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Several properties on Aluthge transformation

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This report is based on the following preprints:

[Y1] T.Yamazaki, *Characterizations of* $\log A \geq \log B$ *and normaloid operators via Heinz inequality*, preprint.

[Y2] T.Yamazaki, *Parallelisms between Aluthge transformation and powers of operators*, to appear in Acta Sci. Math. (Szeged).

[Y3] T.Yamazaki, *An expression of spectral radius via Aluthge transformation*, preprint.

ABSTRACT

In 1990, Aluthge defined an operator transformation $\widetilde{T}$ of T by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T. This transformation has very interesting properties, and many authors call $\widetilde{T}$ *Aluthge transformation* and have studied properties of this transformation.

In this paper, firstly, we shall show properties of Aluthge transformation on operator norm, and a characterization of normaloid operators by giving a definition to n-th Aluthge transformation $\widetilde{T_n} = (\widetilde{\widetilde{T_{n-1}}})$.

Secondly, we shall point out that there are parallelisms between Aluthge transformation and powers of operators. Moreover we shall show $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$ which is a parallel result to $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$.

Lastly, we shall discuss relations between the orders $|\widetilde{T}|^p \geq |T|^p$ and $|T|^{p-1} \geq |T^*|^{p-1}$ for some positive number p.

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. For each $p > 0$, an operator T is said to be *p-hyponormal* if $|T|^{2p} \geq |T^*|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. Especially, an operator T is said to be *hyponormal* if T is 1-hyponormal. It is well known that *"every p-hyponormal operator is q-hyponormal for* $p \geq q > 0$.*"* And it is also well known that *"for each* $q > 0$, *there* *exists a q-hyponormal and non-p-hyponormal operator for any $p > q > 0$.*" Especially, there exists a $\frac{1}{2}$-hyponormal and non-hyponormal operator. Relating to these facts, many authors have studied some operator transformations from $\frac{1}{2}$-hyponormal operator to hyponormal operator. And the following two operator transformations were obtained:

Let $T = U|T|$ be the polar decomposition of T.

(i) $S = U|T|^{\frac{1}{2}}$.

(ii) $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation [1]).

If T is $\frac{1}{2}$-hyponormal, then both S and $\widetilde{T}$ are hyponormal. Moreover, it was shown that $\sigma(T) = \sigma(\widetilde{T})$ in [3, 4, 11], where $\sigma(T)$ is the spectrum of T. So we understand that Aluthge transformation is a better transformation than (i).

In this paper, we shall show several properties of Aluthge transformation as follows: Firstly, it is well known that $\|T\| \geq \|\widetilde{T}\|$ holds for all operator T. Relating to this fact, we shall show a characterization of the condition $\|T\| = \|\widetilde{T}\|$, and generalize this result by giving a definition to "*n-th Aluthge transformation*". An operator T is said to be *normaloid* if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T. It is well known that "*for each $p > 0$, every p-hyponormal operator is normaloid.*" Moreover we shall show a characterization of normaloid operators via Aluthge transformation.

Secondly, we shall show a parallel result to powers of p-hyponormal operators for $p \in [0, 1]$ via n-th Aluthge transformation. And we shall show a new expression of spectral radius via Aluthge transformation.

Lastly, we shall discuss relations between the orders $|T|^p \geq |T^*|^p$ and $|\widetilde{T}|^{p-1} \geq |T|^{p-1}$ for some positive number p.

2. A CHARACTERIZATION OF NORMALOID OPERATORS

Fujii, Izumino and Nakamoto [6] showed the following characterization of normaloid operators via Aluthge transformation as follows:

Theorem A ([6]). *Let $T \in B(H)$. Then the following assertions are mutually equivalent:*

(1) *T is normaloid.*

(2) *$\|T\| = \|\widetilde{T}\|$ and $\widetilde{T}$ is normaloid (i.e., $\|\widetilde{T}\| = r(\widetilde{T})$).*

In this section, we shall discuss the condition under which $\|T\| = \|\widetilde{T}\|$ and normaloidness of T via Aluthge transformation. First, we obtain the following result:

Theorem 1 ([Y1]). *Let $T \in B(H)$. Then for each natural number n, the following assertions are equivalent:*
(1) $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$.
(2) $\|T\| = \|\widetilde{T}^n\|^{\frac{1}{n}}$.

