Powers of class $wA(s, t)$ operators associated with generalized Aluthge transformation

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Abstract

This report is based on the following preprint:


An operator $T = U|T|$ is said to belong to class $wA(s, t)$ for $s, t > 0$ if $|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t}$ and $|T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{s}{s+t}}$, where $\tilde{T}_{s,t} = |T|^s U |T|^t$. We show that if $T$ belongs to class $wA(s, t)$, then $T^n$ belongs to class $wA(\frac{s}{n}, \frac{t}{n})$ for every natural number $n$.

1 Introduction

1.1 An order preserving operator inequality

In this report, an operator means a bounded linear operator on a Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

We begin this report by introducing the following result which is quite useful for the study of the class of operators including normal operators ($\iff T^*T = TT^*$).

Theorem F (Furuta inequality [12]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) \( (B^\frac{r}{2} A^p B^\frac{r}{2})^\frac{1}{q} \geq (B^\frac{r}{2} B^p B^\frac{r}{2})^\frac{1}{q} \)

and

(ii) \( (A^\frac{r}{2} A^p A^\frac{r}{2})^\frac{1}{q} \geq (A^\frac{r}{2} B^p A^\frac{r}{2})^\frac{1}{q} \)

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$. 

\[ \begin{array}{c}
\text{Figure 1}
\end{array} \]
We remark that Theorem F yields L"owner-Heinz theorem ""\(A \geq B \geq 0\) ensures \(A^\alpha \geq B^\alpha\) for any \(\alpha \in [0, 1]\)" when we put \(r = 0\) in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [10][23] and also an elementary one-page proof in [13]. It is shown in [25] that the domain drawn for \(p, q\) and \(r\) in Figure 1 is the best possible for Theorem F.

1.2 Aluthge transformation of \(p\)-hyponormal and log-hyponormal operators

An operator \(T\) is said to be \(p\)-hyponormal for \(p > 0\) if \((T^*T)^p \geq (TT^*)^p\), and \(T\) is said to be log-hyponormal if \(T\) is invertible and \(\log T^*T \geq \log TT^*\). \(p\)-Hyponormality and log-hyponormality were defined as extensions of hyponormality, that is, \(T^*T \geq TT^*\). It is easily seen that every \(q\)-hyponormal operator is \(p\)-hyponormal for \(q \geq p > 0\) by L"owner-Heinz theorem, and every invertible \(p\)-hyponormal operator for some \(p > 0\) is log-hyponormal since \(\log t\) is an operator monotone function. We remark that \(p\)-hyponormality tends to log-hyponormality as \(p \to +0\) since \(\frac{X^p-I}{p} \to \log X\) as \(p \to +0\) for every positive operator \(X\).

The operator \(\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}\) is called Aluthge transformation of an operator \(T\) whose polar decomposition is \(T = U|T|\), where \(|T| = (T^*T)^{\frac{1}{2}}\). Aluthge transformation was first introduced by Aluthge [1], and he showed the following result on Aluthge transformation of \(p\)-hyponormal operators as an application of Theorem F.

**Theorem A ([1]).** Let \(T = U|T|\) be the polar decomposition of a \(p\)-hyponormal operator for \(0 < p < 1\) and \(U\) be unitary. Then

(i) \(\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}\) is \((p + \frac{1}{2})\)-hyponormal if \(0 < p \leq \frac{1}{2}\).

(ii) \(\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}\) is hyponormal if \(\frac{1}{2} \leq p < 1\).

We remark that \(\sigma(\tilde{T}) = \sigma(T)\) holds for any operator \(T\) [4][7], and Theorem A states that \(\tilde{T}\) belongs to a smaller class than a \(p\)-hyponormal operator \(T\) for \(0 < p < 1\).
A generalization of Aluthge transformation of an operator $T = U|T|$ is
\[ \tilde{T}_{s,t} = |T|^s U |T|^t \]
for $s > 0$ and $t > 0$. In fact, it is clear that $\tilde{T}_{1,1} = \tilde{T}$. Huruya [19] and Yoshino [29] showed an extension of Theorem A on generalized Aluthge transformation of $p$-hyponormal operators. Tanahashi [26] showed a parallel result on generalized Aluthge transformation of log-hyponormal operators.

### 1.3 Classes of operators associated with Aluthge transformation

Recently, Aluthge and Wang introduced the class of $w$-hyponormal operators via Aluthge transformation $\tilde{T}$ in [4], and showed an equivalent condition to $w$-hyponormality in [5].

**Definition ([4][5]).**

\[ T : \text{w-hyponormal} \iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \]
\[ \iff (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \text{ and } |T| \geq (|T|^{\frac{1}{2}}|T^*||\tau|^{\frac{1}{2}})^{\frac{1}{2}}, \]

where $\tilde{T}$ is Aluthge transformation of $T$.

