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Extensions of partitions of unity and covers

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1. Introduction

By a space we mean a topological space and γ denotes an infinite cardinal. Let X be a space and A a subspace of X. By Shapiro [13], A is said to be P^{γ}*-embedded* in X if every γ-separable continuous pseudo-metric on A can be extended to a continuous pseudo-metric on X. A subspace A is said to be P*-embedded* in X if A is P^{γ}-embedded in X for every γ. Recently, Dydak [5] defined that A is P^{γ}*(locally-finite)-embedded* in X if for every locally finite partition of unity $\{f_{\alpha} : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a locally finite partition of unity $\{g_{\alpha} : \alpha \in \Omega\}$ on X such that $g_{\alpha}|A = f_{\alpha}$ for every $\alpha \in \Omega$. A subspace A is said to be P*(locally-finite)-embedded* in X if A is P^{γ}(locally-finite)-embedded in X for every γ.

It was proved in [5] that P^{γ}(locally-finite)-embedding implies P^{γ}-embedding. This fact is also verified from characterizations of P^{γ}-embedding and P^{γ}(locally-finite)-embedding as the following. On Theorem 1.1, (1) $\Leftrightarrow$ (2) is well-known (cf. [1]), and (1) $\Leftrightarrow$ (3) is in [5] or [11].

Theorem 1.1 ([1], [5], [11]). *For a space X and a subspace A of X, the following statements are equivalent:*

(1) *A is P^{γ}-embedded in X;*

(2) *for every locally finite cozero-set cover $\{U_{\alpha} : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a locally finite cozero-set cover $\{V_{\alpha} : \alpha \in \Omega\}$ of X such that $V_{\alpha} \cap A \subset U_{\alpha}$ for every $\alpha \in \Omega$;*

(3) *for every locally finite partition of unity $\{f_{\alpha} : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a (not necessarily locally finite) partition of unity $\{g_{\alpha} : \alpha \in \Omega\}$ on X such that $g_{\alpha}|A = f_{\alpha}$ for every $\alpha \in \Omega$.*

Theorem 1.2 ([14]). *For a space X and a subspace A of X, the following statements are equivalent:*

(1) *A is P^{γ}(locally-finite)-embedded in X;*

(2) *for every locally finite cozero-set cover $\{U_{\alpha} : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a locally finite cozero-set cover $\{V_{\alpha} : \alpha \in \Omega\}$ of X such that $V_{\alpha} \cap A = U_{\alpha}$ for every $\alpha \in \Omega$.*

Notice that the space Z given in [11, Example 3] admits a P- but not P^{ω}(locally-finite)-embedded subspace (cf. [14]).

The first purpose of this talk is to characterize P^γ-embedding under the viewpoint of exactly extending cozero-set covers such as in Theorem 1.2. The second one is to investigate for P^ω(point-finite)-embedding (see Section 3 for the definition) under the same viewpoint to Theorem 1.2, and apply it to prove that the rationals $\mathbb{Q}$ of the Michael line $\mathbb{R}_\mathbb{Q}$ is not P^ω(point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$.

A collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous functions $f_\alpha : X \to [0,1]$, $\alpha \in \Omega$, is said to be a *partition of unity* on X if $\sum_{\alpha\in\Omega} f_\alpha(x) = 1$ for every $x \in X$. A partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on X is said to be *locally finite* (resp. *point-finite* [5], or *uniformly locally finite*) if $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is locally finite (resp. point-finite, or uniformly locally finite) in X. Here, a collection $\mathcal{F}$ of subsets of X is said to be *uniformly locally finite* (resp. *uniformly discrete*) in X if there exists a normal open cover $\mathcal{U}$ of X such that every $U \in \mathcal{U}$ meets at most finitely many members (resp. at most one member) of $\mathcal{F}$ ([9], [10], [3]).

2. Exact extensions of cozero-set covers and P-embedding

Our main result in this section is the following; Alò-Shapiro proved in [1] the equivalence (1) $\Leftrightarrow$ (3) assuming that X is normal and A is closed in X.

