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THE $M_3 \Rightarrow M_1$ QUESTION AND FUNCTION SPACES

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0. Introduction.

We give a brief survey of current researches on the classical $M_3 \Rightarrow M_1$ qeustion. In section 1, we give an intruduction to the question. In sections 2 and 3, we discuss the problem whether $C_k(P)$ (the space of real valued functions on the space of irrationals with the compact open topology), which is known to be an M_3-space, is an M_1-space or not.

1. The $M_3 \Rightarrow M_1$ question.

All topological spaces are assumed to be regular T_1.

The class of metrizable spaces is very useful and play an important role in mathematics. However the class is not preserved under closed maps and weak topologies. For example, let Y be the space obtained from the topological sum $X = \bigoplus_{n\in N} I_n$ of countably many copies of the unit interval by identifying all 0's to one point. Such a space is called a CW-complex and often used in algebraic topology. Y is an image of a metrizable space X under a closed map and Y has the weak topology with respect to the family $\{I_n : n \in N\}$. However Y is not metrizable. Indeed 0 doesn't have a countable neighborhood base in Y.

Are there any class of topological spaces which shares useful properties with metrizable spaces and closed under various topological operations, for example, closed images, weak topologies, countable products, and so on? Along this line, many classes of topological spaces have been defined (see [G] and [T] for detailed information).

In 1961, J. Ceder defined three classes of spaces. Let p be a property. Let $\mathcal{A}$ be a family of subsets of X. $\mathcal{A}$ is called $\sigma - p$ if $\mathcal{A}$ can be written as the countalbe union $\mathcal{A} = \cup\{\mathcal{A}_n : n \in N\}$ such that each $\mathcal{A}_n$ has property p. For example, a family $\mathcal{A}$ is σ-locally finite if it is a union of countably many locally fintie families. Recall that Nagata-Smirnov metrization theorem says that a space is *metrizable* if and only if it has a σ-locally finite base. $\mathcal{A}$ is *closure-preserving* if for each subfamily $\mathcal{A}'$ of $\mathcal{A}$, we have

$\mathrm{cl}(\cup\{A : A \in \mathcal{A}\}) = \cup\{\mathrm{cl}\, A : A \in \mathcal{A}\}$. Every locally finite family is closure-preserving because every locally finite family is finite locally and $\mathrm{cl}(\cup\{A_i : i < n\}) = \cup\{\mathrm{cl}\, A_i : i < n\}$ for finite n by the difinition of the closure operator. But the converse is not true. For example let $U_n = (-\infty, 0) \cup (\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}})$ for each $n \in N$. Then $\{U_n : n \in N\}$ is a closure-preserving open family of the real line R which is not locally finite. Note that if $\mathcal{A}$ is a family of closed sets of X, then $\mathcal{A}$ is closure-preserving if and only if $\cup\{A : A \in \mathcal{A}'\}$ is a closed set for any subfamily $\mathcal{A}'$ of $\mathcal{A}$. A space is an M_1*-space* if it has a σ-closure-preserving base. A family $\mathcal{B}$ of subsets is called a *qusi-base* if for any $x \in X$ and a neighborhood U of x, there is $B \in \mathcal{B}$ such that $x \in \mathrm{int}\, B \subset B \in U$. Note that a family is a base if and only if it is a quasi-base consisting of open sets. Ceder's definitions are as follows:

A space is an M_1*-space* if it has a σ-closure-preserving base. A space is an M_2*-space* if it has a σ-closure-preserving quasi-base. A space is an M_3*-space* if it has a σ-cushioned pair base.

Theorem 1 (Ceder). metrizable space $\Rightarrow$ M_1-space $\Rightarrow$ M_2-space $\Rightarrow$ M_3-space.

A space is *stratifiable* if there is a function G which assigns to each $n \in N$ and a closed set F of X, an open set $G(n, F)$ containing F satisfying:

(i) $F = \bigcap_n \mathrm{cl}\, G(n, F)$; and
(ii) $G(n, H) \subset G(n, F)$ whenever $H \subset F$.

