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# An Elementary Construction of a Cantor Set with Arbitrary Hausdorff Dimension

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## Abstract

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree with a distance function as follows.

**Theorem 1** (Kurata).

$$\sup_{x\in X}\left(\liminf_{\substack{B(y_n)\subset B(x)\\ n\to\infty}}\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)}\right)\leq \dim_H(\Omega,\ell)\leq \sup_{\xi\in\Omega}\left(\liminf_{\substack{y_n\in[\xi]\\ n\to\infty}}\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)}\right).$$

In the paper we shall investigate the usefulness of Kurata's formula and obtain the following results.

**Theoerm 2.** There exists a Cantor set for which both sides of Kurata's formula do not coincide.

**Theorem 3.** For each $\gamma,\ 0\leq\gamma\leq\infty,$ there exists a Cantor set E with Hausdorff dimension $\gamma$.

## §1 Introduction

Recently H. Kurata gave an evaluation formura of the Hausdorff dimension of the boundary of a tree and calculated the Hausdorff dimension of certain sets of $\mathbf{R}^n$ by using it.

**Theorem 1** (Kurata's formula [7]). Let $\Omega$ be the boundary of a tree $(X, \mathcal{A}, o)$ with a distance function $\ell$. Then

$$\sup_{x\in X}\left(\liminf_{\substack{B(y_n)\subset B(x)\\ n\to\infty}} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)}\right) \le \dim_H(\Omega,\ell) \le \sup_{\xi\in\Omega}\left(\liminf_{\substack{y_n\in[\xi]\\ n\to\infty}} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)}\right).$$

In the present paper we shall show the following :

**Theoerm 2.** There exists a Cantor set for which the both sides of Kurata's formula do not coincide.

**Theorem 3.** For each $\gamma$, $0 \le \gamma \le \infty$, there exists a Cantor set E with Hausdorff dimension $\gamma$.

Our Cantor sets satisfying the condition in each of Theorems 2 and 3 are not self-similar, in general. So we cannot apply the formula ${c_1}^D + {c_2}^D = 1$ of the Hausdorff dimension $D$, where each $c_i$ denotes the ratio of similarity. We use the Kurata's formula to calculate the Hausdorff dimension of our Cantor sets.

Theorem 3 is known, for example [6], but our construction of required Cantor sets is elementary and geometrical. The ratios of contraction vary in each inductive step in the construction.

Let us recall a tree and the Hausdorff dimension of its boundary with a distance function.

**Definition** (Kurata [7]). Let $(X, \mathcal{A}, o)$ be a tree, i.e. simply connected and locally finite graph. The set $X$ is an infinite set of *points* and the collection $\mathcal{A}$ is a set of *arcs*. The point $o \in X$ is called the *root point*. For $x,\ y \in X$ with $x \ne y$ let $\rho(x, y)$ be the least number of arcs which join $x$ and $y$, and $\rho(x,x) = 0$. Then $\rho$ is a metric on $X$. We assume that $\#\{y \in X : \rho(x,y) = 1\} \ge 2$ for each $x \in X$. We set $X_n = \{x \in X : \rho(o,x) = n\}$ for $n = 0,\ 1,\ 2,\ \cdots$.

Let $\Omega$ be the set of all paths from $o$. A *path* is a sequence of points $(x_0, x_1, x_2, \cdots)$ such that $x_0 = o$, and $\rho(x_n, x_{n+1}) = 1$ for any $x_n \in X_n$, $n = 0,\ 1,\ 2,\ \cdots$. For $\xi = (x_n)_n$, $\eta = (y_n)_n \in \Omega$ we define

$$[\xi] = \{x_0, x_1, x_2, \cdots\} \qquad \text{where } x_0 = o,$$

and

$$P(\xi,\eta) = x_n \qquad \text{if } x_0 = y_0,\ x_1 = y_1,\ \cdots,\ x_n = y_n,\ x_{n+1} \neq y_{n+1}.$$

Now $P(\xi,\xi)$ is not defined. The space $\Omega$ is called the *boundary* of a tree $(X, \mathcal{A}, o)$. Let $\ell$ be a positive function from $X$ to $\mathbf{R}^1$ with the following properties :
For any path $\xi = (x_n)_n$,

(L1) $\ell(x_n)$ is strictly decreasing in $n$,

(L2) $\lim_{n\to\infty} \ell(x_n) = 0$.