As a generalization of the class of $w$-hyponormal operators, Ito [20] introduced class $wA(s, t)$ for $s > 0$ and $t > 0$ via generalized Aluthge transformation $\tilde{T}_{s,t}$. In fact, it is clear that class $wA(\frac{1}{2}, \frac{1}{2})$ coincides with the class of $w$-hyponormal operators.

**Definition ([20]).** For $s > 0$ and $t > 0$,

\[ T \in \text{class } wA(s, t) \iff |\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T^s|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2t}{s+t}} \]
\[ \iff (|T^s|^t|T|^2s|T|^t|)^{\frac{1}{t+s}} \geq |T^s|^{2t} \text{ and } |T^s|^2s \geq (|T|^s|T^s|2t|T|^s)^{\frac{1}{s+t}}, \]

where $\tilde{T}_{s,t}$ is generalized Aluthge transformation of $T$. For the sake of convenience, we call class $wA(1, 1)$ class $wA$ for short.

He also pointed out the following fact.

**Proposition B ([20]).** $T \in \text{class } wA \iff |T|^2 \geq |T|^2 \text{ and } |T^*|^2 \geq |T^2|^2$. 
1.4 Related classes and their inclusion relations

On the other hand, Furuta, Ito and Yamazaki [15] introduced a class of operators called class A.

**Definition ([15]).** \( T \in \text{class A} \iff |T^2| \geq |T|^2 \).

They showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal (\( \iff \|T^2x\| \geq \|Tx\|^2 \) for every unit vector \( x \)). This relations give another proof of the result by Ando [6].

As a generalization of class A, Fujii, D.Jung, S.H.Lee, M.Y.Lee and Nakamoto [11] introduced class \( A(s, t) \) for \( s > 0 \) and \( t > 0 \). In fact, it was pointed out in [28] that class \( A(1, 1) \) coincides with class A.

**Definition ([11]).** For \( s > 0 \) and \( t > 0 \),

(i) \( T \in \text{class A}(s, t) \iff (|T^*|^t|T^{2s}|T^*|^t)^{\frac{s}{s+t}} \geq |T^*|^{2t} \).

(ii) \( T \in \text{class AI}(s, t) \iff T \in \text{class A}(s, t) \) and \( T \) is invertible.

We remark the following inclusion relations:

(\( \bullet \)) \( \text{class A}(s, t) \supseteq \text{class } wA(s, t) \supseteq \text{class AI}(s, t) \)

holds for each \( s > 0 \) and \( t > 0 \). The first relation of (\( \bullet \)) holds obviously, and the second holds by the following lemma.

**Lemma F ([14]).** Let \( A > 0 \) and \( B \) be an invertible operator. Then

\[
(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*
\]

holds for any real number \( \lambda \).

In fact, the first inequality in the definition of class \( wA(s, t) \) yields the second by applying Lemma F in case \( T \) is invertible as follows:

\[
\left( |T|^s|T^*|^{2t}|T|^s \right)^{\frac{1}{s+t}} \\
= |T|^s|T^*|^t\left( |T^*|^t|T|^{2s}|T^*|^t \right)^{\frac{1}{s+t}}|T^*|^t|T|^s \\
\leq |T|^s|T^*|^t |T^*|^{-2t} |T|^s \\
= |T|^{2s}.
\]
We also remark the following results.

**Theorem C.1 ([20]).**

(i) *If an operator $T$ is $p$-hyponormal for some $p > 0$ or log-hyponormal, then $T$ belongs to class $wA(s, t)$ for all $s > 0$ and $t > 0.*

(ii) *Every class $wA(s_1, t_1)$ operator belongs to class $wA(s_2, t_2)$ for each $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2.*

**Theorem C.2 ([11]).**

(i) *An operator $T$ is log-hyponormal if and only if $T$ belongs to class $AI(s, t)$ for all $s > 0$ and $t > 0.*

(ii) *Every class $A(s, t_1)$ operator belongs to class $A(s, t_2)$ for each $0 < t_1 \leq t_2.*

The following diagram shows the inclusion relations among the classes of operators mentioned above.

![Diagram showing the inclusion relations among classes of operators](image-url)
1.5 Results on powers of non-normal operators

Recently, Aluthge and Wang showed results on powers of $p$-hyponormal and log-hyponormal operators in [2][3]. Extensions of the results were shown by Furuta and Yanagida [16][17], Ito [22] and Yamazaki [27].

As continuation of this study, Aluthge and Wang [5] showed the following result on powers of invertible $w$-hyponormal operators. A simplified proof of Theorem D.1 was given by Y.O.Kim [24].

