数理解析研究所講究録
1190 巻 2001 年 136-152

TWINING CHARACTER FORMULA FOR DEMAZURE MODULES

筑波大学・数学系　内藤 聡 (SATOSHI NAITO)
INSTITUTE OF MATHEMATICS,
UNIVERSITY OF TSUKUBA

0. Introduction.

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose the set of positive roots Δ_+ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j\in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I\rangle \subset GL(\mathfrak{h}^*)$ the Weyl group. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$, and let $\mathfrak{h}_{\mathbb{Z}} = \sum_{i\in I} \mathbb{Z}h_i$.

A bijection ω (of order N) of the index set I such that $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i, j \in I$ induces a unique automorphism ω, called a (Dynkin) diagram automorphism, of the Lie algebra $\mathfrak{g}$ such that $\omega(e_{\alpha_i}) = e_{\alpha_{\omega(i)}}$, $\omega(f_{\alpha_i}) = f_{\alpha_{\omega(i)}}$, and $\omega(h_i) = h_{\omega(i)}$ for $i \in I$. We denote by $\langle\omega\rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(\mathfrak{g})$ generated by the diagram automorphism ω. The restriction of ω to $\mathfrak{h}$ induces a transposed map $\omega^*\colon \mathfrak{h}^* \to \mathfrak{h}^*$, which stabilizes the integral weight lattice $\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \quad \text{for all } i \in I\} \simeq \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. We set $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$, $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$, $W^\omega = \{w \in W \mid \omega^* w = w\omega^*\}$, $(\mathfrak{h}^*)^0 = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\} \simeq (\mathfrak{h}^0)^*$, and $(\mathfrak{h}^*_{\mathbb{Z}})^0 = \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid \omega^*(\lambda) = \lambda\}$.

Let $\widehat{\mathfrak{g}}$ be the orbit Lie algebra, which is the dual complex semi-simple Lie algebra of the fixed point (semi-simple) subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}$, i.e., a complex semi-simple Lie algebra with the opposite Dynkin diagram to that of $\mathfrak{g}^0$. Let $\widehat{\mathfrak{h}}$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{b}} \supset \widehat{\mathfrak{h}}$ the Borel subalgebra, and $\widehat{\Delta}_+ \subset \widehat{\mathfrak{h}}^*$ the set of positive roots chosen so that the roots of $\widehat{\mathfrak{b}}$ are $-\widehat{\Delta}_+$. Let $\{\widehat{\alpha}_i \mid i \in \widehat{I}\}$ be the set of simple roots in $\widehat{\Delta}_+$ and $\widehat{W} = \langle \widehat{r}_i \mid i \in \widehat{I}\rangle \subset GL(\widehat{\mathfrak{h}}^*)$ the Weyl group, where the index set $\widehat{I}$ is a set of representatives of the ω-orbits in I. It is known that there exist an isomorphism of groups $\Theta : \widehat{W} \to W^\omega$ and a $\mathbb{C}$-linear isomorphism $P_\omega\colon \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ such that if $P_\omega^*\colon \widehat{\mathfrak{h}}^* \xrightarrow{\sim} (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ is the transposed map of P_ω, then $\Theta(\widehat{w})|_{(\mathfrak{h}^*)^0} = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for all $\widehat{w} \in \widehat{W}$. We set $w_i = \Theta(\widehat{r}_i) \in W^\omega$ for $i \in \widehat{I}$. In particular, $(W^\omega, \{w_i \mid i \in \widehat{I}\})$ forms a Coxeter system.