Remark 1. Put $n = 1$ in Theorem 1, we obtain the following equivalence relation:

$$\|T\| = \|T^2\|^{\frac{1}{2}} \iff \|T\| = \|\widetilde{T}\|. \tag{2.1}$$

To prove Theorem 1, we cite the following norm inequality:

Theorem B ([10]). *Let A and B be positive operators, and $X \in B(H)$. Then the following inequalities hold:*

(i) $\|A^r X B^r\| \le \|AXB\|^r \|X\|^{1-r}$ *for* $r \in [0, 1]$.
(ii) $\|A^r X B^r\| \ge \|AXB\|^r \|X\|^{1-r}$ *for* $r > 1$.

Proof of Theorem 1. Let $T = U|T|$ be the polar decomposition of T.

Proof of (1) $\Longrightarrow$ (2). Assume that $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$. Then we have

$$\begin{aligned} \|T\| &= \|T^{n+1}\|^{\frac{1}{n+1}} \\ &= \||T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^n|T|^{\frac{1}{2}}\|^{\frac{1}{n+1}} \\ &\le \|T\|^{\frac{1}{n+1}}\|\widetilde{T}^n\|^{\frac{1}{n+1}}. \end{aligned}$$

Hence $\|T\| \le \|\widetilde{T}^n\|^{\frac{1}{n}} \le \|\widetilde{T}\| \le \|T\|$ hold.

Proof of (2) $\Longrightarrow$ (1). Assume that $\|T\| = \|\widetilde{T}^n\|^{\frac{1}{n}}$. Then by (i) of Theorem B for $\frac{1}{2} \in [0, 1]$, we have

$$\begin{aligned} \|T\| &= \|\widetilde{T}^n\|^{\frac{1}{n}} \\ &= \|(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^n\|^{\frac{1}{n}} \\ &= \||T|^{\frac{1}{2}}(U|T|)^{n-1}U|T|^{\frac{1}{2}}\|^{\frac{1}{n}} \\ &\le \left\{\||T|(U|T|)^{n-1}U|T|\|^{\frac{1}{2}} \cdot \|(U|T|)^{n-1}U\|^{\frac{1}{2}}\right\}^{\frac{1}{n}} \\ &\le \|T^{n+1}\|^{\frac{1}{2n}} \cdot \|T^{n-1}\|^{\frac{1}{2n}} \\ &\le \|T^{n+1}\|^{\frac{1}{2n}} \cdot \|T\|^{\frac{n-1}{2n}}. \end{aligned}$$

Hence we obtain

$$\|T\| \le \|T^{n+1}\|^{\frac{1}{n+1}} \le \|T\|.$$

Therefore the proof of Theorem 1 is complete. $\square$

By considering the following "*n-th Aluthge transformation*", we obtain another generalization of (2.1).

Definition 1 (n-th Aluthge transformation [Y1]). *Let $T \in B(H)$ and $T = U|T|$ be the polar decomposition of T. Then for each natural number n, n-th Aluthge transformation $\widetilde{T_n}$ of T is defined by $\widetilde{T_n} = \widetilde{(\widetilde{T_{n-1}})}$ and $\widetilde{T_1} = \widetilde{T}$.*

Theorem 2 ([Y1]). *Let $T \in B(H)$. Then for each natural number n, the following assertions are equivalent:*
(1) $\|T\| = \|T^{k+1}\|^{\frac{1}{k+1}}$ *for all* $k = 1, 2, \cdots, n$.
(2) $\|T\| = \|\widetilde{T_k}\|$ *for all* $k = 1, 2, \cdots, n$.

Proof. We shall prove Theorem 2 by induction on n.
In case $n = 1$. Theorem 2 holds by (2.1).
Assume that Theorem 2 holds in case $n = m$.
In case $n = m + 1$.

