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Continuous maps of dendrites with finite branch points and nonwandering sets

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1 Introduction.

Let f be a continuous map from a dendrite X to itself, $\Omega(f)$ the set of nonwandering points for f, $P(f)$ the set of periodic points of f and $EP(f)$ and $\overline{EP(f)}$ the set of eventually periodic points of f and the closure of it, respectively. When X is the interval, in [B], L. Block investigated $\Omega(f)$ and $P(f)$ and showed the followings :

(1) If $\Omega(f)$ is finite, then we have $\Omega(f)=P(f)$ and

(2) $\Omega(f)\subset\overline{EP(f)}$.

Then, after about 20 years, H. Hosaka and H. Kato examined dendrites and in [HK], they proved that (1) and (2) satisfy when X is a tree. And they constructed two dendrites X_1, X_2 and two maps $g_1 : X_1\to X_1, g_2 : X_2\to X_2$ such that $\Omega(g_1)$ is finite, $\Omega(g_1)\neq P(g_1)$, and $\Omega(g_2)\not\subset\overline{EP(g_2)}$. Since the sets of branch points of X_1 and X_2 are infinite, T. Arai asks the following question : When the set of branch points of X is finite, do (1) and (2) hold good?

In [HK], they proved many lemmas to show the above (1). An important Lemma 2.6 in many lemma is able to be extended from a tree to a dendrite with finite branch points.

Theorem 1 *(Invariance of the unstable manifold) Let f be a map from a dendrite X with finite branch points to itself and p a periodic point of f. If $W(p, f)$ is the unstable manifold of p, then $f(W(p, f))=W(p, f)$.*

But, for dendrites with finite branch points which are not trees, the above (1) doesn't always come into being.

Example. Let S be a subspace $\{re^{i\theta} : n=1,2,\cdots,\theta=2\pi/n \text{ and } 0\leq r\leq 1/n\}$ of the complex plane. For each $m>n$, there exists a continuous map $f_{m,n} : S\to S$ such that $|\Omega(f_{m,n})|=m$ and $|P(f_{m,n})|=n$.

But, even if X has finite branch points, the above (2) satisfies.

Theorem 2 *Let f be a map from a dendrite X with finite branch points to itself. Then $\Omega(f)\subset\overline{EP(f)}$.*

2 Notations and definitions.

Let X be a compact metric space and f a continuous map of X into itself. We denote the n-fold composition of f with itself by $f\circ\cdots\circ f$. Let f^0 denote the identity map. A point $x\in X$ is a *periodic point of period* $n \geq 1$ *for* f if $f^n(x) = x$. The least positive integer n for which $f^n(x) = x$ is called the prime period of x. Especially, $x \in X$ is a fixed point for f if $n = 1$. A point $x \in X$ is an *eventually periodic point of period* n *for* f if there exists $m \geq 0$ such that $f^{n+i}(x)=f^i(x)$ for all $i \geq m$. That is, $f^i(x)$ is a periodic point of period n for $i \geq m$. A point $x \in X$ is *nonwandering point for* f if for any open set U containing x there exist $y \in U$ and $n > 0$ such that $f^n(y) \in U$.

We denote the set of fixed points for f, periodic points for f, eventually periodic points for f, and nonwandering points for f by $F(f), P(f), EP(f)$ and $\Omega(f)$, respectively. And $\overline{A}$ denotes the closure of a set A. Notice that $P(f) \subset \Omega(f)$, $P(f) \subset EP(f)$, $f(P(f)) \subset P(f)$, $f(\Omega(f)) \subset \Omega(f)$ and $\Omega(f)$ is closed.