**Theorem D.1 ([5]).** Let $T$ be an invertible $w$-hyponormal operator. Then $T^2$ is also $w$-hyponormal.

Cho, Huruya and Y.O.Kim [8] showed the following result which states that Theorem D.1 remains valid with a weaker condition $N(T) = \{0\}$ than the invertibility of $T$.

**Theorem D.2 ([8]).** Let $T$ be a $w$-hyponormal operator with $N(T) = \{0\}$. Then $T^2$ is also $w$-hyponormal.

On the other hand, Ito [21] showed the following result on powers of invertible class A operators.

**Theorem D.3 ([21]).** Let $T$ be an invertible class A operator. Then the following assertions hold for all positive integer $n$:

(i) $|T^{n+1}|_{\frac{2n}{n+1}} \geq |T^n|^2$ and $|T^{n*}|^2 \geq |T^{n+1*}|_{\frac{2n}{n+1}}$.

(ii) $|T^n|^\frac{2}{n} \geq \cdots \geq |T^2|^2 \geq |T|^2$ and $|T^{*}|^2 \geq |T^{2^{*}}| \geq \cdots \geq |T^{n*}|^\frac{2}{n}$.

(iii) $|T^{2n}| \geq |T^n|^2$ and $|T^{n*}|^2 \geq |T^{2n^{*}}|$, i.e., $T^n$ also belongs to class A.

As an extension of both Theorem D.1 and (iii) of Theorem D.3, Yamazaki [28] showed the following result on powers of class $\text{AI}(s,t)$ operators.

**Theorem D.4 ([28]).** Let $T$ be a class $\text{AI}(s,t)$ operator for $s \in (0,1]$ and $t \in (0,1]$. Then $T^n$ belongs to $\text{AI}(\frac{s}{n}, \frac{t}{n})$ for all positive integer $n$. 
In fact, Theorem D.4 yields Theorem D.1 by putting \( s = t = \frac{1}{2} \) and \( n = 2 \) since class \( \text{AI}(\frac{1}{4}, \frac{1}{4}) \subseteq \text{AI}(\frac{1}{2}, \frac{1}{2}) \) by (ii) of Theorem C.1. Theorem D.4 also yields (iii) of Theorem D.3 by putting \( s = t = 1 \) since class \( \text{Ai}(\frac{1}{n}, \frac{1}{n}) \subseteq \text{A}(1, 1) \) by (ii) of Theorem C.1. It is interesting to remark that Theorem D.4 states that \( T^n \) belongs to a smaller class than a class \( \text{AI}(s, t) \) operator \( T \) for \( s \in (0, 1] \) and \( t \in (0, 1] \).

In this report, we shall show several results on powers of class \( wA(s, t) \) operators as extensions of the results on powers of class \( \text{AI}(s, t) \) operators and \( w \)-hyponormal operators mentioned above.

2 Results

Firstly, we show the following result on powers of class \( wA \) operators.

**Theorem 1.** Let \( T \) be a class \( wA \) operator. Then the following assertions hold for all positive integer \( n \):

(i) \( |T^{n+1}|_{n+1}^{\frac{2n}{n+1}} \geq |T^n|^2 \) and \( |T^{n*}|^2 \geq |T^{n+1*}|_{n+1}^{\frac{2n}{n+1}} \).

(ii) \( |T^n|_{n}^2 \geq \cdots \geq |T^2| \geq |T| \) and \( |T^{*}|^2 \geq |T^{2*}| \geq \cdots \geq |T^{n*}|_{n}^2 \).

Secondly, we show the following result on powers of class \( wA(s, t) \) operators.

**Theorem 2.** Let \( T \) be a class \( wA(s, t) \) operator for \( s \in (0, 1] \) and \( t \in (0, 1] \). Then \( T^n \) belongs to \( wA(\frac{s}{n}, \frac{t}{n}) \) for all positive integer \( n \).

Theorem 1 and Theorem 2 are extensions of Theorem D.3 and Theorem D.4, respectively, since every class \( \text{AI}(s, t) \) operator belongs to class \( wA(s, t) \) by (\( \bullet \)). In other words, Theorem 1 and Theorem 2 state that Theorem D.3 and Theorem D.4 remain valid for class \( wA \) and class \( wA(s, t) \) operators without the invertibility of \( T \), respectively.

Theorem 2 yields the following result as an immediate corollary which is an extension of Theorem D.2.

**Corollary 3.** Let \( T \) be a \( w \)-hyponormal operator. Then \( T^n \) is also \( w \)-hyponormal for all positive integer \( n \).
3 Proofs of the results

In order to give a proof of Theorem 1, we prepare the following results.