For $\xi = (x_n)_n,\ \eta = (y_n)_n \in \Omega$ define

$$d(\xi,\eta) = \begin{cases} \ell(P(\xi,\eta)) & \text{if } \xi \neq \eta, \\ 0 & \text{if } \xi = \eta. \end{cases}$$

Then $d$ is a metric on $\Omega$, and $\Omega$ is a compact space. For $x \in X$ let $B(x) = \{\xi \in \Omega : x \in [\xi]\}$. If we take $\eta \in \Omega$ with $x \in [\eta]$, we have that $B(x) = \{\xi \in \Omega : d(\xi,\eta) \leq \ell(x)\}$. The set $B(x)$ is both open and closed in $\Omega$.

For $K \subset \Omega$ and $\alpha > 0$ we define

$$\Lambda_\alpha^r(K,\ell) = \inf\left\{ \sum_j (\ell(z_j))^\alpha : K \subset \bigcup_j B(z_j),\ \ell(z_j) < r \right\} \quad \text{for } r > 0,$$

and

$$\Lambda_\alpha(K,\ell) = \lim_{r\to+0} \Lambda_\alpha^r(K,\ell) = \sup_{r>0} \Lambda_\alpha^r(K,\ell).$$

We have that $0 \leq \Lambda_\alpha(K,\ell) \leq \infty$. The value $\Lambda_\alpha(K,\ell)$ is called the $\alpha$-dimensional *Hausdorff measure* of $(K,\ell)$. Define the *Hausdorff dimension* of $K$ with a distance function $\ell$ as

$$\dim_H(K,\ell) = \inf\{\alpha : \Lambda_\alpha(K,\ell) = 0\} = \sup\{\alpha : \Lambda_\alpha(K,\ell) = \infty\}.$$

Note that $0 \leq \dim_H(K,\ell) \leq \infty$.

Now we define a function $\varphi(x)$ as follows. Let $\varphi(o) = 1$. For $x \in X_n,\ n > 1$, we take $y \in X_{n-1}$ such that $\rho(x,y) = 1$ and let

$$\varphi(x) = \frac{\varphi(y)}{\#\{z \in X_n : \rho(y,z) = 1\}}.$$

## §2 A construction of a Cantor set with variable ratios of contraction in each inductive step

In this section we construct a Cantor set $E$ with variable ratios of contraction in each inductive step.

For any number $n \geq 1$, let $\{c_j^{(n)}\}_{j=0,1,2,\cdots,2^n-1}$ be a sequence of real numbers with the properties :

(C1) $0 < c_j^{(n)} < 1$ for each $n \geq 1$,

(C2) $\lim_{n\to\infty} a^{(1)}a^{(2)}\cdots a^{(n)} = 0$ where $a^{(n)} = \max\{c_j^{(n)} : j = 0, 1, 2, \cdots, 2^n - 1\}$ for $n \geq 1$.

Let $E_0$ be a bounded closed interval in $\mathbf{R}^1$. Denote the diameter of a set $E \subset \mathbf{R}^1$ by $|E|$. Note that a natural number $j$ can be written by $i_1 i_2 \cdots i_n$ as a number of $n$ figures in a binary notation. For example,

*Case* $n = 2$ : 0=00, 1=01, 2=10, 3=11, in a binary notation ;

*Case* $n = 3$ : 0=000, 1=001, 2=010, 3=011, in a binary notation.

Put $c_{i_1 i_2 \cdots i_n} = c_j^{(n)}$ if $j = i_1 i_2 \cdots i_n$ in a binary notation. Define a family $\{M_{i_1 i_2 \cdots i_n}\}_{i_1 i_2 \cdots i_n}$ of subintervals of $E_0$ indexed by a finite sequence of figures 0, 1 as follows by induction :

(i) For $n = 1$, let $M_0$ and $M_1$ be two closed subintervals of $E_0$ such that

$$E_0 \setminus \text{(a middle open interval)} = M_0 \cup M_1,$$

where $\min M_0 = \min E_0$, $\max M_1 = \max E_0$ and $|M_{i_1}| = |E_0|\, c_{i_1}$ for $i_1 = 0, 1$.

(ii) If $M_{i_1 i_2 \cdots i_n}$ is defined, let $M_{i_1 i_2 \cdots i_n 0}$ and $M_{i_1 i_2 \cdots i_n 1}$ be two closed subintervals of $M_{i_1 i_2 \cdots i_n}$ such that

$$M_{i_1 i_2 \cdots i_n} \setminus \text{(a middle open subinterval)} = M_{i_1 i_2 \cdots i_n 0} \cup M_{i_1 i_2 \cdots i_n 1},$$

where $\min M_{i_1 i_2 \cdots i_n 0} = \min M_{i_1 i_2 \cdots i_n}$, $\max M_{i_1 i_2 \cdots i_n 1} = \max M_{i_1 i_2 \cdots i_n}$ and $|M_{i_1 i_2 \cdots i_n j}| = |M_{i_1 i_2 \cdots i_n}|\, c_{i_1 i_2 \cdots i_n i_{n+1}}$ for $j = i_1 i_2 \cdots i_n i_{n+1}$ in a binary notation.