We may also assume

(iii) $G(m, F) \subset G(n, F)$ for each $n \leq m$.

Every metrizable space (X, d) is stratifiable. Indeed let $G(n, F) = \{x \in X : d(x, F) < \frac{1}{n}\}$.

Theorem 2 (Borges). A space is stratifiable if and only if it is an M_3-space.

Theorem 3 (Gruenhage and Junnila). A space is an M_2-space if and only if it is an M_3-space.

Theorem 4 (Heath and Junnila). For every M_3-space X, there is an M_1-space Y and a perfect retraction $f : Y \to X$.

Idea of the Proof. Note that if $\mathcal{B}$ is a quasi-base for X, then $\{\mathrm{cl}\, B : B \in \mathcal{B}\}$ is also a quasi-base. Furthermore, if $\mathcal{B}$ is a quasi-base, then $\{\mathrm{int}\, B : B \in \mathcal{B}\}$ is a base. Hence a space is an M_3-space if and only if it has a σ-closure-preserving quasi-base consisting of closed sets, and a space is an M_1-space if and only if it has a σ-closure-preserving quasi-base consisting of regular closed sets. Here a subset A of X is *regular closed* if $A = \mathrm{cl}\,\mathrm{int}\, A$, equivalently, there is an open set U such that $A = \mathrm{cl}\, U$.

Let X be an M_3-space with a σ-closure-preserving quasi-base $\mathcal{B}$ consisting of closed sets. Let $[0, \omega]$ be the convergent sequence with the limit point ω. Let Z be the space obtained from $X \times [0, \omega]$ by making all points in $X \times [0, \omega)$ isolated. Then Z is an M_1-space. This follows from the fact that for each $n \in \omega$, the family $\{B \times [n, \omega] : B \in \mathcal{B}\}$ is a closure-preserving family consisting of regular closed sets, because $B \times [n, \omega)$ is an open set of Z and $\mathrm{cl}(B \times [n, \omega)) = B \times [n, \omega]$. By taking a subset Y of Z, we can show that X is a perfect retraction of Y under the projection map. □

Theorem 5 (Ceder, Heath and Junnila). The following conditions are equivalent:

(1) $M_3 = M_1$;
(2) Every closed subspace of an M_1-space is an M_1-space;
(3) Every perfect image of an M_1-space is an M_1-space;
(4) Every closed image of an M_1-space is an M_1-space;

Theorem 6 (Ito). Every first countable M_3-space is an M_1-space.

Theorem 7 (Mizokami, Shimane). Every M_3-k-space is an M_1-space.

Recently, Mizokami, Shimane and Kitamura obtained a result more general than Theorem 7 (see [MSK]).

A space is *F_σ-metrizable* if it is the union of countably may closed metrizable subspaces. A space is a *μ-space* if it can be embedded in the product of countably many paracompact F_σ-metrizable spaces.

Theorem 8 (Mizokami). Every M_3, μ-space is an M_1-space.

A family $\mathcal{A}$ of subsets of a space X is called *mosaical* if the partition $\mathcal{P}$ induced by $\mathcal{A}$ can be refined by a σ-discrete closed cover $\mathcal{F}$. Note that $\mathcal{F}$ need not be a partition and $\mathcal{F}$ is called a refinement if it is a refinement as a cover, i.e., for any $F \in \mathcal{F}$, there is $P \in \mathcal{P}$ satisfying $F \subset P$. $\mathcal{P}$ is defined by $\{\cap \mathcal{A}' - \cup(\mathcal{A} - \mathcal{A}') : \mathcal{A}' \subset \mathcal{A}\}$.

The following theorem essentially due to Siwiec and Nagata is most important in the theory of stratifiable spaces.

Theorem 9 (Siwiec and Nagata). Every closure-preserving closed family of a stratifiable space (more generally, a semi-stratifiable space) is mosaical.