For dominant $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$, let $L(\lambda)$ be the simple $\mathfrak{g}$-module of highest weight λ. It admits a unique $\mathbb{C}$-linear $\langle\omega\rangle$-action such that $\omega \cdot (xv) = \omega(x)(\omega \cdot v)$ for each $x \in \mathfrak{g}$, $v \in L(\lambda)$ and such that $\omega \cdot v_\lambda = v_\lambda$, where v_λ is a (nonzero) highest weight vector of $L(\lambda)$. So therefore does its dual module $L(\lambda)^* \simeq L(-w_0(\lambda))$ with w_0 the longest element in W. Let $\mathfrak{U}(\mathfrak{b})$ be the universal enveloping algebra of $\mathfrak{b}$, and for each $w \in W^\omega$, let $J_w(\lambda) = \mathfrak{U}(\mathfrak{b})v^*_{w(\lambda)} \subset L(\lambda)^*$ be Joseph's module of highest weight $-w(\lambda)$ in $L(\lambda)^*$, with $v^*_{w(\lambda)}$ a (nonzero) weight vector in $L(\lambda)^*$ of weight $-w(\lambda)$. Since $w \in W^\omega$, the weight vector $v^*_{w(\lambda)} \in L(\lambda)^*$ turns out to be fixed by the action of $\langle\omega\rangle$, and hence Joseph's module $J_w(\lambda) \subset L(\lambda)^*$ is $\langle\omega\rangle$-invariant. In the talk we will prove a formula of Demazure type for the twining character $\mathrm{ch}^\omega(J_w(\lambda))$ of $J_w(\lambda)$ defined by

$$\mathrm{ch}^\omega(J_w(\lambda)) = \sum_{\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^0} \mathrm{Tr}(\omega|_{J_w(\lambda)_\mu})\, e(\mu)$$

in the group algebra $\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ over $\mathbb{C}$ of $(\mathfrak{h}_{\mathbb{Z}}^*)^0$ with basis $e(\mu)$, $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$. As a corollary, we will find a striking relation:

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\Big(\mathrm{ch}\, \widehat{J}_{\widehat{w}}(\widehat{\lambda})\Big),$$

where $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$, $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$, and $\mathrm{ch}\, \widehat{J}_{\widehat{w}}(\widehat{\lambda}) \in \mathbb{C}[\widehat{\mathfrak{h}}^*]$ is the ordinary character of Joseph's module $\widehat{J}_{\widehat{w}}(\widehat{\lambda})$ of highest weight $-\widehat{w}(\widehat{\lambda})$ over the orbit Lie algebra $\widehat{\mathfrak{g}}$.

Although our problem can be stated purely algebraically as above, it seems very difficult (at least for me) to solve it only by algebraic methods. Hence we resort to (algebro-) geometric methods. For that purpose, we introduce more notation. Let G be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$ such that $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(T) = \mathfrak{h}$, and $\mathrm{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \mathrm{Hom}(T, GL_1)$ of T may be identified with $\mathfrak{h}_{\mathbb{Z}}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle \lambda, \alpha_i^\vee \rangle = (\mathrm{d}\lambda)(h_i)$, where $\alpha_i^\vee \in \mathrm{Hom}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. There exists an automorphism of G whose differential at the identity element coincides with the diagram automorphism ω of $\mathfrak{g}$ above. By abuse of notation, we will denote by ω this automorphism of G and by $\langle\omega\rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(G)$ generated by ω. We will also denote the induced action of $\omega \in \langle\omega\rangle$ on Λ by the same letter ω, and set $\Lambda^\omega = \{\lambda \in \Lambda \mid \omega \cdot \lambda = \lambda\}$, $\Lambda_+^\omega = \{\lambda \in \Lambda^\omega \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \quad \text{for all } i \in I\}$.