Proposition 4. Let $A$ and $B$ be positive operators. Then the following assertions hold:

(i) If $(B^\frac{\alpha_0}{\alpha_0+\beta_0} A^{\alpha_0} B^\frac{\beta_0}{\alpha_0+\beta_0}) \geq B^{\beta_0}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[
(B^\frac{\beta}{\alpha_0+\beta} A^{\alpha_0} B^\frac{\beta}{\alpha_0+\beta}) \geq B^{\beta}
\]

holds for any $\beta \geq \beta_0$, and

\[
A^{\frac{\alpha_0}{\alpha_0+\beta_0}} B^\beta A^{\frac{\alpha_0}{\alpha_0+\beta_0}} \geq (A^{\frac{\alpha}{\alpha+\beta}} B^{\beta_0} A^{\frac{\alpha_0}{\alpha_0+\beta_0}})^{\frac{\alpha}{\alpha+\beta_0}}
\]

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$.

(ii) If $A^{\alpha_0} \geq (A^{\frac{\alpha}{\alpha+r}} B^{\beta_0} A^{\frac{\alpha}{\alpha+r}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[
A^{\alpha} \geq (A^{\frac{\alpha}{\alpha+r}} B^{\beta_0} A^{\frac{\alpha}{\alpha+r}})^{\frac{\alpha}{\alpha+r}}
\]

holds for any $\alpha \geq \alpha_0$, and

\[
(B^{\frac{\alpha_0}{\alpha_0+\beta_0}} A^{\alpha_2} B^{\frac{\beta_0}{\alpha_0+\beta_0}})^{\frac{\alpha_1+\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0} A^{\frac{\alpha_1}{\alpha_1}} B^{\frac{\alpha_2}{\alpha_2}}
\]

holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Lemma 5. Let $A$, $B$ and $C$ be positive operators. Then the following assertions holds for each $p \geq 0$ and $r \in (0, 1]$:

(i) If $(B^{\frac{\alpha}{\alpha+r}} A^{p} B^{\frac{\beta}{\beta+r}})^{\frac{p+r}{p+r}} \geq B^{r}$ and $B \geq C$, then $(C^{\frac{\alpha}{\alpha+r}} A^{p} C^{\frac{\beta}{\beta+r}})^{\frac{p+r}{p+r}} \geq C^{r}$.

(ii) If $A \geq B$, $B^{r} \geq (B^{\frac{\alpha}{\alpha+r}} C^{p} B^{\frac{\beta}{\beta+r}})^{\frac{p+r}{p+r}}$ and the condition

(*) if \( \lim_{n \to \infty} B^{\frac{1}{2}} x_n = 0 \) and \( \lim_{n \to \infty} A^{\frac{1}{2}} x_n \) exists, then \( \lim_{n \to \infty} A^{\frac{1}{2}} x_n = 0 \) hold, then $A^{r} \geq (A^{\frac{\alpha}{\alpha}} C^{p} A^{\frac{\beta}{\beta}})^{\frac{p+r}{p+r}}$. 
Proof of Proposition 4.

Proof of (i). Put $A_1 = (B^2 A^\alpha B^\beta A^\alpha + \beta_0)^{\frac{\beta_0}{\alpha_0 + \beta_0}}$ and $B_1 = B^\beta_0$, then $A_1 \geq B_1 \geq 0$ by the hypothesis. By applying (i) of Theorem F to $A_1$ and $B_1$, we have

\[(3.3) \quad (B^\frac{r_1}{2} A^{p_1} B^\frac{r_1}{2})^{\frac{1+\rho_1}{p_1+\rho_1}} \geq B_1^{1+\rho_1} \text{ for any } p_1 \geq 1 \text{ and } \rho_1 \geq 0.\]

Put $p_1 = \frac{\alpha_0 + \beta_0}{\beta_0} \geq 1$ and $\beta = (1 + \rho_1)\beta_0 \geq \beta_0$ in (3.3), then we have

\[(3.1) \quad (B^\frac{\beta}{2} A^\alpha B^\beta A^\alpha + \beta_0)^{\frac{\beta}{\alpha_0 + \beta}} \geq B^\beta \text{ for any } \beta \geq \beta_0.\]

By applying L"owner-Heinz theorem to (3.1), we have

\[(3.4) \quad (B^\frac{\beta}{2} A^\alpha B^\beta A^\alpha + \beta_0)^{\frac{v}{\alpha_0 + \beta}} \geq B^v \text{ for any } \beta \geq \beta_0 \text{ and } v \text{ such that } \beta \geq v \geq 0.\]

Put $f_{\beta_1}(\beta) = (A^\alpha B^\beta A^\alpha + \beta_0)^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta}}$. For any $\beta$, $\beta_1$ and $v$ such that $\beta \geq \beta_1 \geq \beta_0$ and $\beta \geq v \geq 0$, we have

\[
f_{\beta_1}(\beta) = (A^\alpha B^\beta A^\alpha + \beta_0)^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta}}
\]
\[
= \{(A^\alpha B^\beta A^\alpha)^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta}}\}^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta + v}}
\]
\[
= \{A^\alpha B^\beta A^\alpha\}^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta + v}}
\]
\[
\geq \{A^\alpha B^\beta A^\alpha\}^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta + v}}
\]
\[
= (A^\alpha B^\beta + \beta_0 A^\alpha)^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta + v}}
\]
\[
= f_{\beta_1}(\beta + v).
\]