Example 1. Let R be the real line with the usual topology and let A be an uncountable set of positive real numbers. Let $\mathcal{A} = \{[0, r) : r \in A\}$. Then $\mathcal{A}$ is neither closure-preserving nor mosaical.

Proof. First we show the following claim:

Claim 1. The partition $\mathcal{P}$ (resp. $\mathcal{P}'$) induced by $\mathcal{A}$ (resp. $\text{cl}\,\mathcal{A} = \{[0,r] : r \in A\}$) is uncountable.

Proof. Let $A_1 = \{x \in A :$ there is $\epsilon > 0$ such that $(x-\epsilon, x] \cap A = \emptyset\}$ and $A_2 = \{x \in A :$ there is $\epsilon > 0$ such that $[x, x+\epsilon) \cap A = \emptyset\}$. Then $A' = A - (A_1 \cup A_2)$ is uncountable because A_1 and A_2 are contable.

Then for every $r, s \in A'$ with $r \neq s$, we have that r and s is in different members of $\mathcal{P}$ (resp. $\mathcal{P}'$). Indeed, we may assume that $r < s$. Then there is $t \in A$ with $r < t < s$. Hence $r \in [0,t)$ (resp. $r \in [0,t]$ but $s \notin [0,t)$ (resp. $s \notin [0,t]$). □

Claim 2. Let $\mathcal{A}$ be a mosaical family of a Lindelöf space X. Then the partition $\mathcal{P}$ induced by $\mathcal{A}$ is countable.

Proof. This follows from the fact that every σ-discrete closed family of a Lindelöf space is countable. □

By the two claims above, $\mathcal{A}$ and $\text{cl}\,\mathcal{A}$ are not mosaical.

Next, suppose that $\mathcal{A}$ is closure-preserving, then $\text{cl}\,\mathcal{A}$ is a closure-preserving closed family, hence by Theorem 9, it must be mosaical, a contradiction.

One can prove that $\mathcal{A}$ is not closure-preserving in another way. Since A is uncountable, There is an increasing convergent sequence $\{r_n : n \in N\}$ in A. Then $\{[0, r_n) : n \in N\}$ is not closure-preserving. □

Theorem 10 (Mizokami, Junnila and Tamano). An M_3-space is a μ-space if and only if it has a σ-mosaical base.

2. $C_k(P)$ is an M_3-space.

Let $P = N^N$ be the space of irrational numbers with the usual topology. Let $C_k(P)$ (resp. $C_k(P,2)$) be the space of real valued (resp. 2-valued) continuous functions on P with the compact open topology, where $2 = \{0,1\}$ with the discrete topology. The base of $C_k(X)$ consists of sets of the form

$$B(f, K, \epsilon) = \{g \in C_k(X) : |g(x) - f(x)| < \epsilon \text{ for any } x \in K\},$$

where $f \in C_k(X)$, K is a compact set of X, and $\epsilon > 0$.

Gartside and Reznichenko proved the following theorem:

Theorem 11 (Gartside and Reznichenko [GR]).

(a) $C_k(P,2)$ is an M_0-space, i.e., a space with a σ-closure-preserving clopen base, hence $C_k(P,2)$ is an M_1-space.

(b) $C_k(P)$ is an M_3-space.

The following question is very interesting:

Question 1 (Gartside and Reznichenko). Is $C_k(P)$ an M_1-space?

The following question remains open:

Question 2 (Mizokami, Junnila and Tamano). Is there an M_3-space which is not a μ-space?

Compare with the following:

Example 2 (Tamano). There is a Lindelöf σ-space which is not a μ-space.

Every Lindelöf σ-space can be embedded in $C_p(M)$ (the space of real valued continuous functions on M with the topology of pointwise convergence) for some separable metrizable space M. So the following question might be interesting:

Question 3.

(1) Is $C_p(P)$ a μ-space?
(2) Is $C_k(P)$ a μ-space?