By a $G \rtimes \langle\omega\rangle$-module M, we will mean a finite-dimensional rational G-module that admits a $\mathbb{C}$-linear $\langle\omega\rangle$-action such that $\omega \cdot (g\, m) = \omega(g)(\omega \cdot m)$ for each $g \in G$ and $m \in M$. Regarding the semi-direct product $G \rtimes \langle\omega\rangle$ of G and $\langle\omega\rangle$ as a linear algebraic group, this is the same as a rational $G \rtimes \langle\omega\rangle$-module. Likewise for $B \rtimes \langle\omega\rangle$- and $T \rtimes \langle\omega\rangle$-modules. Let $\mathbb{C}[\Lambda^\omega]$ be the group algebra over $\mathbb{C}$ of Λ^ω with basis $e(\mu)$, $\mu \in \Lambda^\omega$. For a $T \rtimes \langle\omega\rangle$-module

V, we define the twining character $\mathrm{ch}^{\omega}(V) \in \mathbb{C}[\Lambda^{\omega}]$ of V to be

$$\mathrm{ch}^{\omega}(V) = \sum_{\mu \in \Lambda^{\omega}} \mathrm{Tr}(\omega|_{V_\mu})\, e(\mu),$$

where $V_\mu = \{v \in V \mid tv = \mu(t)v \quad \text{for all} \quad t \in T\}$ is the μ-weight space of V.

Recall that $W \simeq N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Fix $w \in W^{\omega}$, and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, which is the Zariski closure in the flag variety G/B of the Bruhat cell $B\dot{w}B/B$, where $\dot{w}$ denotes a right coset representative of w in $N_G(T)$ fixed by $\omega \in \mathrm{Aut}(G)$. If M is a $B \rtimes \langle\omega\rangle$-module, then the B-equivariant $\mathcal{O}_{X(w)}$-module $\mathcal{L}_{X(w)}(M)$ associated to M carries a structure of "$(B, \langle\omega\rangle)$-equivariant" ($\doteqdot B \rtimes \langle\omega\rangle$-equivariant) sheaf, so that its cohomology groups $H^{\bullet}(X(w), \mathcal{L}_{X(w)}(M))$ are $B \rtimes \langle\omega\rangle$-modules. (For the precise definition of a $(B, \langle\omega\rangle)$-equivariant sheaf, see our preprint [KN].)

For each $\lambda \in \Lambda^{\omega}$, we let $\mathbb{C}_\lambda$ denote the one-dimensional $B \rtimes \langle\omega\rangle$-module on which B acts via λ through the quotient $B \to T$ and $\langle\omega\rangle$ trivially. We call $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$ for $\lambda \in \Lambda^{\omega}_{+}$ a Demazure module. Joseph's module $J_w(\lambda)$ admits a structure of $B \rtimes \langle\omega\rangle$-module, and we have an isomorphism of $B \rtimes \langle\omega\rangle$-modules

$$J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)),$$

where $J_w(\lambda)^*$ is the dual $B \rtimes \langle\omega\rangle$-module of $J_w(\lambda)$.

For $i \in \widehat{I}$, we define the ω-Demazure operator $\widehat{D}_i$ to be a $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\Lambda^{\omega}]$ such that

$$\widehat{D}_i(e(\mu)) = \frac{e(\mu) - e(-s_i\beta_i)e(w_i(\mu))}{1 - e(-s_i\beta_i)} \quad \text{for } \mu \in \Lambda^{\omega},$$

where $\beta_i = \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}$ and $s_i = 2/\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)}$ with N_i the number of elements of the ω-orbit of $i \in I$.

The following is our main result.

THEOREM 0.1. *Let M be a finite-dimensional rational $B \rtimes \langle\omega\rangle$-module, $w \in W^{\omega}$, and let $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ be a reduced expression in the Coxeter system $(W^{\omega}, \{w_i \mid i \in \widehat{I}\})$. Then we have in $\mathbb{C}[\Lambda^{\omega}]$,*

$$\begin{aligned}\chi_w^{\omega}(M) &= \sum_{j \geq 0} (-1)^j \mathrm{ch}^{\omega}(H^j(X(w), \mathcal{L}_{X(w)}(M))) \\ &= \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n}\Big(\mathrm{ch}^{\omega}(M)\Big).\end{aligned}$$

