On metrizable spaces in dimension theory

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Abstract

We present a small variation of Mrowka’s recent technique for producing metrizable spaces with non-coinciding dimensions. This variation has several uses. First, it is easier to verify many of the important properties of spaces constructed this way. Secondly, it is more general, allowing for each complete separable metric space $X$, a zero-dimensional and metrizable space space $M(X)$ with, consistently, the same covering dimension as $X$. As an application, we consistently produce, for each $n \in N$, a zero-dimensional metrizable space $X_n$ satisfying $n = \dim X_n = \dim (X_n)^\omega$.

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The first example of a metrizable space for which the inductive dimensions disagreed was given by Roy in [R]. Subsequently, there were several other such examples ([M1], [K3], [K4], [O]), but none exhibited a spread between dim and ind which is greater than one.

However recently, in [M2], Mrowka gives a remarkable example of a metrizable space which he calls $\nu \mu_0$ and in [M3] he shows it is consistent that $\dim \nu \mu_0^2 = 2$ while $\text{ind } \nu \mu_0^2 = 0$. In [K5] it is shown that $\dim \nu \mu_0^n = n$. Thus, at least consistently, the spread between dim and ind can be arbitrarily large.
Readers of these papers have commented that it is difficult to understand \( \nu \mu_0 \) and verify its properties. In this note we present a slight variation of Mrowka's example which is somewhat easier to understand and more general. Given a complete and separable metric space \( X \), we produce a zero-dimensional metric space \( M(X) \) which satisfies \( \dim M(X) = \dim X \). The example is presented in a way which makes certain properties, including metrizability and zero-dimensionality, more transparent. Mrowka's space is, essentially, \( M(I) \) where \( I \) is the unit interval. As an application, we show that, for each \( n \in \mathbb{N} \), there is a zero-dimensional metric space \( X_n \) such that \( \dim X_n = \dim X_n^k = n \) for all \( k \in \mathbb{N} \). This gives a nonseparable analogue for some results concerning separable spaces.

The basic set theoretic assumption we use, \( S(\mathrm{c}) \), is due to Mrowka (in [M2]), and is shown consistent in [D]. We note that \( S(\mathrm{c}) \) is a large cardinal assumption.

\( S(\mathrm{c}) \): The space \( A^\mathbb{N} \), where \( A \) is a discrete space of cardinality \( \mathfrak{c} \), cannot be written as a countable union of closed sets each of which is countable on all lines parallel to some axis.

For basics not defined here, the reader is referred to [E1] and [E2].

1 The construction of \( M(X) \).

The construction proceeds in steps. We will define a factor space, a subspace of the factor space, and finally the example, \( M(X) \), which will be a subspace in the product of countably many factor spaces.

Fix \( X \), a complete and separable metric space, and \( K \) a closed subset of \( X \). The factor space, \( M(X, K) \) will have \( X \times C \) as its point set, where \( C \) denotes the usual Cantor Set. For \( (x, c) \in K \times C \) basic neighborhoods are usual product neighborhoods \( O \times J \) where \( p \in O, O \) is open in \( X, c \in J \) and \( J \) is open in \( C \). For \( (x, c) \in X \setminus K \times C \) basic neighborhoods are like product neighborhoods if \( C \) is assumed discrete; so a basic neighborhood at \( (x, c) \) is \( O \times \{ c \} \) where \( x \in O \) with \( O \) open in \( X \setminus K \).

Trivially, \( M(X, K) \) is regular. We check that it is metrizable, by showing it has a \( \sigma \)-discrete base and applying the Nagata-Bing-Smirnov Metrization
Theorem. Let \( \{ V_i : i \in N \} \) be the discrete collections forming a \( \sigma \)-discrete base for \( X \), and let \( \{ J_i : i \in N \} \) be a countable base for \( C \). For \( i, j \in N \), let \( V_{i,j} = \{ v \times \{ c \} : v \in V_i, v \cap K = \emptyset \text{ and } c \in C \} \cup \{ v \times J_j : v \in V_i, v \cap K \neq \emptyset \} \). It is easy to see that \( M(X, K) \) is regular and that \( \{ V_{i,j} : i, j \in N \} \) is a \( \sigma \)-discrete collection whose union forms a base for \( M(X, K) \).

