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On metrizable spaces in dimension theory

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Abstract

We present a small variation of Mrowka's recent technique for producing metrizable spaces with non-coinciding dimensions. This variation has several uses. First, it is easier to verify many of the important properties of spaces constructed this way. Secondly, it is more general, allowing for each complete separable metric space X, a zero-dimensional and metrizable space space $M(X)$ with, consistently, the same covering dimension as X. As an application, we consistently produce, for each $n \in N$, a zero-dimensional metrizable space X_n satisfying $n = \dim X_n = \dim (X_n)^{\omega}$.

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The first example of a metrizable space for which the inductive dimensions disagreed was given by Roy in [R]. Subsequently, there were several other such examples ([M1], [K3], [K4], [O]), but none exhibited a spread between dim and ind which is greater than one.

However recently, in [M2], Mrowka gives a remarkable example of a metrizable space which he calls $\nu\mu_0$ and in [M3] he shows it is consistent that $\dim \nu\mu_0^2 = 2$ while ind $\nu\mu_0^2 = 0$. In [K5] it is shown that $\dim \nu\mu_0^n = n$. Thus, at least consistently, the spread between dim and ind can be arbitrarily large.

Readers of these papers have commented that it is difficult to understand $\nu\mu_0$ and verify its properties. In this note we present a slight variation of Mrowka's example which is somewhat easier to understand and more general. Given a complete and separable metric space X, we produce a zero-dimensional metric space $M(X)$ which satisfies dim $M(X) =$ dim X. The example is presented in a way which makes certain properties, including metrizability and zero-dimensionality, more transparent. Mrowka's space is, essentially, $M(I)$ where I is the unit interval. As an application, we show that, for each $n \in N$, there is a zero-dimensional metric space X_n such that dim $X_n =$ dim $X_n^k = n$ for all $k \in N$. This gives a nonseparable analogue for some results concerning separable spaces.

The basic set theoretic assumption we use, $S(\mathbf{c})$, is due to Mrowka (in [M2]), and is shown consistent in [D]. We note that $S(\mathbf{c})$ is a large cardinal assumption.

$S(\mathbf{c})$: The space A^N, where A is a discrete space of cardinality $\mathbf{c}$, cannot be written as a countable union of closed sets each of which is countable on all lines parallel to some axis.

For basics not defined here, the reader is referred to [E1] and [E2].

1 The construction of $M(X)$.

The construction proceeds in steps. We will define a factor space, a subspace of the factor space, and finally the example, $M(X)$, which will be a subspace in the product of countably many factor spaces.

Fix X, a complete and separable metric space, and K a closed subset of X. The factor space, $M(X,K)$ will have $X \times C$ as its point set, where C denotes the usual Cantor Set. For $(x,c) \in K \times C$ basic neighborhoods are usual product neighborhoods $O \times J$ where $p \in O$, O is open in X, $c \in J$ and J is open in C. For$(x,c) \in X\backslash K \times C$ basic neighborhoods are like product neighborhoods if C is assumed discrete; so a basic neighborhood at (x,c) is $O \times \{c\}$ where $x \in O$ with O open in $X\backslash K$.

Trivially, $M(X,K)$ is regular. We check that it is metrizable, by showing it has a σ-discrete base and applying the Nagata-Bing-Smirnov Metrization

Theorem. Let $\{V_i : i \in N\}$ be the discrete collections forming a σ-discrete base for X, and let $\{J_i : i \in N\}$ be a countable base for C. For $i, j \in N$, let $V_{i,j} = \{v \times \{c\} : v \in V_i, v \cap K = \emptyset \text{ and } c \in C\} \cup \{v \times J_j : v \in V_i, v \cap K \neq \emptyset\}$. It is easy to see that $M(X, K)$ is regular and that $\{V_{i,j} : i, j \in N\}$ is a σ-discrete collection whose union forms a base for $M(X, K)$.

It is easy to check that $M(X, K)$ is completely metrizable if X is.

For $B \subset C$, let $M(X, K, B)$ denote the subset $(K \times B) \cup (X \backslash K \times C \backslash B)$ of $M(X, K)$. In what follows, B will be an n-Bernstein set (introduced in [M4]) for all $n \in N$.

For completeness, we include the following. A subset B of a complete metric space M is *Bernstein* provided that $B \cap K \neq \emptyset$ and $(M \backslash B) \cap K \neq \emptyset$ whenever K is a perfect subset of M. A subset S of $\prod_{i \in I} X_i$ is *oblique* if whenever σ and τ are in S, $\sigma(i) \neq \tau(i)$ for all $i \in I$. The Bernstein set $B \subset C$ is *n-Bernstein* provided B^n intersects each oblique perfect subset of C^n. It is not difficult construct, by transfinite induction, a subset of C which is n-Bernstein for all n, but not all Bernstein sets are n-Bernstein.