Proof of (1) $\Longrightarrow$ (2). Suppose that

$$\|T\| = \|T^2\|^{\frac{1}{2}} = \cdots = \|T^{m+2}\|^{\frac{1}{m+2}}.$$

Then we have

$$\|T\| = \|\widetilde{T}\| = \cdots = \|\widetilde{T}^{m+1}\|^{\frac{1}{m+1}} \tag{2.2}$$

by Theorem 1. Put $S = \widetilde{T}$ in (2.2). Then (2.2) asserts

$$\|S\| = \|S^2\|^{\frac{1}{2}} = \cdots = \|S^{m+1}\|^{\frac{1}{m+1}}.$$

By the induction hypothesis for the case $n = m$, we have

$$\|\widetilde{T}\| = \|S\| = \|\widetilde{S}\| = \cdots = \|\widetilde{S_m}\| = \|\widetilde{(\widetilde{T_m})}\| = \|\widetilde{T_{m+1}}\|. \tag{2.3}$$

Hence we obtain

$$\|T\| = \|\widetilde{T}\| = \cdots = \|\widetilde{T_{m+1}}\|$$

by (2.2) and (2.3).

Proof of (2) $\Longrightarrow$ (1). Suppose that

$$\|T\| = \|\widetilde{T}\| = \cdots = \|\widetilde{T_{m+1}}\| = \|\widetilde{(\widetilde{T})_m}\|. \tag{2.4}$$

Put $S = \widetilde{T}$ in (2.4). Then we have

$$\|S\| = \|\widetilde{S}\| = \cdots = \|\widetilde{S_m}\|$$

By the induction hypothesis for the case $n = m$, we have

$$\|T\| = \|\widetilde{T}\| = \|S\| = \|S^2\|^{\frac{1}{2}} = \cdots = \|S^{m+1}\|^{\frac{1}{m+1}} = \|\widetilde{T}^{m+1}\|^{\frac{1}{m+1}}.$$

Hence we have

$$\|T\| = \|T^k\|^{\frac{1}{k}} \quad \text{for all } k = 1, 2, \cdots, m+2$$

by Theorem 1.

Therefore the proof of Theorem 2 is complete. □

By Theorem 2, we obtain the following Corollary 3 which is a characterization of normaloid operators, immediately.

Corollary 3 ([Y1]). *Let $T \in B(H)$. Then the following assertions are equivalent:*
(1) *T is normaloid.*
(2) *$\|T\| = \|\widetilde{T_n}\|$ for all natural number n.*

By Corollary 3, we can obtain Theorem A, easily as follows:

$$\begin{aligned} & T \text{ is normaloid} \\ \Longleftrightarrow \quad & \|T\| = \|\widetilde{T}\| = \|\widetilde{T_n}\| = \|\widetilde{(\widetilde{T})_{n-1}}\| \text{ for all natural number } n \text{ by Corollary 3} \\ \Longleftrightarrow \quad & \|T\| = \|\widetilde{T}\| \text{ and } \widetilde{T} \text{ is normaloid by Corollary 3.} \end{aligned}$$

Proof of Corollary 3. We recall the following well-known result:

$$T \textit{ is normaloid} \quad \Longleftrightarrow \quad \|T\| = \|T^n\|^{\frac{1}{n}} \textit{ for all positive integer } n.$$

Hence we obtain Corollary 3 by Theorem 2. □

3. Parallel results between Aluthge transformation and powers of operators

It was shown that *"there exists a hyponormal operator T such that T^2 is not hyponormal"* in [9]. Relating to this fact, Aluthge and Wang [2] showed that *"if T is a p-hyponormal operator for $p \in (0,1]$, then T^n is $\frac{p}{n}$-hyponormal for all natural number n."* As an extension of this result, the following result was shown in [8]:

Theorem C ([8]). *Let T be a p-hyponormal operator for $p \in (0,1]$. Then for each natural number n, the following inequalities hold:*
(i) $|T|^{2(p+1)} \leq |T^2|^{p+1} \leq \cdots \leq |T^n|^{\frac{2(p+1)}{n}}$.
(ii) $|T^*|^{2(p+1)} \geq |T^{*2}|^{p+1} \geq \cdots \geq |T^{*n}|^{\frac{2(p+1)}{n}}$.

