completes the proof. 

Remark. In [4, Example 2], Ikenaga gave an example of a Lindelöf space $X$ and a separable space $Y$ such that $X \times Y$ is not star-Lindelöf. By contrast, as far as the author knows, it is open whether the product of an $\mathcal{L}$-starcompact space and a separable space is $\mathcal{L}$-starcompact.

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