Theorem 2.1 (Main). *For a space X and a subspace A of X, the following statements are equivalent:*

(1) *A is P^γ-embedded in X;*

(2) *for every uniformly locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha | A = f_\alpha$ for every $\alpha \in \Omega$;*

(3) *for every uniformly locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$.*

We apply Theorem 2.1 to give another characterization of P-embedding by exactly extending zero-set collections. Blair [3] essentially proved that: *A subspace A of a space X is P^γ-embedded in X if and only if for every uniformly discrete zero-set collection $\{Z_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a uniformly discrete zero-set collection $\{F_\alpha : \alpha \in \Omega\}$ of X such that $F_\alpha \cap A = Z_\alpha$ for every $\alpha \in \Omega$.* In our case, we give the following:

Theorem 2.2. *For a space X and a subspace A of X, the following statements are equivalent:*

(1) *A is P^γ-embedded in X;*

(2) *every uniformly locally finite zero-set collection* $\{Z_\alpha : \alpha \in \Omega\}$ *of* A *with* $|\Omega| \leq \gamma$*, there exists a uniformly locally finite zero-set collection* $\{F_\alpha : \alpha \in \Omega\}$ *of* X *such that* $F_\alpha \cap A = Z_\alpha$ *for every* $\alpha \in \Omega$.

As another application of Theorem 2.1, we give some results concerning locations of spaces around functionally Katětov spaces. Let γ, κ be infinite cardinals. In [15], a space X is said to be (γ, κ)-*Katětov* if X is normal and for every closed subspace A of X and every locally finite κ^+-open cover $\{U_\alpha : \alpha < \gamma\}$ of A, there exists a locally finite κ^+-open cover $\{V_\alpha : \alpha < \gamma\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha < \gamma$. Here, a subspace U of X is said to be κ^+-*open set* if U can be expressed as the union of κ many cozero-sets of X. When X is (γ, ω)-Katětov for every γ, X is said to be *functionally Katětov* (cf. [7], [11], [15]). Similarily, when X is (γ, κ)-Katětov for every γ and κ (resp. (ω, κ)-Katětov for every κ, or (ω, ω)-Katětov), X is said to be *Katětov* (resp. *countably Katětov*, or *countably functionally Katětov*). Note that γ-collectionwise normal countably paracompactness implies being (γ, κ)-Katětov, and the latter implies γ-collectionwise normality (cf. [7], [15]). Moreover they were proved in [11] that every hereditarily normal space is countably Katětov, and that Rudin's Dowker space is functionally Katětov but not countably Katětov. In [11], they were essentially proved that every collectionwise normal P-space is functionally Katětov and that every normal P-space is countably functionally Katětov; here a space is said to be a *P-space* if every cozero-set is closed. A space X is said to be *hereditarily basically disconnected* if for every subspace A of X, the closure of a cozero-set of A in A is open in A.

With the aid of Theorem 2.1, we slightly generalize the result mentioned above in the following:

Lemma 2.3. *Let* X *be a* γ*-collectionwise normal space. Assume that for every closed subspace* A *of* X*, every locally finite* κ^+*-open cover, with card* $\leq \gamma$*, of* A *is uniformly locally finite in* A*. Then,* X *is* (γ, κ)*-Katětov.*

Hence we have:

Theorem 2.4. *Every* γ*-collectionwise normal and hereditarily basically disconnected space is* (γ, ω)*-Katětov.*

It also follows from Lemma 2.3 that: *If* X *is a collectionwise normal and hereditarily extremally disconnected space, then* X *is Katětov;* where X is said to be *hereditarily extremally disconnected* if for every subspace A of X, the closure of an open set of A in A is open in A. The auther does not know the assumption of X above implies countable paracompactness of X.

3. P(point-finite)-embeddings and covers

Let X be a space and A a subspace of X. On exactly extending partitions of unity, consider the following conditions:

(i) for every partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$;

(ii) for every point-finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a point-finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$;

(iii) for every locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$;

(iv) for every uniformly locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$.