It is easy to check that \( M(X, K) \) is completely metrizable if \( X \) is.

For \( B \subset C \), let \( M(X, K, B) \) denote the subset \( (K \times B) \cup (X \backslash K \times C \backslash B) \) of \( M(X, K) \). In what follows, \( B \) will be an \( n \)-Bernstein set (introduced in [M4]) for all \( n \in N \).

For completeness, we include the following. A subset \( B \) of a complete metric space \( M \) is Bernstein provided that \( B \cap K \neq \emptyset \) and \( (M \backslash B) \cap K \neq \emptyset \) whenever \( K \) is a perfect subset of \( M \). A subset \( S \) of \( \Pi_{i \in I} X_i \) is oblique if whenever \( \sigma \) and \( \tau \) are in \( S \), \( \sigma(i) \neq \tau(i) \) for all \( i \in I \). The Bernstein set \( B \subset C \) is \( n \)-Bernstein provided \( B^n \) intersects each oblique perfect subset of \( C^n \). It is not difficult construct, by transfinite induction, a subset of \( C \) which is \( n \)-Bernstein for all \( n \), but not all Bernstein sets are \( n \)-Bernstein.

**Remark.** Until now the construction has been rather general, but could in fact have been even more general. For example, any metric space could have been used in place of \( C \) (with a small change in the proof of metrizability).

Now we get more specific. Let \( \dim X = n \); then there is a countable base, \( \{ b_i : i \in N \} \) for \( X \) such that, letting \( d_i \) denote the boundary of \( b_i \), no point is in more than \( n \) elements of \( \{ d_i : i \in n \} \).

The example \( M(X) \) is a subspace of the countable product \( \Pi_{i \in N} M(X, d_i, B) \). For \( \sigma \in \Pi_{i \in N} M(X, d_i) \), we write \( \sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots) \) where \( \sigma_i = (x_{\sigma_i}, c_{\sigma_i}) \) is an element of \( M(X, d_i) \), and we say that \( \sigma \) is \( X \)-constant if \( x_{\sigma_i} = x_{\sigma_j} \) for all \( i, j \in N \). Then \( M(X) = \{ \sigma \in \Pi_{i \in M} M(X, d_i, B) : \sigma \text{ is } X \text{-constant} \} \). For an \( X \)-constant \( \sigma \), \( x_\sigma \) will denote the fixed \( X \)-coordinate of \( \sigma \).

**Remark.** The construction can be carried out in the event that \( \dim X \) is infinite, with the modification that it is not assumed that there is a
restriction on the number of $d_i$'s a point may lie on. That condition only is used to guarantee that $\dim M(X) \leq n$.

$M(X)$ is metrizable. Since it is contained in a countable product of metrizable spaces, it is obvious that $M(X)$ is metrizable.

$M(X)$ is zero-dimensional. To see that $M(X)$ is zero-dimensional, consider the following: Fix $j \in N$ and $s_1, s_2, \ldots, s_j$ where, for $i < j, s_i$ is either a one element set from $C \setminus B$ or else is a clopen set in $C$, and $s_j$ is a one element set in $\in C \setminus B$. Then $(\Pi_{i \leq j} (b_j \times s_i) \times \Pi_{i > j} M(X, d_i, B)) \cap M(X)$ is a clopen set in $M(X)$, and the collection of all such sets forms a basis for $M(X)$. Such a set is clopen because if $\sigma$ were to be on its boundary, then $x_\sigma$ would have to be in $k_j$, but then $\sigma_j = (x_\sigma, s_j)$ (actually not $s_j$ but the one point in $s_j$), and there are no such points in $M(X, d_j, B)$.