We discuss about partial negative answers to Question 1 and Question 3 in the next section. Here we only show the idea of the proof of Theorem 11 (a).

Idea of the Proof. Take

(1) a family $\mathcal{K}$ of compact sets of X such that for any compact set C of X there is $K \in \mathcal{K}$ with $C \subset K$; and
(2) a function $m : CO(P) \to [P]^{<\omega}$ which assigns to each clopen set U of P, a finite subset $m(U)$ of U such that $m(U) \cap K \neq \emptyset$ whenever $U \cap K \neq \emptyset$ for $U \in CO(P)$ and $K \in \mathcal{K}$.

Now assume that $\mathcal{K}$ and $m : CO(P) \to [P]^{<\omega}$ above have already taken. We show that $\mathcal{B} = \{B(\mathbf{0}, K) : K \in \mathcal{K}\}$ is a closure-preserving clopen neighborhood base of $\mathbf{0}$, which is sufficient to show that X is an M_0-space because $C_k(P, 2)$ is a separable topological group. Here $\mathbf{0}$ is the constant function with the value 0, and $B(f, K) = \{g \in C_k(P, 2) : g(x) = f(x) \text{ for any } x \in K\}$.

Claim 1. $\mathcal{B}$ is an neighborhood base of $\mathbf{0}$.

Proof. This is because $\mathcal{K}$ satisfies (1). □

Claim 2. Each member of $\mathcal{B}$ is clopen.

Proof. Easy. □

Claim 3. $\mathcal{B}$ is closure-preserving.

Proof. Let $\mathcal{K}'$ be a subfamily of $\mathcal{K}$. It suffices to show that $\cup\{B(\mathbf{0},K) : K \in \mathcal{K}'\}$ is closed. Suppose that $f \notin \cup\{B(\mathbf{0},K) : K \in \mathcal{K}'\}$. Then there is a clopen set U of P such that f is equal to the characteristic function χ_U of U. Note that for each $K \in \mathcal{K}'$, we have $f \notin B(\mathbf{0},K)$ if and only if $U \cap K \neq \emptyset$ if and only if $m(U) \cap K \neq \emptyset$ (which follows from (2)). Hence $B(f, m(U))$ is an open neighborhood of f which misses $\cup\{B(\mathbf{0},K) : K \in \mathcal{K}'\}$. □

3. $C_k(P)$ might be a non-M_1-space.

Let X be a space. An open set of $C_k(X)$ is called *basic* if it is of the form $\cap\{[K_i,(r_i,s_i)] : i \leq n\}$, where $n \in N$; each K_i is a compact set of X, $r_i, s_i \in R$ and $r_i < s_i$ for each $i \leq n$. Here $[K,(r,s)] = \{f \in C_k(X) : f(K) \subset (r,s)\}$. Gruenhage and Tamano essentially proved the following:

Theorem 12. Suppose that X is a separable metric space which is not σ-compact. Then

(a) (Gruenhage [G$_2$]). Any family of basic open sets of $C_k(P)$ cannot be a σ-closure-preserving base.
(b) (Tamano [T$_2$]). Any family of basic open sets of $C_k(P)$ cannot be a σ-mosaical base.

Idea of the Proof. We assume that X is zero-dimensional (in order to show the proof easily) and we only show that the family of the form $\mathcal{B} = \{[K_\alpha,(r_\alpha,s_\alpha)] : \alpha \in A\}$ cannot be a σ-closure-preserving (resp. σ-mosaical) neighborhood base at $\mathbf{0}$.

Suppose the contrary and assume that $\mathcal{B}$ above is a σ-closure-preserving (resp. σ-mosaical) neighborhood base at $\mathbf{0}$.

Claim 1. $\{r_\alpha : \alpha \in A\} \cup \{s_\alpha : \alpha \in A\}$ is countable.