We remark that as a generalization of the result by Aluthge and Wang, Ito [12] showed that *"if T is a p-hyponormal operator for $p > 0$, then T^n is* $\min\{1, \frac{p}{n}\}$*-hyponormal for all natural number n."* And he showed an extension of Theorem C. As a parallel result to Theorem C, we obtain the following result:

Theorem 4 ([Y2]). *Let T be a $\frac{p}{2}$-hyponormal operator for $p \in (0,1]$. Then for each natural number n, the following inequalities hold:*

(i) $|T|^{p+1} \leq |\widetilde{T}|^{p+1} \leq \cdots \leq |\widetilde{T_n}|^{p+1}$.

(ii) $|T^*|^{p+1} \geq |\widetilde{T^*}|^{p+1} \geq \cdots \geq |\widetilde{T_n^*}|^{p+1}$.

To prove Theorem 4, we prepare the following results and definition:

Theorem D ([1, 7, 11, 16]). *Let T be a $\frac{p}{2}$-hyponormal operator for $p > 0$ (i.e., $|T|^p \geq |T^*|^p$). Then the following inequalities hold:*

(i) *In case $p \in (0,1]$. $|\widetilde{T}|^{p+1} \geq |T|^{p+1} \geq |(\widetilde{T})^*|^{p+1}$*
(i.e., $\widetilde{T}$ is $\frac{p+1}{2}$-hyponormal).

(ii) *In case $p \in [1,2]$. $|\widetilde{T}|^2 \geq |T|^2 \geq |(\widetilde{T})^*|^2$ (i.e., $\widetilde{T}$ is hyponormal).*

Theorem D was shown in [1] when U is unitary, where $T = U|T|$ is the polar decomposition of T. And Theorem D was shown in [11, 16]. Moreover, a generalization of Theorem D was shown in [7, 11, 16].

Definition 2 ($*$-Aluthge transformation [Y2]). *Let $T = U|T|$ be the polar decomposition of an operator T. Then $*$-Aluthge transformation of T is defined as follows:*

(i) $\widetilde{T}^{(*)} \overset{\text{def}}{=} (\widetilde{T^*})^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ *($*$-Aluthge transformation).*

(ii) *For each natural number n,*
$\widetilde{T_n}^{(*)} \overset{\text{def}}{=} \left(\widetilde{\widetilde{T_{n-1}}^{(*)}}\right)^{(*)} = (\widetilde{T_n^*})^*$ *and* $\widetilde{T_1}^{(*)} \overset{\text{def}}{=} \widetilde{T}^{(*)}$
(n-th $*$-Aluthge transformation).

As relations between $\widetilde{T}$ and $\widetilde{T}^{(*)}$, we obtain the following results, immediately.

Theorem 5 ([Y2]). *Let $T \in B(H)$. Then the following assertions hold:*

(i) $\sigma(\widetilde{T}) = \sigma(\widetilde{T}^{(*)}) = \sigma(T)$.

(ii) $w(\widetilde{T}) = w(\widetilde{T}^{(*)})$, *where $w(T)$ is the numerical radius of T.*

(iii) $\|\widetilde{T}\| = \|\widetilde{T}^{(*)}\|$.

We remark that (i) of Theorem 5 asserts more generalization form of $\sigma(\widetilde{T}) - \{0\} = \sigma(\widetilde{T}^{(*)}) - \{0\} = \sigma(T) - \{0\}$. And $\sigma(\widetilde{T}) = \sigma(T)$ has been already shown in [3, 4, 11].

Proposition 6 ([Y2]). *Let $T \in B(H)$. Then for each $p > 0$,*

$$\widetilde{T} \text{ is } p\text{-hyponormal} \Longleftrightarrow \widetilde{T}^{(*)} \text{ is } p\text{-hyponormal.}$$

Remark 2. If T is $\frac{p}{2}$-hyponormal for $p \in (0,2]$, then we obtain the following assertions, easily. $\widetilde{T_n}$ is $\frac{p}{2}$-hyponormal for all natural number n by using Theorem D several times, and also $\widetilde{T_n}^{(*)}$ is $\frac{p}{2}$-hyponormal for all natural number n by Theorem D and Proposition 6 several times.