Dydak proved in [5] that (i) equals that A is P^γ-embedded in X, and Theorem 2.1 shows that (iv) also equals that A is P^γ-embedded in X. The condition (iii) is precisely the definition of P^γ(locally-finite)-embedding; as was already commented in the introduction, (iii) is strictly stronger than the P^γ-embedding. By Dydak [5], the above condition (ii) is said to be that A is *P^γ(point-finite)-embedded* in X and it is proved in [5] that this condition is also strictly stronger than the P^γ-embedding (cf. Theorem 3.4 below).

Recall Theorem 1.2 and (2) $\Leftrightarrow$ (3) of Theorem 2.1. Then, we see that P^γ-embedding and P^γ(locally-finite)-embedding can be stated by extensions of cozero-set covers as well as extensions of partitions of unity. On the other hand, for P^γ(point-finite)-embedding, we have the following theorem and examples.

Theorem 3.1 (Main). *For a space X and a subspace of A, the following statements are equivalent:*

(1) *A is P^ω(point-finite)-embedded in X;*

(2) *for every point-finite countable cozero-set cover $\{U_n : n \in \mathbb{N}\}$ of A, there exists a point-finite countable cozero-set cover $\{V_n : n \in \mathbb{N}\}$ of X such that $V_n \cap A = U_n$ for every $n \in \mathbb{N}$.*

The following examples show that Theorem 3.1 need not hold on uncountable cardinal cases.

Example 3.2. *Let γ be an uncountable cardinal. There exist a space X and a closed subspace A of X such that every point-finite cozero-set cover* *of A can be extended to a point-finite cozero-set cover of X, but A is not P^γ-embedded in X.*

Sketch of the construction. We use notations as in [2] and [8]. In particular, we assume the uncountable set P in [2] as $|P| = \gamma$. Let F, f_p and F_p be the same as in [2]. Let G be the space in [8], namely,

$$G = F_p \cup \{f \in F : f(q) = 0 \text{ except for finitely many } q \in Q\}.$$

Consider the space introduced in the last part of [8, Example 2] and denote it X, namely,

$$X = (F_p \times \{0\}) \cup (G \times \{1/i : i \in \mathbb{N}\})$$

taking as a base at a point $(y, 0)$ the sets $\{(y, 0)\} \cup (U \times \{1/i : i \geq j\})$, where U is a neighborhood of y in G and $j \in \mathbb{N}$, and other points be isolated. Let $A = F_p \times \{0\}$.

Example 3.3. *There exist a space X and a closed subspace A of X such that A is P(point-finite)-embedded in X, but that A has a point-finite cozero-set cover which can not be extended to a cozero-set cover of X.*

Sketch of the construction. Consider the product space $Z = L(\omega_1) \times (\omega + 1) \times (\omega_2 + 1)$, where $L(\omega_1)$ is the set $\omega_1 + 1$ taking a base at the point ω_1 as $\{[\beta, \omega_1] : \beta < \omega_1\}$ and other points be isolated; and $\omega + 1$ and $\omega_2 + 1$ have the usual order topology. Let $X = Z - \{(\omega_1, \omega, \omega_2)\}$ and $A = L(\omega_1) \times (\omega + 1) \times \{\omega_2\} - \{(\omega_1, \omega, \omega_2)\}$ a subspace of X.

We give an application of Theorem 3.1. Let $\mathbb{R}_\mathbb{Q}$ be the Michael line and $\mathbb{Q}$ be the rationals. Dydak commented in [5] that "we do not know if $\mathbb{Q}$ is P(point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$" and contstructed his own example of a P-embedding which is not P(point-finite)-embedding. Answering his question, we have the following:

Theorem 3.4. *$\mathbb{Q}$ is not P^ω(point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$.*

Finally we give a result that three extension properties equal under a condition only for the subspace A.

Theorem 3.5. *Let X be a space, A a subspace of X and γ an infinite cardinal. If A is a P-space, then the following statements are equivalent:*

(1) *A is P^γ-embedded in X;*
(2) *A is P^γ(locally-finite)-embedded in X;*
(3) *A is P^γ(point-finite)-embedded in X.*

Note that every closed subspace of Rudin's Dowker space is P(point-finite)-embedded; it can be proved by combining some results in [5], [6] and [12]. This fact can also be seen by the above theorem directly.

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