An *arc* is any space which is homeomorphic to the closed interval $[0,1]$. A *continuum* is nonempty, compact and connected metric space. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are disjoint or disintersect only in one or both of their end points. From now on, X denotes a *tree* by which we mean a graph which contains no simple closed curve. A *dendrite* is a locally connected, uniqly arcwise connected continuum. We say subcontinuum A of a continuum X is of order less than or equal to β in X, written $\mathrm{Ord}(A, X) \leq \beta$, provided that for each open subset U of X with $A \subset U$ there exist an open subset V of X such that $A \subset V \subset U$ and $|\mathrm{Bd}(V)| \leq \beta$, where $\mathrm{Bd}(V)$ means the boundary of V. We say that A is of order β in X, written $\mathrm{Ord}(A, X) = \beta$, if $\mathrm{Ord}(A, X) \leq \beta$ and $\mathrm{Ord}(A, X) \not\leq \alpha$ for any cardinal number $\alpha < \beta$. A point $x \in X$ is called a *branched point of* X provided that $\mathrm{Ord}(x,X) \geq 3$. Let $\mathrm{B} = \{b_1, b_2, ..., b_n\}$ be the set of branched points of a dendrite X. For $x \in X \setminus B$, there exists an open neighborhood V of x such that V is homeomorphic to $(0,1)$ or $(0,1]$.

And the *unstable manifolds* $W(p, f)$ for some periodic point p is as follows:

$$W(p, f) = \{x \in X \mid \text{for any neighborhood } V \text{of } p, x \in f^n(V) \text{ for some } n > 0\}$$

Let X be a dendrite and Y a subspace of X. We denote the minimal connected set containing Y by $[Y]$. Particularly, if $Y = \{x, y\}$, then write $[Y] = [x, y]$.

3 Lemmas.

By the proof of [Y, Lemma 2.8], we have the following.

Lemma 1 *Let X be a dendrite, f a continuous map from X into itself and $X \setminus B = \bigcap_{j=1}^{\infty} I_j$. If an open interval $J \subset I_j$ for some $j = 1, 2, \cdots$ satisfies $J \cap P(f) = \emptyset$, then $J \cap f^n(J \cap \Omega(f)) = \emptyset$ for any positive integer n.*

By the proof of [HK, Lemma 2.4], we have the following.

Lemma 2 *Let f be a continuous map from a dendrite X to itself and p a fixed point of f. Then $W(p, f)$ is connected.*

Lemma 3 *Let f be a continuous map from a dendrite X with finite branch points to itself and p a fixed point of f. Then $f(W(p, f)) = W(p, f)$.*

Proof. By the definition, we see that $f(W(p, f)) \subset W(p, f)$. We show that $f(W(p, f)) \supset W(p, f)$. It suffuices to show that $f^{-1}(z) \cap W(p, f) \neq \emptyset$ for each $z \in W(p, f)$. We suppose that $f^{-1}(z) \cap W(p, f) = \emptyset$ for some $z \in W(p, f)$.

Since $z \in W(p,f)$, there exist an increasing sequence $n_1, n_2, \cdots$ and $x_i \in X(i = 1, 2, \cdots)$ such that $f^{n_i}(x_i) = z$ for each $i = 1, 2, \cdots$ and $x_i \to p(i \to \infty)$. We notice that $Y = \{f^{n_i - 1}(x_i) : i = 1, 2, \cdots\} \subset f^{-1}(z)$. We suppose that $|\{i : y = f^{n_i-1}(x_i)\}| = \infty$ for some $y \in f^{-1}(z)$. Since p is a fixed point of f, we have $y \in W(p,f)$ and this is a contradiction. We may assume that $f^{n_i-1}(x_i) \neq f^{n_j-1}(x_j)(i \neq j)$. Moreover we may assume that $y_i = f^{n_i-1}(x_i) \to x_0(i \to \infty)$.

Since $f^{-1}(z)$ is closed, we have $x_0 \in f^{-1}(z)$. We suppose that $x_0 \in \overline{W(p,f)}$. Since p is a fixed point of f, $Y \cap W(p,f) = \emptyset$ and $W(p,f)$ is connected, we have $x_0 \in \bigcap_{i=1}^{\infty}[p, y_i] \subset W(p,f)$ and this is a contradiction. We may assume that $x_0 \notin \overline{W(p,f)}$. We have that $Y \cup \{x_0\}$ is contained in a component C of $X \backslash \overline{W(p,f)}$. There exists the component C_0 of $C \backslash \{x_0\}$ such that $\overline{C_0} \cap \overline{W(p,f)} \neq \emptyset$. If $C_0 \cap Y$ is finite, we have $x_0 \in W(p,f)$ and a contradiction. We may assume that $C_0 \cap Y$ is infinite. But since $y_i \to x_0(i \to \infty)$, $Y \cap [p, x_0] \cap C_0$ is infinite and is contained in $W(p,f)$. This is contadiction.