The following definitions and comments are useful in each of the next two parts. For $Y \subseteq M(X)$, let $\Pi_X Y = \{x_\sigma : \sigma \in Y\}$. For $F \in [N]^{\omega}$, let $S_F = \{x \in X : x \in d_i \text{ if and only if } i \in F\}$, and let $T_F = \{\sigma \in M(X) : x_\sigma \in S_F\}$. Note that $S_F$ and $T_F$ are empty if $|F| > n$, due to the conditions we imposed on the base. It is easy to see that $T_F$ is homeomorphic to a a subset of the product of $S_F$ with $|F|$ copies of $B$ (with the Cantor set topology) and $\omega$ copies of $C \setminus B$ (with the discrete topology); since each of these is strongly zero-dimensional, $\dim T_F = 0$. It is clear that $\bigcup_{F \in [N]^{\omega}} T_F = M(X)$.

To help clarify the above, Let $L(C)$ denote the space with point set $C$ which is discrete at points of $C \setminus B$ and has usual neighborhoods at points of $B$. If $Z$ is an $X$-constant set in $\Pi_{i \in N}M(X, d_i, B)$, then $h : Z \rightarrow X \times L(C)^N$ defined by $h(\sigma) = (x_\sigma, c_{\sigma_1}, c_{\sigma_2}, \ldots)$ is easily seen to be a homeomorphism, and the $C$ coordinates of a point of $T_F$ are associated with a point of $B$ precisely on those coordinates $j$ where $j \in F$.

$\dim M(X) \leq \dim X$. Each $T_F$ is closed in $\bigcup_{|J|=|F|} T_J$, so $\dim \bigcup_{|F|=k} T_F = 0$. As $M(X) = \bigcup_{0 \leq k \leq n} \bigcup_{|F|=k} T_F$, $M(X)$ is a union of $n + 1$ sets of $\dim = 0$, hence, $\dim M(X) \leq n$.

$\dim M(X) \geq \dim X$. We need to use the following theorem, which can be found in [K5], where it is Theorem 3.1.
Theorem 1.1 Suppose $k \in N$, $T$ is a complete and separable zero-dimensional metric space, and $F$ is a closed subset of $(C \setminus B)^N \times C^k \times T$, where each copy of $C \setminus B$ is assumed to have the discrete topology, which does not intersect $(C \setminus B)^N \times B^k \times T$. Then there are closed sets $\{K_i : i \in N\}$ of $(C \setminus B)^N \times C^k$ each of which is countable on all lines parallel to some axis, and such that the projection of $F$ to $(C \setminus B)^N \times C^k$ is contained in $\bigcup\{K_i\}$.

Now, $M(X)$ is a subset of the completely metrizable space $\Pi_{i \in N} M(X, d_i)$. By the Lavrentieff theorem, every completion of $M(X)$ must contain a subset homeomorphic to a $G_\delta$ subset of $\Pi_{i \in N} M(X, d_i)$ which contains $M(X)$. With that in mind, we show that for an $F_\sigma$ set $H = \bigcup_{j \in N} H_j$ in $\Pi_{i \in N} M(X, d_i)$ which does not intersect $M(X)$, the complement of $H$ must contain a copy of $M(X)$. Since every metric space admits a completion which preserves dim, it follows that $\dim M(X) \geq \dim X$.

Fix a closed set $H$ in $\Pi_{i \in N} M(X, d_i)$ which does not intersect $M(X)$. For each $F \in [N]^\omega$, let $W_F = \{\sigma \in \Pi_{i \in N} M(X, d_i) : \sigma$ is $X$-constant and $x_\sigma \in S_F\}$. Now, $T_F \subset W_F$ and $H \cap T_F = \emptyset$. Letting $|F| = k$, as $T_F$ can be viewed as $(C \setminus B)^N \times B^k \times S_F$, and $W_F$ can be viewed as containing the product $(C \setminus B)^N \times C^k \times S_F$ which contains $T_F$ in the natural way. The theorem can then be applied to find the closed sets $\{K_i : i \in N\}$ with properties as stated in the theorem, relative to the closed set $H$. Note that the intersection of $K_i$ with $(C \setminus B)^N \times (C \setminus B)^k$ is also closed and countable on all lines parallel to some axis.