Proof of Theorem 4. We shall prove Theorem 4 by induction on n.
Proof of (i). (a) By (i) of Theorem D, we have $|\widetilde{T}|^{p+1} \geq |T|^{p+1}$.

(b) Assume that $|\widetilde{T_{n-1}}|^{p+1} \geq \cdots \geq |\widetilde{T}|^{p+1} \geq |T|^{p+1}$.

(c) Proof of $|\widetilde{T_n}|^{p+1} \geq |\widetilde{T_{n-1}}|^{p+1}$.

Put $S = \widetilde{T_{n-1}}$. Then S is also $\dfrac{p}{2}$-hyponormal by Remark 2. Hence we have

$$|\widetilde{T_n}|^{p+1} = |\widetilde{S}|^{p+1} \geq |S|^{p+1} = |\widetilde{T_{n-1}}|^{p+1} \quad \text{by (i) of Theorem D.}$$

Proof of (ii). Let $T = U|T|$ be the polar decomposition of T.
(a) Proof of $|T^*|^{p+1} \geq |\widetilde{T^*}|^{p+1}$.

$$\begin{aligned}
|T^*|^{p+1} &= U|T|^{p+1}U^* \\
&\geq U|(\widetilde{T})^*|^{p+1}U^* \quad \text{by (i) of Theorem D} \\
&= U(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{\frac{p+1}{2}}U^* \\
&= (U|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}U^*)^{\frac{p+1}{2}} \\
&= (|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}})^{\frac{p+1}{2}} \\
&= |\widetilde{T^*}|^{p+1}.
\end{aligned}$$

(b) Assume that $|T^*|^{p+1} \geq |\widetilde{T^*}|^{p+1} \geq \cdots \geq |\widetilde{T^*_{n-1}}|^{p+1}$.

(c) Proof of $|\widetilde{T^*_{n-1}}|^{p+1} \geq |\widetilde{T^*_n}|^{p+1}$.
Put $S = (\widetilde{T^*_{n-1}})^* = \widetilde{T_{n-1}}^{(*)}$. Then S is also $\dfrac{p}{2}$-hyponormal by Remark 2. By (a), we obtain

$$|\widetilde{T^*_{n-1}}|^{p+1} = |S^*|^{p+1} \geq |\widetilde{S^*}|^{p+1} = |\widetilde{T^*_n}|^{p+1}.$$

Therefore the proof of Theorem 4 is complete. □

By considering Theorem 4, we can understand that n-th Aluthge transformation and powers of operators have similar properties. On the other hand, $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$ is a very famous and useful result. So we shall show the parallel result to this one as follows:

Theorem 7 ([Y3]). *Let $T \in B(H)$. Then* $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$.

To prove Theorem 7, we prepare the following lemmas:

Lemma 8 ([Y3]). *For a natural number n and $k = 0, 1, \cdots, n+1$, let*

$$ {}_nD_k = \frac{n!(n-2k+1)}{k!(n-k+1)!}. \tag{3.1}$$

Then the following assertions hold:

(i) ${}_nD_0 = 1$ *for all natural number n.*

(ii) ${}_nD_k + {}_nD_{k+1} = {}_{n+1}D_{k+1}$ *for all natural number n and $k = 0, 1, \cdots, n$.*

(iii) ${}_{2n+1}D_n = {}_{2n+2}D_{n+1}$ *for all natural number n.*

(iv) $\displaystyle\sum_{k=0}^{[\frac{n}{2}]}(n-2k+1)_nD_k = 2^n$,
where $[\frac{n}{2}]$ is the largest integer satisfying $[\frac{n}{2}] \le \frac{n}{2}$.

(v) $\displaystyle\lim_{n\to\infty} \frac{(n-2k+1)_nD_k}{2^n} = 0$ *for all positive integer k.*

Lemma 9 ([Y3]). *Let $T \in B(H)$. Then*

$$\|\widetilde{T}^n\| \le \|T^{n+1}\|^{\frac{1}{2}}\|T^{n-1}\|^{\frac{1}{2}}$$

holds for all natural number n.