Proof. Suppose not. then there is an uncountable subset A' such that $\mathcal{B}' = \{[K_\alpha,(r_\alpha,s_\alpha)] : \alpha \in A'\}$ is closure-preserving (resp. mosaical) and $\{r_\alpha : \alpha \in A'\} \cup \{s_\alpha : \alpha \in A'\}$ is uncountable.

Let $\mathbf{R} = \{\mathbf{r} : r \in R\}$, where $\mathbf{r}$ is the constant function with value r. Then $\mathbf{R} \subset C_k(P)$ is homeomorphic to R. Consider $\mathcal{B}'|\mathbf{R}$. By using the same argument as Example 1, we can show that $\mathcal{B}'$ is not closure-preserving (resp. not mosaical), a contradiction. □

Claim 2. $\cup\{K_\alpha : \alpha \in A\} = X$.

Proof. This follows from the fact that $\mathcal{B}$ is a neighborhood base at $\mathbf{0}$. □

Claim 3. There are an uncountable subset A'' of A, r and s such that

(1) for any $\alpha \in A''$, we have $(r_\alpha,s_\alpha) = (r,s)$;
(2) $\{K_\alpha : \alpha \in A''\}$ is not included in any σ-compact set; and
(3) $\mathcal{B}'' = \{[K_\alpha,(r,s)] : \alpha \in A''\}$ is closure-preserving (resp. mosaical).

Proof. This follows from Claim 1 and Claim 2. □

Claim 4. Let $\mathcal{K}$ be a family of compact sets of P whose union is not included in any σ-compact sets. Then there are a subfamily $\{K_n : n \in N\}$ of $\mathcal{K}$, a family $\{G_n : n \in N\}$ of clopen sets of P (zero-dimensionality of X is used only here) satisfying

(1) $\{G_n : n \in N\}$ is discrete in P, or there is a point $p \in P - \cup\{K_n : n \in N\}$ such that $\{G_n : n \in N\}$ converges to p, i.e., for any neighborhood U of p, there is m such that $G_n \subset U$ for any $n \geq m$;

(2) $G_n \cap K_n \neq \emptyset$ for any $n \in N$; and

(3) $G_n \cap K_m = \emptyset$ for any n, $m \in N$ with $n \neq m$.

Now apply Claim 4 to the family $\mathcal{K} = \{K_\alpha : \alpha \in A''\}$ in Claim 3. We only show the case that $\{G_n : n \in N\}$ converges to p in Claim 4 (1). Take an arbitrary function $f \in C_k(X)$ satisfying $f(x) \in (r, s)$ for each $x \neq p$ and $f(p) = s$. For each subset S of N, define $f_S \in C_k(X)$ by $f_S(x) = s + \frac{1}{n}$ for any $x \in G_n$ with $n \in S$; and $f_S(x) = f(x)$ for any $x \in X - (\cup\{G_n : n \in S\})$. Observe that $f_S \in [K_n, (r, s)]$ if and only if $n \notin S$.

Note that $f_N \notin \cup\{[K_n, (r, s)] : n \in N\}$. To show Theorem 12 (a), we show that $f_N \in \mathrm{cl}(\cup\{[K_n, (r, s)] : n \in N\})$, which contradicts Claim 3 (3). Indeed, f_N is in the closure with respect to the topology of uniform convergence. Let $\epsilon > 0$. Take n_0 such that $|f_N(x) - f(x)| < \epsilon$ for any $x \in G_{n_0}$. Then $f_{(N-\{n_0\})} \in [K_{n_0}, (r, s)]$ and $|f_N - f_{(N-\{n_0\})}| < \epsilon$.

To show Theorem 12 (b), note that for any subsets S and S' of N, f_S and $f_{S'}$ are in different members of the partition $\mathcal{P}$ induced by $\{[K_n : (r, s)] : n \in N\}$. Hence $\mathcal{P}$ is uncountable, which contradicts Claim 3 (3) and Example 1, Claim 2. □

References.

We give here small number of references. See the references of the following papers for fuller information.

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