Then the family $\{M_{i_1 i_2 \cdots i_n}\}_{i_1 i_2 \cdots i_n}$ satisfies the following :

(M1) For any infinite sequence $i_1 i_2 \cdots i_n \cdots$ in $\{0, 1\}$,

$$M_{i_1} \supset M_{i_1 i_2} \supset \cdots \supset M_{i_1 i_2 \cdots i_n} \supset M_{i_1 i_2 \cdots i_n i_{n+1}} \supset \cdots.$$

(M2) If $i_1 i_2 \cdots i_n \neq k_1 k_2 \cdots k_n$, then $M_{i_1 i_2 \cdots i_n} \cap M_{k_1 k_2 \cdots k_n} = \emptyset$ .

(M3) $|M_{i_1 i_2 \cdots i_n}| = |E_0| \, c_{i_1} c_{i_1 i_2} \cdots c_{i_1 i_2 \cdots i_n}$.

(M4) For any infinite sequence $i_1 i_2 \cdots i_n \cdots$ in $\{0, 1\}$,

$$\lim_{n \to \infty} |M_{i_1 i_2 \cdots i_n}| = 0.$$

Hence, $\bigcap_{n=1}^{\infty} M_{i_1 i_2 \cdots i_n} =$ one point.

Let

$$E_n = \bigcup_{n=1}^{\infty} \{M_{i_1 i_2 \cdots i_n} : i_1 i_2 \cdots i_n \text{ is a sequence in } \{0,\ 1\} \text{ with length } n\} \quad \text{for } n \geq 1.$$

Then the set $E = \bigcap_{n=1}^{\infty} E_n$ is a Cantor set in $\mathbf{R}^1$.

**Remark.** The 1/3-*Cantor set* is a set $E$ with

$$c_j^{(n)} = \frac{1}{3} \quad \text{for } n \geq 1 \text{ and } j = 0,\ 1,\ \cdots,\ 2^n - 1.$$

Next we define a tree $(X, \mathcal{A}, o)$ corresponding to the Cantor set $E$ as follows :

(T1) $X = X_0 \cup \bigcup_{n=1}^{\infty} X_n$, where $X_0 = \{o\}$, $X_1 = \{0, 1\}$, $\cdots$, and

$$X_n = \{i_1 i_2 \cdots i_n : \text{ a sequence in } \{0,1\} \text{ with length } n\} \quad \text{for } n \geq 1.$$

(T2) $\mathcal{A} = \{[o, 0],\ [o, 1]\} \cup$

$$\bigcup_{n=1}^{\infty} \{\, [x_n,\ y_{n+1}] : x_n \in X_n,\ y_{n+1} \in X_{n+1},\ x_n = i_1 i_2 \cdots i_n,\ y_{n+1} = i_1 i_2 \cdots i_n i_{n+1}\},$$

where $[x, y]$ means the arc joining $x$ and $y$ in $X$.

Then $\varphi(x_n) = \dfrac{1}{2^n}$ for $x_n \in X_n$.

Define $\ell(x_n) = c_{i_1} c_{i_1 i_2} \cdots c_{i_1 i_2 \cdots i_n}$ for $x_n = i_1 i_2 \cdots i_n$.
Then the function $\ell$ satisfies the requirements in the definition of the boundary of a tree.

We have a bijection $g : \Omega \longrightarrow E$ defined by

$$g(\xi) = s \qquad \text{where } \{s\} = \bigcap_{n=1}^{\infty} M_{i_1 i_2 \cdots i_n}$$

for $\xi = (o,\, y_1,\, y_2,\, \cdots,\, y_n,\, \cdots)$ with $y_n = i_1 i_2 \cdots i_n,\ n \geq 1$.
Then $\dim_H(\Omega, \ell) = \dim_H E$.

## §3 Proofs

Example 1 in the following shows Theorem 2.