The above inequality holds by (3.4) and L"owner-Heinz theorem since $\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta + v} \in [0, 1]$. Therefore for each $\beta_1 \geq \beta_0$, $f_{\beta_1}(\beta)$ is decreasing for $\beta \geq \beta_1$, so that

\[A^\alpha B^\beta A^\alpha \geq f_{\beta_1}(\beta_1) \geq f_{\beta_1}(\beta_2) = (A^\alpha B^\beta A^\alpha)^{\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta_2}}\]

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$, hence we have (3.2).

(ii) can be proved in the same way as (i), so that we omit the proof. \[\square\]
Lemma 5 can be obtained as an application of the following results.

**Theorem E.1 ([9])**. Let $A$ and $B$ be bounded linear operators on a Hilbert space $H$. The following statements are equivalent;

1. $R(A) \subseteq R(B)$;
2. $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and
3. there exists a bounded linear operator $C$ on $H$ so that $A = BC$.

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator $C$ so that

(a) $\|C\|^2 = \inf \{\mu | AA^* \leq \mu BB^*\}$;
(b) $N(A) = N(C)$; and
(c) $R(C) \subseteq \overline{R(B^*)}$.

**Theorem E.2 ([18])**. Let $X$ and $A$ be bounded linear operators on a Hilbert space $H$. We suppose that $X \geq 0$ and $\|A\| \leq 1$. If $f$ is an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$, then

$$A^* f(X) A \leq f(A^* X A).$$

We remark that the condition (c) in Theorem E.1 is equivalent to the condition $(c') \overline{R(C)} \subseteq \overline{R(B^*)}$. Here we consider when the equality of $(c')$ holds.

**Lemma 6.** Let $A$ and $B$ be operators which satisfy (1), (2) and (3) of Theorem E.1, and $C$ be the operator which is given in (3) and determined uniquely by (a), (b) and (c) of Theorem E.1. Then the following assertions are mutually equivalent:

(i) $\overline{R(C)} = \overline{R(B^*)}$.

(ii) If $\lim_{n \to \infty} A^* x_n = 0$ and $\lim_{n \to \infty} B^* x_n$ exists, then $\lim_{n \to \infty} B^* x_n = 0$. 
Proof. (i) is equivalent to \(N(C^*) = N(B)\) and
\[
N(C^*) = N(B) \oplus (N(B)^\perp \cap N(C^*)) = N(B) \oplus (\overline{R(B^*)} \cap N(C^*))
\]
since \(N(C^*) \supseteq N(B)\) by (c) of Theorem E.1, so that (i) is equivalent to the following (3.5):

(3.5) \(\overline{R(B^*)} \cap N(C^*) = \{0\}\).

Noting that when \(y = \lim_{n \to \infty} B^* x_n\) for some \(\{x_n\} \subseteq H\),
\[
C^* y = C^* \left( \lim_{n \to \infty} B^* x_n \right) = \lim_{n \to \infty} C^* B^* x_n = \lim_{n \to \infty} A^* x_n
\]
holds by (3) of Theorem E.1, so that we have
\[
\overline{R(B^*)} \cap N(C^*) = \{y \mid \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{n \to \infty} B^* x_n \text{ and } C^* y = 0\} = \{y \mid \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{n \to \infty} B^* x_n \text{ and } \lim_{n \to \infty} A^* x_n = 0\},
\]
hence (3.5) is equivalent to (ii). \(\square\)

We also require the following lemma in order to give a proof of Lemma 5.

**Lemma 7.** Let \(S\) be a positive operator and \(\alpha \in (0, 1]\). If \(\lim_{n \to \infty} S x_n = 0\) and \(\lim_{n \to \infty} S^\alpha x_n\) exists, then \(\lim_{n \to \infty} S^\alpha x_n = 0\).