This procedure can be carried out over all pairs in $\{H_j : j \in N\} \times [N]^\omega$, to get that the projection of $H$ to $(C \setminus B)^N$ (view as the $(C \setminus B)^N$ from the second coordinate factors of $\Pi_{i \in N} M(X, d_i)$) is contained in a countable union of closed sets (viewing each factor as discrete, since the topology on $C$ is weaker than the discrete topology), each of which is countable on all lines parallel to some axis. By $S(c)$, this is not all of $(C \setminus B)^N$. Fixing $(r_1, r_2, r_3, \ldots) \in (C \setminus B)^N$ but not in the projection of $H$, then for the set $L = \{\sigma \in \Pi_{i \in N} M(X, d_i) : c_{\sigma_i} = r_i \text{ for all } i \in N\}$, $L \cap H = \emptyset$ and the projection $p : L \to X$ given by $p(\sigma) = x_\sigma$ is a homeomorphism, so the complement of $H$ contains a copy of $X$.
2 An Application.

For each $n \in N$, there is a zero-dimensional metrizable space $X_n$ such that $\dim X_n = \dim X_n^k = n$ for all $k \in N$. The author thanks Professor Y. Hattori for a discussion leading to this result.

Fix $n \in N$. From [K4], there is a complete and separable metric space $Y_n$ satisfying $n = \dim Y_n = \dim Y_n^\omega$. Then $M(Y_n)$ will be the example.

We need the following simple lemma:

**Lemma 2.1** For any $k \in N$, and closed sets $\{d_1, d_2, d_3, \ldots, d_k\}$ in $Y_n$, $\dim [M(Y_n, d_1) \times M(Y_n, d_2) \times \ldots \times M(Y_n, d_k)] \leq n$.

**Proof.** For simplicity, assume the $d_i$'s are all the same set $D$. Choose closed sets $\{F_i : i \in N\}$ such that $Y_n \setminus D = \bigcup_{i \in N} F_i$. For each $j \in N$ let $E_j = F_j \times C$; then $E_j$ is closed in $M(Y_n, D)$, and in $M(Y_n, D)$ is the product of $E_j$ with a discrete space. Also $D \times C$ is closed in $M(Y_n, D)$, and $M(Y_n, D) = (D \times C) \cup E_1 \cup E_2 \cup \ldots$. Thus $M(Y_n, D)$ is a countable union of closed sets, each of which is the product of a closed set in $Y_n$ with a strongly zero-dimensional metric space. It follows that $M(Y_n, D)^k$ is a countable union of closed sets each of which is a product of $k$ sets chosen from $\{D \times C, E_1, E_2, \ldots\}$. Each such set is easily seen to be a product of $2k$ sets, with $k$ of them subsets of $Y_n$ and the other $k$ strongly zero-dimensional. Since $\dim Y_n^k = n$, these sets all have $\dim \leq n$.

Now $M(Y_n) \subset \prod_{i \in N} M(Y_n, k_i)$ (with $k_i$'s as in the previous section), and so it is easy to see that $\dim M(Y_n)^\omega$ is bounded by the $\dim$ of finite products of sets taken from $\{M(Y_n, d_i : i \in N\}$. Applying the lemma, this gives $\dim M(Y_n)^\omega \leq n$. We are done since we already know from the previous section that $n \leq \dim M(Y_n) \leq \dim M(Y_n)^\omega$.

**Remark.** With a great deal more work, it can also be shown that zero-dimensional metrizable spaces with covering dimension following the allowable sequences from [K2] can be obtained. However the entire argument needs reworking since the separable spaces in that paper are not complete.
References