**Example 1.** For each $n$, define

$$c_j^{(n)} = \begin{cases} \frac{1}{3} & : \ j = 0,\, 2,\, \cdots,\, 2^n - 2, \\ \frac{1}{9} & : \ j = 1,\, 3,\, \cdots,\, 2^n - 1, \end{cases}$$

and

$$\begin{aligned} \ell(y_n) &= c_{i_1} c_{i_1 i_2} \cdots c_{i_1 i_2 \cdots i_n} \qquad \text{for } y_n = i_1 i_2 \cdots i_n \\ &= c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_n}^{(n)}, \end{aligned}$$

where $\quad j_r = 2^{r-1} i_1 + 2^{r-2} i_2 + \cdots + 2\, i_{r-1} + i_r\,, \quad r = 1,\, 2,\, \cdots,\, n.$

Then, the resulting Cantor set E gives an example of Theorem 2 (see Fig. 1).

(1) The right side of Kurata's formura $= \dfrac{\log 2}{\log 3}$ .

In fact, take a path $\xi = (o, y_1, y_2, \cdots, y_n, \cdots) \in \Omega$ with $y_n = 00 \cdots 0$ for any $n$. We have that

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{\log 3} \qquad \text{for any } n.$$

Hence,

$$\liminf_{\substack{y_n \in [\xi] \\ n \to \infty}} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{\log 3}\,.$$

(2) The left side of Kurata's formula $= \dfrac{\log 2}{2 \log 3}$ .

In fact, take any $x \in X$ with $x = i_1 i_2 \cdots i_n$. Let $y_n$ be any point in $X$ such that $B(y_n) \subset B(x)$. For any $n > m$, set $y_n = i_1 i_2 \cdots i_m i_{m+1} \cdots i_n$ and $i_{m+1} = \cdots = i_n = 1$. Then, for any $n > m$

$$\ell(y_n) = c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_m}^{(m)} \left(\frac{1}{9}\right)^{n-m}$$

and

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2(1-\frac{m}{n})\log 3 - \frac{1}{n}\log c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_n}^{(n)}}\ .$$

Hence,

$$\liminf_{\substack{B(y_n)\subset B(x)\\ n\to\infty}} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2\log 3}\ . \quad \square$$

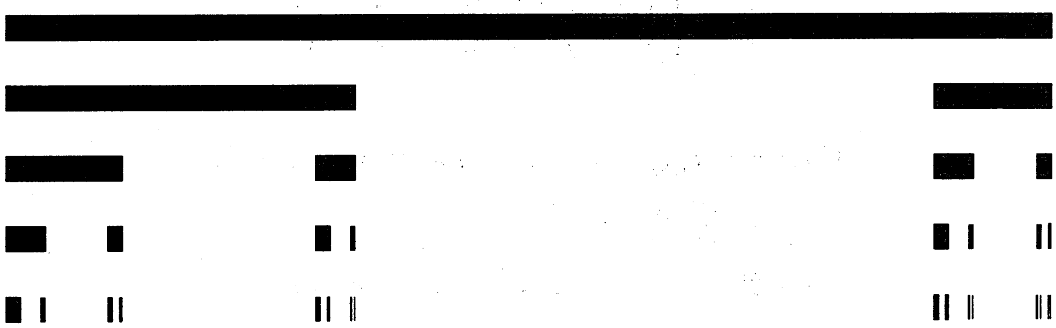

Fig. 1

Theorem 3 is established by Examples 2 - 6 in the following.

**Example 2.** *Case* : $\gamma = 0$. For each $n$, define

$$c_j^{(n)} = \left(\frac{1}{3}\right)^n \qquad \text{for } j = 0,\ 1,\ \cdots,\ 2^n - 1\ .$$

Then, the resulting Cantor set E has Hausdorff dimension 0.

In fact, take any $y_n \in \Omega$ with $y_n = i_1 i_2 \cdots i_n$. Then

$$\ell(y_n) = \left(\frac{1}{3}\right)^1 \left(\frac{1}{3}\right)^2 \cdots \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^{\frac{1}{2}n(n+1)}.$$

The function $\ell$ satisfies the conditions (L1) - (L2).
Since

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{2\log 2}{(n+1)\log 3} \longrightarrow 0 \quad \text{as } n \to \infty ,$$

we have that $\quad \dim_H E = 0 \quad$ from Theorem 1. □

**Example 3.** *Case* : $\gamma = 1$. For each $n$, define

$$c_j^{(n)} = \frac{1}{4}\,\frac{2^n+1}{2^{n-1}+1} \qquad \text{for } j = 0,\, 1,\, \cdots,\, 2^n - 1 .$$

Then, the resulting Cantor set E has Hausdorff dimension 1.