**Proof.** \(\lim_{n \to \infty} S^\alpha x_n \in \overline{R(S^\alpha)} \cap N(S^{1-\alpha}) = \overline{R(S)} \cap N(S) = \{0\}\) for \(\alpha \in (0, 1)\) since \(S^{1-\alpha} \left( \lim_{n \to \infty} S^\alpha x_n \right) = \lim_{n \to \infty} S x_n = 0\) by the hypothesis. \(\square\)

**Proof of Lemma 5.**

**Proof of (i).** \(B \geq C\) ensures \(B^r \geq C^r\) for \(r \in (0, 1]\) by Löwner-Heinz theorem. By Theorem E.1, there exists an operator \(X\) such that

(3.6) \(B^{\frac{r}{2}} X = X^* B^{\frac{r}{2}} = C^{\frac{r}{2}}\),

(3.7) \(\|X\| \leq 1\).
Then we have
\[(C^\frac{r}{2}A^pC^\frac{r}{2})^{\frac{r}{p+r}} = (X^*B^\frac{r}{2}A^pB^\frac{r}{2}X)^{\frac{r}{p+r}} \quad \text{by (3.6)}\]
\[\geq X^*(B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{r}{p+r}}X \quad \text{by Theorem E.2 and (3.7)}
\[\geq X^*B^rX \quad \text{by the hypothesis}
\[= C^r \quad \text{by (3.6).}\]

**Proof of (ii).** \(A \geq B\) ensures \(A^r \geq B^r\) for \(r \in (0, 1]\) by Löwner-Heinz theorem. By Theorem E.1, there exists an operator \(Y\) such that
\[(3.8) \quad A^\frac{r}{2}Y = Y^*A^\frac{r}{2} = B^\frac{r}{2},\]
\[(3.9) \quad \|Y\| \leq 1.\]

Then we have
\[Y^*(A^\frac{r}{2}C^pA^\frac{r}{2})^{\frac{r}{p+r}}Y \leq (Y^*A^\frac{r}{2}C^pA^\frac{r}{2}Y)^{\frac{r}{p+r}} \quad \text{by Theorem E.2 and (3.9)}
\[= (B^\frac{r}{2}C^pB^\frac{r}{2})^{\frac{r}{p+r}} \quad \text{by (3.8)}
\[\leq B^r \quad \text{by the hypothesis}
\[= Y^*A^rY \quad \text{by (3.8)},\]
so that \(A^r \geq (A^\frac{r}{2}C^pA^\frac{r}{2})^{\frac{r}{p+r}}\) holds on \(\overline{R(Y)}\). On the other hand, (*) implies the following condition:

\[(**) \quad \text{if } \lim_{n \to \infty} B^\frac{r}{2}x_n = 0 \text{ and } \lim_{n \to \infty} A^\frac{r}{2}x_n \text{ exists, then } \lim_{n \to \infty} A^\frac{r}{2}x_n = 0\]
since if \(\lim_{n \to \infty} B^\frac{r}{2}x_n = 0\) and \(\lim_{n \to \infty} A^\frac{r}{2}x_n\) exists, then
\[\lim_{n \to \infty} B^\frac{1}{2}x_n = B^\frac{1-r}{2} \left( \lim_{n \to \infty} A^\frac{r}{2}x_n \right) = 0\]
and \(\lim_{n \to \infty} A^\frac{1}{2}x_n = A^\frac{1-r}{2} \left( \lim_{n \to \infty} A^\frac{r}{2}x_n \right)\) exists, so that \(\lim_{n \to \infty} A^\frac{1}{2}x_n = 0\) by (*), and \(\lim_{n \to \infty} A^\frac{r}{2}x_n = 0\) by Lemma 7. (**) ensures \(\overline{R(Y)} = \overline{R(A^\frac{r}{2})}\) by Lemma 6, hence we have
\[N((A^\frac{r}{2}C^pA^\frac{r}{2})^{\frac{r}{p+r}}) = N(A^\frac{r}{2}C^pA^\frac{r}{2}) \supseteq N(A^\frac{r}{2}) = N(A^r) = N(Y^*),\]
so that \(A^r = (A^\frac{r}{2}C^pA^\frac{r}{2})^{\frac{r}{p+r}} = 0\) on \(N(Y^*)\). Consequently the proof is complete since \(H = \overline{R(Y)} \oplus N(Y^*)\). \(\square\)
**Proof of Theorem 1.** Put $A_n = |T^n|^\frac{2}{n}$ and $B_n = |T^{n*}|^\frac{2}{n}$ for each integer $n$.

By the definition, $T$ belongs to class $wA$ if and only if

\[(3.10) \quad (B_1^\frac{1}{2}A_1B_1^\frac{1}{2})^\frac{1}{2} = (|T^*||T|^2|T^*|)^\frac{1}{2} \geq |T^*|^2 = B_1\]

and

\[(3.11) \quad A_1 = |T|^2 \geq (|T||T^*|^2|T|)^\frac{1}{2} = (A_1^\frac{1}{12}B_1A_1^\frac{1}{12})^\frac{1}{2}.