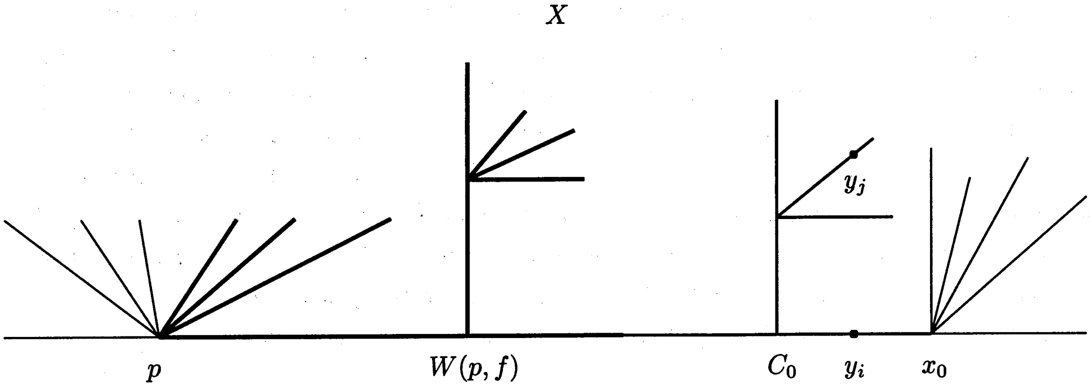

Figure 1

Lemma 4 *Let f be a continuous map from a dendrite X to itself and p a point of X with $f^n(p) = p(n > 1)$. Then $f(W(p, f^n)) = W(f(p), f^n)$.*

Proof. By the definition, we have $f(W(p, f^n)) \subset W(f(p), f^n)$. Thinking of p as $f^k(p)$ $(k = 1, 2, \cdots, n)$, we have $f(W(f^k(p), f^n)) \subset W(f^{k+1}(p), f^n)$. We see that $f^n(W(f(p), f^n)) \subset f^{n-1}(W(f^2(p), f^n)) \subset \cdots \subset f(W(f^n(p), f^n)) = f(W(p, f^n))$. Since $f(p)$ is a fixed point of f^n, by Lemma 3, we have that $f^n(W(f(p), f^n) = W(f(p), f^n)$. Thus it holds that $W(f(p), f^n) \subset f(W(p, f^n))$. We conclude that $f(W(p, f^n)) = W(f(p), f^n)$.

4 Proofs.

Proof of Theorem 1. Let p be an n-periodic point of f. We have
$f(W(p,f))$
$= f(W(p, f^n)) \cup f(W(f(p), f^n)) \cup \cdots f(W(f^{n-1}(p), f^n))$ (by [HK,Lemma 2.5])
$= W(f(p), f^n) \cup W(f^2(p), f^n) \cup \cdots W(f^n(p), f^n)$ (by Lemma 4)
$= f(W(p,f))$ (by [HK,Lemma 2.5])

In [HK, Example 1.5], for each point $p \in P(g_1)$ we have $f(W(p,f)) = W(p,f)$.

Question. Let f be a map from a dendrite X to itself and p a periodic point of f. Do we have $f(W(p,f)) = W(p,f)$?

Example. Let S be a subspace $\{re^{i\theta} : n = 1,2,\cdots,\theta = 2\pi/n \text{ and } 0 \le r \le 1/n\}$ of the complex plane. Take integers $m > n$. We construct a continuous map $f_{m,n} : S \to S$ such that $|\Omega(f_{m,n})| = m$ and $|P(f_{m,n})| = n$.