Remark. Until now the construction has been rather general, but could in fact have been even more general. For example, any metric space could have been used in place of C (with a small change in the proof of metrizability).

Now we get more specific. Let dim $X = n$; then there is a countable base, $\{b_i : i \in N\}$ for X such that, letting d_i denote the boundary of b_i, no point is in more than n elements of $\{d_i : i \in n\}$.

The example M(X)is a subspace of the countable product $\Pi_{i \in N} M(X, d_i, B)$. For $\sigma \in \Pi_{i \in N} M(X, d_i)$, we write $\sigma = (\sigma_1, \sigma_2, \sigma_3, ...)$ where $\sigma_i = (x_{\sigma_i}, c_{\sigma_i})$ is an element of $M(X, d_i)$, and we say that σ is *X-constant* if $x_{\sigma_i} = x_{\sigma_j}$ for all $i, j \in N$. Then $M(X) = \{\sigma \in \Pi_{i \in M} M(X, d_i, B) : \sigma \text{ is } X\text{-constant}\}$. For an X-constant σ, x_σ will denote the fixed X-coordinate of σ.

Remark. The construction can be carried out in the event that dim X is infinite, with the modification that it is not assumed that there is a restriction on the number of d_i's a point may lie on. That condition only is used to guarantee that dim $M(X) \leq n$.

$M(X)$ **is metrizable.** Since it is contained in a countable product of metrizable spaces, it is obvious that $M(X)$ is metrizable.

$M(X)$ **is zero-dimensional.** To see that $M(X)$ is zero-dimensional, consider the following: Fix $j \in N$ and $s_1, s_2, ..., s_j$ where, for $i < j, s_i$ is either a one element set from $C\backslash B$ or else is a clopen set in C, and s_j is a one element set in $\in C\backslash B$. Then $(\Pi_{i\leq j}(b_j \times s_i) \times \Pi_{i>j} M(X, d_i, B)) \bigcap M(X)$ is a clopen set in $M(X)$, and the collection of all such sets forms a basis for $M(X)$. Such a set is clopen because if σ were to be on its boundary, then x_σ would have to be in k_j, but then $\sigma_j = (x_\sigma, s_j)$ (actually not s_j but the one point in s_j), and there are no such points in $M(X, d_j, B)$.

The following definitions and comments are useful in each of the next two parts. For $Y \subseteq M(X)$, let $\Pi_X Y = \{x_\sigma : \sigma \in Y\}$. For $F \in [N]^{<\omega}$, let $S_F = \{x \in X : x \in d_i$ if and only if $i \in F\}$, and let $T_F = \{\sigma \in M(X) : x_\sigma \in S_F\}$. Note that S_F and T_F are empty if $|F| > n$, due to the conditions we imposed on the base. It is easy to see that T_F is homeomorphic to a a subset of the product of S_F with $|F|$ copies of B (with the Cantor set topology) and ω copies of $C\backslash B$ (with the discrete topology); since each of these is strongly zero-dimensional, dim $T_F = 0$. It is clear that $\bigcup_{F\in[N]^{<\omega}} T_F = M(X)$.

To help clarify the above, Let $L(C)$ denote the space with point set C which is discrete at points of $C\backslash B$ and has usual neighborhoods at points of B. If Z is an X-constant set in $\Pi_{i\in N} M(X, d_i, B)$, then $h : Z \to X \times L(C)^N$ defined by $h(\sigma) = (x_\sigma, c_{\sigma_1}, c_{\sigma_2}, \ldots)$ is easily seen to be a homeomorphism, and the C coordinates of a point of T_F are associated with a point of B precisely on those coordinates j where $j \in F$.

dim $M(X) \leq$ **dim** X. Each T_F is closed in $\bigcup_{|J|=|F|} T_J$, so dim $\bigcup_{|F|=k} T_F = 0$. As $M(X) = \bigcup_{0\leq k\leq n} \bigcup_{|F|=k} T_F$, $M(X)$ is a union of $n+1$ sets of dim $= 0$, hence, dim $M(X) \leq n$.

dim $M(X) \geq$ **dim** X. We need to use the following theorem, which can be found in [K5], where it is Theorem 3.1.

Theorem 1.1 *Suppose $k \in N$, T is a complete and separable zero-dimensional metric space, and F is a closed subset of $(C\backslash B)^N \times C^k \times T$, where each copy of $C\backslash B$ is assumed to have the discrete topology, which does not intersect $(C\backslash B)^N \times B^k \times T$. Then there are closed sets $\{K_i : i \in N\}$ of $(C\backslash B)^N \times C^k$ each of which is countable on all lines parallel to some axis, and such that the projection of F to $(C\backslash B)^N \times C^k$ is contained in $\bigcup\{K_i\}$.*

Now, $M(X)$ is a subset of the completely metrizable space $\Pi_{i\in N} M(X, d_i)$. By the Lavtentieff theorem, every completion of $M(X)$ must contain a subset homeomorphic to a G_δ subset of $\Pi_{i\in N} M(X, d_i)$ which contains $M(X)$. With that in mind, we show that for an F_σ set $H = \bigcup_{j\in N} H_j$ in $\Pi_{i\in N} M(X, d_i)$ which does not intersect $M(X)$, the complement of H must contain a copy of $M(X)$. Since every metric space admits a completion which preserves dim, it follows that dim $M(X) \geq$ dim X.