We shall prove

\[(3.12) \quad A_{n+1}^n = |T^{n+1}|^\frac{2n}{n+1} \geq |T^n|^2 = A_n^n\]

and

\[(3.13) \quad B_n^n = |T^{n*}|^2 = |T^{n+1*}|^\frac{2n}{n+1} = B_{n+1}^n\]

hold for all positive integer $n$ by induction. (3.12) and (3.13) hold for $n = 1$ by Proposition B. Assume (3.12) holds for $n = 1, 2, \cdots, k - 1$. Then $A_{n+1} \geq A_n$ holds by Löwner-Heinz theorem for $\frac{1}{n} \in [0, 1]$, so that we have

\[(3.14) \quad A_k \geq A_{k-1} \geq \cdots \geq A_2 \geq A_1.

We remark that $A_1$ and $A_k$ satisfy the condition

\[\text{(★) if } \lim_{n \to \infty} A_1^\frac{1}{2}x_n = 0 \text{ and } \lim_{n \to \infty} A_k^\frac{1}{2}x_n \text{ exists, then } \lim_{n \to \infty} A_k^\frac{1}{2}x_n = 0 \]

since

\[\lim_{n \to \infty} A_1^\frac{1}{2}x_n = 0 \iff \lim_{n \to \infty} |T|x_n = 0 \iff \lim_{n \to \infty} Tx_n = 0 \iff \lim_{n \to \infty} T^k x_n = 0 \]

\[\iff \lim_{n \to \infty} |T^k|x_n = 0 \iff \lim_{n \to \infty} A_k^\frac{1}{2}x_n = 0 \iff \lim_{n \to \infty} A_k^\frac{1}{2}x_n = 0.\]

The last implication holds by Lemma 7. By applying (ii) of Lemma 5 to (3.11) and (3.14), we have

\[(3.15) \quad A_k \geq (A_k^\frac{1}{2}B_1A_k^\frac{1}{2})^\frac{1}{2}.\]
By applying (ii) of Proposition 4 to (3.15),

\[(B_{1}^{1/2}A_{k}^{\alpha_{2}}B_{1}^{1/2})^{\alpha_{2}+1/(\alpha_{2}+1)} \geq B_{1}^{1/2}A_{k}^{\alpha_{1}}B_{1}^{1/2}\]

holds for any \(\alpha_{1}\) and \(\alpha_{2}\) such that \(\alpha_{2} \geq \alpha_{1} \geq 1\), so that we have

\[(B_{1}^{1/2}A_{k}^{1/2}B_{1}^{1/2})^{k/(k+1)} \geq B_{1}^{1/2}A_{k}^{k-1}B_{1}^{1/2} \geq B_{1}^{1/2}A_{k-1}^{k-1}B_{1}^{1/2},\]

since the first inequality is obtained by putting \(\alpha_{1} = k - 1\) and \(\alpha_{2} = k\) in (3.16), and the second holds since (3.12) holds for \(n = k - 1\) by the inductive assumption. (3.17) yields the following (3.18):

\[(|T^{*}||T^{k}|^{2}|T^{*}|)^{k/(k+1)} \geq |T^{*}|^{2}|T^{k-1}|^{2}|T^{*}|.\]

Let \(T = U|T|\) be the polar decomposition of \(T\), then \(T^{*} = U^{*}|T^{*}|\) is the polar decomposition of \(T^{*}\). Here we have

\[|T^{k+1}|^{2/(k+1)} = (T^{*}|T^{k}|^{2}T)^{k/(k+1)}\]
\[= (U^{*}|T^{*}|^{2/T^{k}}|T^{*}|^{2}U)^{k/(k+1)}\]
\[= U^{*}(|T^{*}|^{2/T^{k}}|T^{*}|)^{k/(k+1)}U\]
\[\geq U^{*}|T^{*}|^{2/T^{k-1}}|T^{*}|U \quad \text{by (3.18)}\]
\[= T^{*}|T^{k-1}|^{2}T\]
\[= |T^{k}|^{2},\]

so that it is proved that (3.12) holds for \(n = k\). (3.13) can be proved in the same way as (3.12), so that we omit the proof.

**Proof of (ii).** The first inequality of (ii) has been already proved in (3.14), and the second can be proved in the same way as the first. \(\square\)

**Proof of Theorem 2.** Put \(A_{n} = |T^{n}|^{2/\alpha_{n}}\) and \(B_{n} = |T^{n*}|^{2/\alpha_{n}}\) for each integer \(n\), then \(T\) belongs to class \(wA(s, t)\) if and only if

\[(B_{1}^{1/2}A_{1}^{s}B_{1}^{1/2})^{1/(s+t)} = (|T^{*}||T^{2}|^{s}T^{*}T)^{1/(s+t)} \geq |T^{*}|^{2t} = B_{1}^{t}\]

and

\[A_{1}^{s} = |T|^{2s} \geq (|T^{*}|^{2t}|T^{s}|)^{1/(s+t)} = (A_{1}^{s}B_{1}^{t}A_{1}^{s})^{1/(s+t)}\]
by the definition. Now $T$ belongs to class $wA$ since

$$\text{class } wA = \text{class } wA(1, 1) \supseteq \text{class } wA(s, t)$$

for $s \in (0, 1]$ and $t \in (0, 1]$ by (ii) of Theorem C.1, so that by (ii) of Theorem 1,