Proof. Let $T = U|T|$ be the polar decomposition of T. Then we have

$$\begin{aligned}
\|\widetilde{T}^n\| &= \|(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^n\| = \||T|^{\frac{1}{2}}(U|T|)^{n-1}U|T|^{\frac{1}{2}}\| \\
&\le \||T|(U|T|)^{n-1}U|T|\|^{\frac{1}{2}}\|(U|T|)^{n-1}U\|^{\frac{1}{2}} \quad \text{by (i) of Theorem B} \\
&\le \|T^{n+1}\|^{\frac{1}{2}}\|T^{n-1}\|^{\frac{1}{2}}.
\end{aligned}$$

□

Lemma 10 ([Y3]). *Let $T \in B(H)$ and $m = [\frac{n}{2}]$. Then*

$$\|\widetilde{T_n}\| \le \|T^{n+1}\|^{\frac{{}_nD_0}{2^n}}\|T^{n-1}\|^{\frac{{}_nD_1}{2^n}} \cdots \|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}} \cdots \|T^{n-2m+1}\|^{\frac{{}_nD_m}{2^n}}.$$

Proof. We shall prove Lemma 10 by induction on n.

(a) $\|\widetilde{T}\| \le \|T^2\|^{\frac{1}{2}}$ holds by Lemma 9.

(b) Assume that

$$\begin{aligned}
\|\widetilde{T_{n-1}}\| \le{}& \|T^n\|^{\frac{{}_{n-1}D_0}{2^{n-1}}}\|T^{n-2}\|^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\times \cdots \times \|T^{n-2k}\|^{\frac{{}_{n-1}D_k}{2^{n-1}}} \cdots \|T^{n-2m}\|^{\frac{{}_{n-1}D_m}{2^{n-1}}},
\end{aligned} \tag{3.2}$$

where $m = [\frac{n-1}{2}]$.

(c-1) In case $n = 2m+1$ for $m = 1, 2, \cdots$. Then $[\frac{n}{2}] = [\frac{n-1}{2}] = m$. Hence by (3.2), we have

$$
\begin{aligned}
\|\widetilde{T_n}\| &= \|\widetilde{(\widetilde{T})_{n-1}}\| \\
&\leq \|\widetilde{T}^n\|^{\frac{{}_{n-1}D_0}{2^{n-1}}} \|\widetilde{T}^{n-2}\|^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \|\widetilde{T}^{n-2k+2}\|^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \|\widetilde{T}^{n-2k}\|^{\frac{{}_{n-1}D_k}{2^{n-1}}} \cdots \|\widetilde{T}^3\|^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \|\widetilde{T}\|^{\frac{{}_{n-1}D_m}{2^{n-1}}} \\
&\leq \left(\|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_0}{2^{n-1}}} \left(\|T^{n-1}\|^{\frac{1}{2}} \|T^{n-3}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \left(\|T^{n-2k+3}\|^{\frac{1}{2}} \|T^{n-2k+1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \left(\|T^{n-2k+1}\|^{\frac{1}{2}} \|T^{n-2k-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_k}{2^{n-1}}} \\
&\qquad \times \cdots \times \left(\|T^4\|^{\frac{1}{2}} \|T^2\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \|T^2\|^{\frac{{}_{n-1}D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{{}_{n-1}D_0}{2^n}} \|T^{n-1}\|^{\frac{{}_{n-1}D_0 + {}_{n-1}D_1}{2^n}} \\
&\quad \times \cdots \times \|T^{n-2k+1}\|^{\frac{{}_{n-1}D_{k-1} + {}_{n-1}D_k}{2^n}} \cdots \|T^2\|^{\frac{{}_{n-1}D_{m-1} + {}_{n-1}D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{{}_nD_0}{2^n}} \|T^{n-1}\|^{\frac{{}_nD_1}{2^n}} \cdots \|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}} \cdots \|T^2\|^{\frac{{}_nD_m}{2^n}}
\end{aligned}
$$

by (i) and (ii) of Lemma 8, and the last inequality holds by Lemma 9.