First, we construct a continuous map $f : S \to S$ such that $\Omega(f) = \{(0,0),(1/2,0)\}$ and $P(f) = \{(0,0)\}$. Denote $I_n = \{re^{2\pi i/n} : 0 \le r \le 1/n\} \subset S$, $J_n = \{(x,0) : 1/2 + 1/2n \le x \le 1/2 + 1/2(n-1)\}$ for each $n = 2,3,\cdots$ and $J = \{(x,0) : 1/2 < x \le 1\} = \bigcup_{n=2}^{\infty} J_n$.

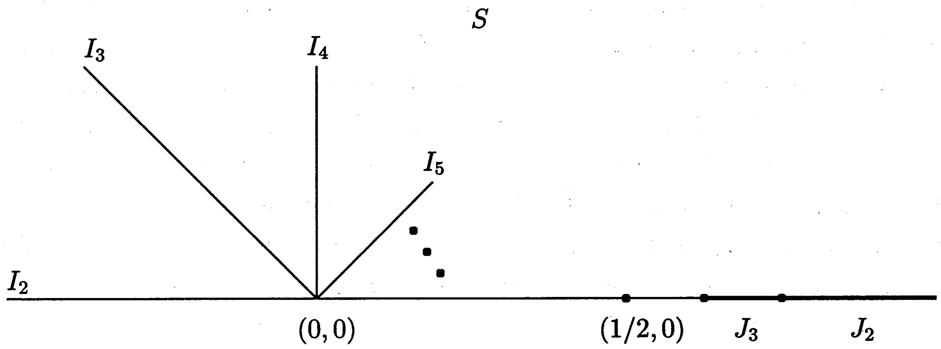

Figure 2

Define $f(\{(x,0) : 0 \le x \le 1/2 \text{ or } x = 1/2 + 1/2n \text{ for each } n = 2,3,\cdots\}) = \{(0,0)\}$, $f(I_n) = I_{n-1}$ for each $n > 2$, $f(I_2) = \{(x,0) : 0 \le x \le 1/2\}$ and $f(J_n) = I_n$ for each $n = 2,3,\cdots$. Since $f^n(I_n) = \{(0,0)\}$ for each $n = 2,3,\cdots$, we have $\Omega(f) \cap I_n = \{(0,0)\}$ for each $n = 2,3,\cdots$. And we see that $\Omega(f) \cap \{(x,0) : 0 < x \le 1/2\} = \{(1/2,0)\}$.

Since $f^m(J) \cap J = \emptyset$ for each $m \ge 1$, we have $\Omega(f) \cap J_n = \emptyset$. We conclude that $\Omega(f) = \{(0,0),(1/2,0)\}$ and $P(f) = \{(0,0)\}$.

There exists a continuous map $g : [0,1] \to [0,1]$ such that $\Omega(g) = P(g) = \{0,1\}$. See Figure 3.

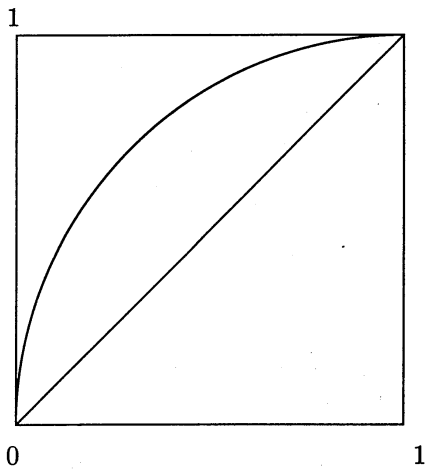

Figure 3

Denote the space $S \cup_{(0,0)=0} [0,1]$ attached by a point (0,0) of S and a point 0 of $[0,1]$. We see that $S \cup_{(0,0)=0} [0,1]$ is homeomorphic to S. Define $f_{3,2} = f \cup g : S \cup_{(0,0)=0} [0,1] \to S \cup_{(0,0)=0} [0,1]$. We have $|\Omega(f_{3,2})| = 3$ and $|P(f_{3,2})| = 2$.

Denote the space $S \cup_{(0,0)=(0,0)} S$ attached by a point (0,0) of S and a point (0,0) of an another spase S. We see that $S \cup_{(0,0)=(0,0)} S$ is homeomorphic to S. Define $f_{3,1} = f \cup f : S \cup_{(0,0)=(0,0)} S \to S \cup_{(0,0)=(0,0)} S$. We have $|\Omega(f_{3,1})| = 3$ and $|P(f_{3,1})| = 1$.