Fix a closed set H in $\Pi_{i\in N} M(X, d_i)$ which does not intersect $M(X)$. For each $F \in [N]^{<\omega}$, let $W_F = \{\sigma \in \Pi_{i\in N} M(X, d_i) : \sigma$ is X-constant and $x_\sigma \in S_F\}$. Now, $T_F \subset W_F$ and $H \cap T_F = \emptyset$. Letting $\mid F \mid = k$, as T_F can be viewed as $(C\backslash B)^N \times B^k \times S_F$, and W_F can be viewed as containing the product $(C\backslash B)^N \times C^k \times S_F$ which contains T_F in the natural way. The theorem can then be applied to find the closed sets $\{K_i : i \in N\}$ with properties as stated in the theorem, relative to the closed set H. Note that the intersection of K_i with $(C\backslash B)^N \times (C\backslash B)^k$ is also closed and countable on all lines parallel to some axis.

This procedure can be carried out over all pairs in $\{H_j : j \in N\} \times [N]^{<\omega}$, to get that the projection of H to $(C\backslash B)^N$ (view as the $(C\backslash B)^N$ from the second coordinate factors of $\Pi_{i\in N} M(X, d_i)$) is contained in a countable union of closed sets (viewing each factor as discrete, since the topology on C is weaker than the discrete topology), each of which is countable on all lines parallel to some axis. By $S(\mathbf{c})$, this is not all of $(C\backslash B)^N$. Fixing $(r_1, r_2, r_3, \ldots) \in (C\backslash B)^N$ but not in the projection of H, then for the set $L = \{\sigma \in \Pi_{i\in N} M(X, d_i) : c_{\sigma_i} = r_i$ for all $i \in N\}$, $L \cap H = \emptyset$ and the projection $p : L \to X$ given by $p(\sigma) = x_\sigma$ is a homeomorphism, so the complement of H contains a copy of X.

2 An Application.

For each $n \in N$, there is a zero-dimensional metrizable space X_n such that dim X_n =dim $X_n^k = n$ for all $k \in N$. The author thanks Professor Y. Hattori for a discussion leading to this result.

Fix $n \in N$. From [K4], there is a complete and separable metric space Y_n satisfying $n =$ dim $Y_n =$ dim Y_n^ω. Then $M(Y_n)$ will be the example.

We need the following simple lemma:

Lemma 2.1 *For any $k \in N$, and closed sets $\{d_1, d_2, d_3, \ldots, d_k\}$ in Y_n, dim $[M(Y_n, d_1) \times M(Y_n, d_2) \times \ldots \times M(Y_n, d_k)] \leq n$.*

Proof. For simplicity, assume the d_i's are all the same set D. Choose closed sets $\{F_i : i \in N\}$ such that $Y_n \backslash D = \bigcup_{i \in N} F_i$. For each $j \in N$ let $E_j = F_j \times C$; then E_j is closed in $M(Y_n, D)$, and in $M(Y_n, D)$ is the product of E_j with a discrete space. Also $D \times C$ is closed in $M(Y_n, D)$, and $M(Y_n, D) = (D \times C) \cup E_1 \cup E_2 \cup \ldots$. Thus $M(Y_n, D)$ is a countable union of closed sets, each of which is the product of a closed set in Y_n with a strongly zero-dimensional metric space. It follows that $M(Y_n, D)^k$ is a countable union of closed sets each of which is a product of k sets chosen from $\{D \times C, E_1, E_2, \ldots\}$. Each such set is easily seen to be a product of $2k$ sets, with k of them subsets of Y_n and the other k strongly zero-dimensional. Since dim$Y_n^k = n$, these sets all have dim $\leq n$.

Now $M(Y_n) \subset \Pi_{i \in N} M(Y_n, k_i)$ (with k_i's as in the previous section), and so it is easy to see that dim $M(Y_n)^\omega$ is bounded by the dim of finite products of sets taken from $\{M(Y_n, d_i) : i \in N\}$. Applying the lemma, this gives dim $M(Y_n)^\omega \leq n$. We are done since we already know from the previous section that $n \leq$ dim $M(Y_n) \leq$ dim $M(Y_n)^\omega$.

Remark. With a great deal more work, it can also be shown that zero-dimensional metrizable spaces with covering dimension following the allowable sequences from [K2] can be obtained. However the entire argument needs reworking since the separable spaces in that paper are not complete.

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