In fact, take any $y_n \in \Omega$ with $y_n = i_1 i_2 \cdots i_n$. Then

$$\ell(y_n) = \left(\frac{1}{4}\right)^n \frac{2^n+1}{2} = \frac{2^n+1}{2^{2n+1}}.$$

The $\ell$ satisfies the conditions (L1) - (L2).
Since

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{1}{(2+\frac{1}{n}) - \frac{\log(2^n+1)}{\log 2^n}} \longrightarrow 1 \quad \text{as } n \to \infty ,$$

we have that $\quad \dim_H E = 1 \quad$ from Theorem 1. □

**Example 4.** *Case* : $0 < \gamma < 1$. For each $n$, define

$$c_j^{(n)} = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \qquad \text{for } j = 0,\, 1,\, \cdots,\, 2^n - 1 .$$

Then, the resulting Cantor set $E$ has Hausdorff dimension $\gamma$.
In fact, the both sides of Kurata's formula are equal to $\gamma$. □

**Example 5.** *Case* : $1 < \gamma < \infty$. For some integer $N \geq 2$ with $\gamma \leq N$, we can obtain a Cantor set $E$ in $\mathbf{R}^N$ with $\dim_H E = \gamma$ by appropriate modifications to that of § 2. We explain how to construct such a Cantor set $E$ in $\mathbf{R}^2$ for $N = \gamma = 2$.

Let $E_0$ be a closed regular square in $\mathbf{R}^2$. For each $n$, define

$$c_j^{(n)} = \frac{1}{4}\,\frac{2^n+1}{2^{n-1}+1} \qquad \text{for } j = 0,\, 1,\, \cdots,\, 2^n - 1 ,$$

and

$$c_{i_1 i_2 \cdots i_n} = c_j^{(n)} \quad \text{for} \quad j = i_1 i_2 \cdots i_n \text{ in a 4-ary notation.}$$

Define a family $\{M_{i_1 i_2 \cdots i_n}\}_{i_1 i_2 \cdots i_n}$ of closed subsquares of $E_0$ indexed by a finite sequence of figures 0, 1, 2, 3 with the properties (M1) - (M4). Analogously in § 2 we have a Cantor set $E \subset \mathbf{R}^2$ with $\dim_H E = 2$ (Fig. 2). □

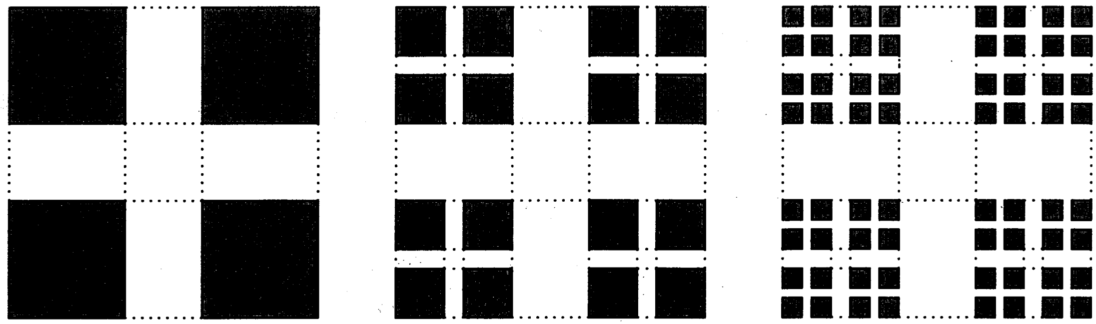

Fig. 2

**Example 6.** *Case* : $\gamma = \infty$. We construct a Cantor set $E$ in the Hilbelt cube $Q$ with $\dim_H E = \infty$. The Hilbert cube means a space

$$Q = \{(t_i) : 0 \leq \frac{1}{t_i} \leq \frac{1}{i} \quad \text{for} \quad i = 1, 2, 3, \cdots\}$$

with the metric

$$d(s,t) = \sqrt{\sum_{n=1}^{\infty} (s_i - t_i)^2} \qquad \text{for} \quad s = (s_i),\ t = (t_i).$$

Define a set $E \subset Q$ as follows :

$$E = \bigcup_{n=1}^{\infty} A_n \cup \{a_0\},$$

where $a_0 = (0, 0, 0, \cdots)$, and for any $n$, $A_n$ is a Cantor set such that

(A1) $A_n \subset \left[ \frac{1}{n+1}, \frac{1}{n} \right]^n \times \{0\} \times \{0\} \times \cdots ,$

(A2) $\dim_H A_n = n$,

(A3) $A_m \cap A_n = \emptyset \quad$ if $\; m \neq n$.

Since $E$ is a totally disconnected compact metric space with no isolated points, it is a Cantor set. We have that

$$\dim_H E = \sup_n \dim_H A_n = \infty. \quad \square$$

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