In particular, for $\lambda \in \Lambda_+^\omega$, we have

$$\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n}(e(\lambda)).$$

There is thus revealed a striking relation between twining characters for $\mathfrak{g}$ and ordinary characters for the orbit Lie algebra $\widehat{\mathfrak{g}}$. Let $\widehat{\mathfrak{h}}_\mathbb{Z} = \sum_{i\in\widehat{I}} \mathbb{Z}\,\widehat{h}_i$ and $\widehat{\mathfrak{h}}_\mathbb{Z}^* = \mathrm{Hom}(\widehat{\mathfrak{h}}_\mathbb{Z}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^*$. For dominant $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, let $\widehat{L}(\widehat{\lambda})$ be the simple $\widehat{\mathfrak{g}}$-module of highest weight $\widehat{\lambda}$, and for each $\widehat{w} \in \widehat{W}$, let $\widehat{J}_{\widehat{w}}(\widehat{\lambda}) = \mathfrak{U}(\widehat{\mathfrak{b}})\,\widehat{v}^*_{\widehat{w}(\widehat{\lambda})} \subset \widehat{L}(\widehat{\lambda})^*$ be Joseph's module of highest weight $-\widehat{w}(\widehat{\lambda})$, with $\widehat{v}^*_{\widehat{w}(\widehat{\lambda})} \in \widehat{L}(\widehat{\lambda})^*$ a (nonzero) weight vector of weight $-\widehat{w}(\widehat{\lambda})$.

Corollary 0.2. *Let $\lambda \in (\mathfrak{h}_\mathbb{Z}^*)^0$ be dominant and $w \in W^\omega$. We set $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$. Then we have in $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\left(\mathrm{ch}\,\widehat{J}_{\widehat{w}}(\widehat{\lambda})\right),$$

where P_ω^* on the right-hand side is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*] \tilde{\to} \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ defined by $P_\omega^*(e(\widehat{\mu})) = e(P_\omega^*(\widehat{\mu}))$ for each basis element $e(\widehat{\mu})$, $\widehat{\mu} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, of the group algebra $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ over $\mathbb{C}$ of $\widehat{\mathfrak{h}}_\mathbb{Z}^*$.

1. Preliminaries.

1.1. **Diagram automorphisms.** Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose the set of positive roots Δ_+ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j\in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I\rangle \subset GL(\mathfrak{h}^*)$ the Weyl group. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$, and let $\mathfrak{h}_\mathbb{Z} = \sum_{i\in I} \mathbb{Z}h_i$.

We fix a bijection $\omega \colon I \to I$ of the index set I such that

$$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for all } i, j \in I.$$

Let N be the order of ω, and N_i the number of elements of the ω-orbit of $i \in I$. This ω can be extended in a unique way to an automorphism (also denoted by ω) of order N of the Lie algebra $\mathfrak{g}$ in such a way that

$$\begin{cases} \omega(e_{\alpha_i}) = e_{\alpha_{\omega(i)}}, & i \in I, \\ \omega(f_{\alpha_i}) = f_{\alpha_{\omega(i)}}, & i \in I, \\ \omega(h_i) = h_{\omega(i)}, & i \in I. \end{cases}$$

Note that the restriction of ω to the Cartan subalgebra $\mathfrak{h}$ induces a transposed map $\omega^*\colon \mathfrak{h}^* \to \mathfrak{h}^*$ such that $\omega^*(\lambda)(h) = \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$. We set

$$(\mathfrak{h}^*)^0 = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\} \quad \text{and} \quad (\mathfrak{h}^*_{\mathbb{Z}})^0 = \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid \omega^*(\lambda) = \lambda\},$$

where $\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\} \simeq \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. Note that the Weyl vector $\rho = (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha$ is in $(\mathfrak{h}^*_{\mathbb{Z}})^0$.