(c-2) In case $n = 2m+2$ for $m = 0, 1, 2, \cdots$. Then $[\frac{n}{2}] = m+1$ and $[\frac{n-1}{2}] = m$. Hence by (3.2), we have

$$
\begin{aligned}
\|\widetilde{T_n}\| &= \|\widetilde{(\widetilde{T})_{n-1}}\| \\
&\leq \|\widetilde{T}^n\|^{\frac{{}_{n-1}D_0}{2^{n-1}}} \|\widetilde{T}^{n-2}\|^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \|\widetilde{T}^{n-2k+2}\|^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \|\widetilde{T}^{n-2k}\|^{\frac{{}_{n-1}D_k}{2^{n-1}}} \cdots \|\widetilde{T}^4\|^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \|\widetilde{T}^2\|^{\frac{{}_{n-1}D_m}{2^{n-1}}} \\
&\leq \left(\|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_0}{2^{n-1}}} \left(\|T^{n-1}\|^{\frac{1}{2}} \|T^{n-3}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \left(\|T^{n-2k+3}\|^{\frac{1}{2}} \|T^{n-2k+1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \left(\|T^{n-2k+1}\|^{\frac{1}{2}} \|T^{n-2k-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_k}{2^{n-1}}} \\
&\qquad \times \cdots \times \left(\|T^5\|^{\frac{1}{2}} \|T^3\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \left(\|T^3\|^{\frac{1}{2}} \|T\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_m}{2^{n-1}}} \\
&= \|T^{n+1}\|^{\frac{{}_{n-1}D_0}{2^n}} \|T^{n-1}\|^{\frac{{}_{n-1}D_0 + {}_{n-1}D_1}{2^n}} \\
&\quad \times \cdots \times \|T^{n-2k+1}\|^{\frac{{}_{n-1}D_{k-1} + {}_{n-1}D_k}{2^n}} \cdots \|T^3\|^{\frac{{}_{n-1}D_{m-1} + {}_{n-1}D_m}{2^n}} \|T\|^{\frac{{}_{n-1}D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{{}_nD_0}{2^n}} \|T^{n-1}\|^{\frac{{}_nD_1}{2^n}} \cdots \|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}} \cdots \|T^3\|^{\frac{{}_nD_m}{2^n}} \|T\|^{\frac{{}_nD_{m+1}}{2^n}}
\end{aligned}
$$

by (i), (ii) and (iii) of Lemma 8, and the last inequality holds by Lemma 9. □

Lemma 11 ([Y3]). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying $\lim_{n\to\infty} a_n = a$, and for each natural number n, $\{c_{n,k}\}_{k=1}^{n}$ be a positive sequence satisfying*

$$c_{n,1} + \cdots + c_{n,k} + \cdots + c_{n,n} = 1 \quad \textit{for all natural number } n \tag{3.3}$$

and $\lim_{n\to\infty} c_{n,k} = 0$ for fixed $k = 1, 2, \cdots$. Then

$$\lim_{n\to\infty} (c_{n,1}a_1 + \cdots + c_{n,k}a_k + \cdots + c_{n,n}a_n) = a.$$

Proof of Theorem 7. Let $m = [\frac{n}{2}]$. Then by Lemma 10, (iv) of Lemma 8 and Arithmetic mean-Geometric mean inequality, we have

$$\begin{aligned} r(T) = r(\widetilde{T_n}) &\le \|\widetilde{T_n}\| \le \|T^{n+1}\|^{\frac{nD_0}{2^n}} \|T^{n-1}\|^{\frac{nD_1}{2^n}} \\ &\qquad \cdots \|T^{n-2k+1}\|^{\frac{nD_k}{2^n}} \cdots \|T^{n-2m+1}\|^{\frac{nD_m}{2^n}} \\ &\le \frac{(n+1)_n D_0}{2^n} \|T^{n+1}\|^{\frac{1}{n+1}} + \frac{(n-1)_n D_1}{2^n} \|T^{n-1}\|^{\frac{1}{n-1}} \\ &\quad + \cdots + \frac{(n-2k+1)_n D_k}{2^n} \|T^{n-2k+1}\|^{\frac{1}{n-2k+1}} \\ &\qquad + \cdots + \frac{(n-2m+1)_n D_m}{2^n} \|T^{n-2m+1}\|^{\frac{1}{n-2m+1}} \\ &\longrightarrow r(T) \quad \text{as } n \to \infty \end{aligned}$$

by $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$, (iv) and (v) of Lemma 8 and Lemma 11. □