(3.21) \hspace{1cm} A_n \geq A_1

and

(3.22) \hspace{1cm} B_1 \geq B_n

hold for all positive integer $n$. Hence we have

(3.23) \hspace{1cm} A_n^s \geq (A_n^s B_1^t A_n^s)^{s \over s+t} \geq (A_n^s B_n^t A_n^s)^{s \over s+t}.

The first inequality in (3.23) is obtained by applying (ii) of Lemma 5 to (3.20) and (3.21) since $A_1$ and $A_n$ satisfy the condition

(\star) \hspace{1cm} \lim_{k \to \infty} A_1^{1 \over 2} x_k = 0 \text{ and } \lim_{k \to \infty} A_n^{1 \over 2} x_k \text{ exists, then } \lim_{k \to \infty} A_n^{1 \over 2} x_k = 0,

and the second holds by (3.22) and Löwner-Heinz theorem. (3.23) yields the following (3.24):

(3.24) \hspace{1cm} |T^n|^{2s \over n} \geq (|T^n|^{s \over n} |T^m|^{s \over n} |T^n|^{s \over n})^{s \over s+t}.

The following (3.25) can be obtained in the same way as (3.24):

(3.25) \hspace{1cm} (|T^m|^{|s \over n}|T^n|^{|2s \over n}|T^m|^{|s \over n})^{s \over s+t} \geq |T^m|^{|s \over n},

so that $T^n$ belongs to class $wA(s \over n, t \over n)$ by the definition.

Proof of Corollary 3. If $T$ belongs to class $wA(s \over 2n, t \over 2n)$, then $T^n$ belongs to class $wA(1 \over 2n, 1 \over 2n)$ by Theorem 2, so that $T^n$ belongs to class $wA(1 \over 2, 1 \over 2)$ by (ii) of Theorem C.1. Hence the proof is complete since class $wA(1 \over 2, 1 \over 2)$ coincides with the class of $w$-hyponormal operators.
4 Concluding remarks

Remark 1. \((B^\frac{\alpha_0}{\alpha_0+\beta_0} A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0}) \geq B^\beta_0\) and \(A^\alpha \geq (A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0} A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0}) \geq \overline{p} \geq B^\beta_0\) in the assumptions of (i) and (ii) of Proposition 4 are mutually equivalent in case both \(A\) and \(B\) are invertible. In fact, by applying Lemma F to the right-hand side of the second inequality, we have

\[
A^\alpha \geq (A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0} A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0}) \geq A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0} \geq B^\beta_0,
\]

so that the first inequality is obtained. But it is pointed out in [20] that they are not equivalent in general if either \(A\) or \(B\) are not invertible. In fact, \(A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) satisfy the second inequality, but do not satisfy the first.

Remark 2. Lemma 5 can be proved easily in case \(A\), \(B\) and \(C\) are invertible. In fact, (i) can be proved as follows: By Lemma F, \((B^\frac{\alpha_0}{\alpha_0+\beta_0} A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0}) \geq B^\beta_0\) and \((C^\frac{\alpha_0}{\alpha_0+\beta_0} A^\alpha C^\frac{\alpha_0}{\alpha_0+\beta_0}) \geq C^\beta_0\) are equivalent to \(A^\alpha \geq (A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0} A^\alpha B^\frac{\alpha_0}{\alpha_0+\beta_0}) \geq \overline{p} \geq B^\beta_0\), respectively, so that the first inequality implies the second by the assumption \(B \geq C\) and Löwner-Heinz theorem. (ii) can be proved similarly.

And one might expect that (ii) of Lemma 5 holds without the condition \((*)\). But there exists a counterexample. Put

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},
\]

then \(A \geq B\) and \(N(A) \subseteq N(B)\), so that \(A\) and \(B\) do not satisfy the condition \((*)\). And for each \(p > 0\) and \(r \in (0, 1]\),

\[
B^r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq \overline{p} = (B^\frac{r}{2} C^p B^\frac{r}{2}) \geq (B^\frac{r}{2} C^p B^\frac{r}{2}) \geq B^\beta_0,
\]

but

\[
A^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \not\geq \overline{pr} = (A^\frac{r}{2} C^p A^\frac{r}{2}) \geq (A^\frac{r}{2} C^p A^\frac{r}{2}) \geq B^\beta_0.
\]
References

[12] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq p + 2r$*, Proc. Amer. Math. Soc. 101 (1987), 85–88.