1.2. **Orbit Lie algebras.** We choose and fix a set $\widehat{I}$ of representatives of the ω-orbits in I, and set $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$, where $\widehat{a}_{ij}$ is given by

$$\widehat{a}_{ij} = s_j \times \sum_{k=0}^{N_j - 1} a_{i,\omega^k(j)} \quad \text{for } i, j \in \widehat{I} \quad \text{with} \quad s_j = \frac{2}{\sum_{k=0}^{N_j-1} a_{j,\omega^k(j)}} \quad \text{for } j \in \widehat{I}.$$

Set for each $i \in \widehat{I}$, $I_i = \{\omega^k(i) \mid 0 \le k \le N_i - 1\} \subset I$. We know that for each $i \in \widehat{I}$,

$$\sum_{k \in I_i} a_{ik} = 1 \text{ or } 2.$$

Moreover, there are only two possibilities:

(a) if $\sum_{k \in I_i} a_{ik} = 1$, then N_i is even and the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is of type $A_2 \times \cdots \times A_2$ (where A_2 appears $N_i/2$ times);
(b) if $\sum_{k \in I_i} a_{ik} = 2$, then the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is totally disconnected and of type $A_1 \times \cdots \times A_1$ (where A_1 appears N_i times).

The orbit Lie algebra associated to the diagram automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ is defined to be the complex semi-simple Lie algebra $\widehat{\mathfrak{g}}$ associated to the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ with the Cartan subalgebra $\widehat{\mathfrak{h}}$, the Borel subalgebra $\widehat{\mathfrak{b}} \supset \widehat{\mathfrak{h}}$, the set of positive roots $\widehat{\Delta}_+ \subset \widehat{\mathfrak{h}}^*$ chosen so that the roots of $\widehat{\mathfrak{b}}$ are $-\widehat{\Delta}_+$, the set of simple roots $\{\widehat{\alpha}_i \mid i \in \widehat{I}\} \subset \widehat{\mathfrak{h}}^*$, the set of simple coroots $\{\widehat{h}_i \mid i \in \widehat{I}\} \subset \widehat{\mathfrak{h}}$, and the Weyl group $\widehat{W} = \langle \widehat{r}_i \mid i \in \widehat{I} \rangle \subset GL(\widehat{\mathfrak{h}}^*)$.

Remark 1.2.1. We can easily deduce that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the dual complex semi-simple Lie algebra of the fixed point (semi-simple) subalgebra $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$ of $\mathfrak{g}$, i.e., a complex semi-simple Lie algebra which has the opposite Dynkin diagram to that of $\mathfrak{g}^0$.

We set $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$. Then there exists a linear isomorphism $P_\omega \colon \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ given by

$$P_\omega\left(\sum_{k\in I_i} h_k\right) = N_i \widehat{h}_i \quad \text{for each } i \in \widehat{I}.$$

This map $P_\omega \colon \mathfrak{h}^0 \tilde{\to} \widehat{\mathfrak{h}}$ induces a transposed map $P_\omega^* \colon \widehat{\mathfrak{h}}^* \tilde{\to} (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ such that

$$P_\omega^*(\widehat{\lambda})(h) = \widehat{\lambda}(P_\omega(h)) \quad \text{for } \widehat{\lambda} \in \widehat{\mathfrak{h}}^*,\ h \in \mathfrak{h}^0.$$

Note that, if $\widehat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{i\in\widehat{I}} \mathbb{Z}\,\widehat{h}_i$ and $\widehat{\mathfrak{h}}_{\mathbb{Z}}^* = \operatorname{Hom}(\widehat{\mathfrak{h}}_{\mathbb{Z}}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^*$, then $P_\omega^*(\widehat{\mathfrak{h}}_{\mathbb{Z}}^*) = (\mathfrak{h}_{\mathbb{Z}}^*)^0$.