By the above, we have a continuous map $f_{m,n} : S \to S$ such that $|\Omega(f_{m,n})| = m$ and $|P(f_{m,n})| = n$.

Proof of Theorem 2. We suppose that $\Omega(f) \not\subset \overline{EP(f)}$, i.e. $V \cap \Omega(f) \neq \emptyset$, where $V = X \setminus \overline{EP(f)}$. Let x be an element of $V \cap \Omega(f)$ and W the component of V containing x. Since V is open, W is a neighborhood of x. Since $x \in \Omega(f)$, there exists a positive integer n such that $f^n(W) \cap W \neq \emptyset$. Denote $g = f^n$ and $T = \bigcup_{i=0}^{\infty} g^i(W)$ which is connected containing x. We see that $Y = \{g^i(x) : i = 0, 1, \cdots\} \subset T \supset g(T)$, that $T \cap EP(f) = \emptyset$ and that $\overline{T}$ is a dendrite.

Let B be the set of branch points of X. By [HK, Theorem 1.2], we may assume that $\bigcup_{j=1}^{\infty} I_j = \overline{T} \setminus B$, where each I_j is a component of $\overline{T} \setminus B$. If there exist disjoint integers i_1, i_2 and $j = 0, 1, \cdots$ such that $g^{i_1}(x), g^{i_2}(x) \in I_j$, by Lemma 1, then $I_j \cap P(f) \neq \emptyset$ and we have a contradiction. We may assume that $|Y \cap I_j| \leq 1$ for each j. This shows that $\overline{Y} \setminus Y \subset B \cap T$. Since $g(Y) \subset Y$ and $g(\overline{Y}) \subset \overline{Y}$, we have $g(\overline{Y} \setminus Y) \subset \overline{Y} \setminus Y$.

We have $n(1) < n(2) < \cdots$ and $b \in B \cap T$ that $|Y \cap I_{n(j)}| = 1$ for each j and that $\{b\} = \bigcap_{j=1}^{\infty} \overline{I_{n(j)}}$. Since B is finite, we have $b \notin Y$ and $b \in EP(f)$. And since $|Y \cap I_{n(j)}| = 1$ for each j and $\{b\} = \bigcap_{j=1}^{\infty} \overline{I_{n(j)}}$, we have $b \in T$. This contradicts because $T \cap EP(f) = \emptyset$.

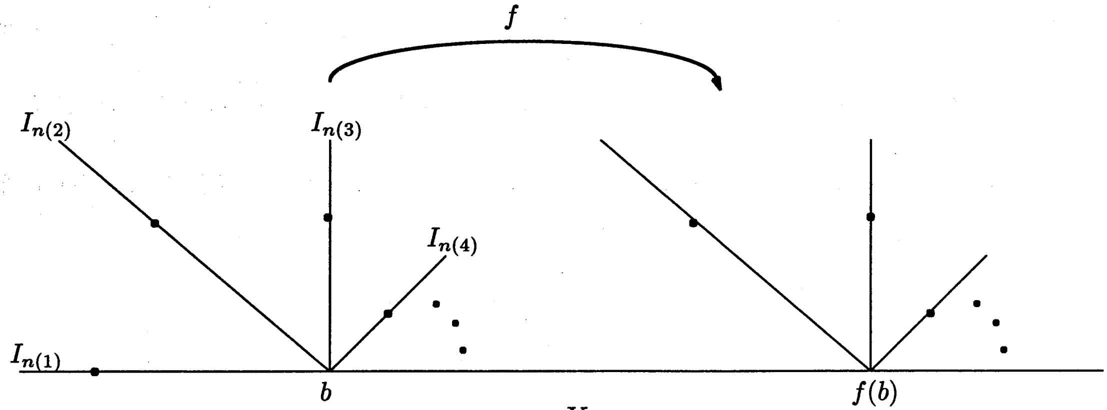

Figure 4

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