4. Relations between the orders $|\widetilde{T}|^{p-1} \ge |T|^{p-1}$ and $|T|^p \ge |T^*|^p$

In this section, we shall discuss properties of the order $|\widetilde{T}|^p \ge |T|^p$ for some $p > 0$. Relating to this order, Theorem D is very famous. As a converse of Theorem D, we obtain the following result:

Theorem 12 ([Y2]). *Let T be an invertible operator. Then the following assertions hold:*

(i) *For each $p \in [2, 4]$, $|\widetilde{T}|^p \ge |T|^p$ ensures $|T|^{p-1} \ge |T^*|^{p-1}$.*
(ii) *For each $p \ge 4$, $|\widetilde{T}|^p \ge |T|^p$ ensures $|T|^3 \ge |T^*|^3$.*

To prove Theorem 12, we need the following result:

Theorem E ([5, 13, 14, 15]). *Let A and B be positive invertible operators. Then the following assertions hold:*

(i) *$A \ge B > 0$ ensures $(B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^{\frac{1-t}{p-t}} \ge B^{1-t}$ for $1 \ge p \ge \dfrac{1}{2}$ with $p > t \ge 0$.*
(ii) *$A \ge B > 0$ ensures $(B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^{\frac{2p-t}{p-t}} \ge B^{2p-t}$ for $\dfrac{1}{2} \ge p > t \ge 0$.*

Proof of Theorem 12. Let $T = U|T|$ be the polar decomposition of T. Then U is unitary since T is invertible.
Proof of (i). By applying (i) of Theorem E to $|\widetilde{T}|^p \geq |T|^p$, we have

$$(4.1) \qquad (|T|^{\frac{-t_1 p}{2}}|\widetilde{T}|^{pp_1}|T|^{\frac{-t_1 p}{2}})^{\frac{1-t_1}{p_1-t_1}} \geq |T|^{p(1-t_1)}$$

for $1 \geq p_1 \geq \frac{1}{2}$ with $p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.1). Then we have

$$(|T|^{\frac{-1}{2}}|\widetilde{T}|^2|T|^{\frac{-1}{2}})^{p-1} \geq |T|^{p-1}.$$

It is equivalent to

$$(|T|^{\frac{-1}{2}}|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}|T|^{\frac{-1}{2}})^{p-1} \geq |T|^{p-1},$$

that is, $U^*|T|^{p-1}U \geq |T|^{p-1}$ since U is unitary. Hence we have $|T|^{p-1} \geq U|T|^{p-1}U^* = |T^*|^{p-1}$.

Proof of (ii). By applying (ii) of Theorem E to $|\widetilde{T}|^p \geq |T|^p$, we have

$$(4.2) \qquad (|T|^{\frac{-t_1 p}{2}}|\widetilde{T}|^{pp_1}|T|^{\frac{-t_1 p}{2}})^{\frac{2p_1-t_1}{p_1-t_1}} \geq |T|^{p(2p_1-t_1)}$$

for $\frac{1}{2} \geq p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.2). Then we have

$$(|T|^{\frac{-1}{2}}|\widetilde{T}|^2|T|^{\frac{-1}{2}})^3 \geq |T|^3.$$

It is equivalent to

$$(|T|^{\frac{-1}{2}}|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}|T|^{\frac{-1}{2}})^3 \geq |T|^3,$$

that is, $U^*|T|^3U \geq |T|^3$ since U is unitary. Hence we have $|T|^3 \geq U|T|^3U^* = |T^*|^3$. □

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