We now define the subgroup W^ω of W by

$$W^\omega = \{w \in W \mid \omega^* w = w\omega^*\}.$$

It is known that there exists an isomorphism of groups $\Theta : \widehat{W} \to W^\omega$ from the Weyl group $\widehat{W}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ onto the group W^ω such that the following diagram commutes for each $\widehat{w} \in \widehat{W}$:

$$\begin{array}{ccc} \widehat{\mathfrak{h}}^* & \xrightarrow{P_\omega^*} & (\mathfrak{h}^*)^0 \\ {\scriptstyle \widehat{w}}\downarrow & & \downarrow{\scriptstyle \Theta(\widehat{w})|_{(\mathfrak{h}^*)^0}} \\ \widehat{\mathfrak{h}}^* & \xrightarrow[P_\omega^*]{} & (\mathfrak{h}^*)^0. \end{array}$$

For each $i \in \widehat{I}$, set $w_i = \Theta(\widehat{r}_i) \in W^\omega$. Explicitly,

$$w_i = \begin{cases} \displaystyle\prod_{k=0}^{N_i/2-1} \left(r_{\omega^k(i)}\, r_{\omega^{k+N_i/2}(i)}\, r_{\omega^k(i)}\right) & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \displaystyle\prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases}$$

Hence each w_i is the longest element of the subgroup W_{I_i} of the Weyl group W generated by the r_k's for $k \in I_i$. Furthermore, $(W^\omega, \{w_i \mid i \in \widehat{I}\})$ forms a Coxeter system as $(\widehat{W}, \{\widehat{r}_i \mid i \in \widehat{I}\})$ is. We will denote the length function of the Coxeter system $(W, \{r_i \mid i \in I\})$ (resp. $(W^\omega, \{w_i \mid i \in \widehat{I}\})$) by $\ell \colon W \to \mathbb{Z}_{\geq 0}$ (resp. $\widehat{\ell} \colon W^\omega \to \mathbb{Z}_{\geq 0}$).

Remark 1.2.2. Note that the longest element $w_0 \in W$ belongs to W^ω. In fact, we can easily show that the isomorphism $\Theta \colon \widehat{W} \tilde{\to} W^\omega$ maps the longest element $\widehat{w}_0 \in \widehat{W}$ to the longest element $w_0 \in W$.

1.3. **The ω-Demazure operators.** Recall the ordinary Demazure operator D_i for $i \in I$ on the group ring $\mathbb{Z}[\mathfrak{h}_{\mathbb{Z}}^*] = \coprod_{\lambda\in\mathfrak{h}_{\mathbb{Z}}^*} \mathbb{Z}\, e(\lambda)$:

$$D_i \colon e(\lambda) \mapsto \frac{e(\lambda) - e(-\alpha_i)e(r_i(\lambda))}{1 - e(-\alpha_i)}.$$

Let $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*]$ be the group algebra over $\mathbb{C}$ of $\widehat{\mathfrak{h}}_{\mathbb{Z}}^*$ with basis $e(\widehat{\lambda})$, $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*$. Define likewise the Demazure operator $D_{\widehat{r}_i}$, $i \in \widehat{I}$, on $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*]$ to be the $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*]$ given by

$$D_{\widehat{r}_i}(e(\widehat{\lambda})) = \frac{e(\widehat{\lambda}) - e(-\widehat{\alpha}_i)e(\widehat{r}_i(\widehat{\lambda}))}{1 - e(-\widehat{\alpha}_i)}.$$

Then transfer $D_{\widehat{r}_i}$ via P_ω^* onto the group algebra $\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ to define the ω-Demazure operator

$$\widehat{D}_i = P_\omega^* \circ D_{\widehat{r}_i} \circ (P_\omega^*)^{-1} \quad \text{for } i \in \widehat{I}. \tag{1.3.1}$$

Thus we can easily check the following.

LEMMA 1.3.1. *Let $i \in \widehat{I}$. For each $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$, we have*

$$\widehat{D}_i(e(\lambda)) = \frac{e(\lambda) - e(-s_i\beta_i)e(w_i(\lambda))}{1 - e(-s_i\beta_i)},$$

and moreover

$$\widehat{D}_i(e(\lambda)) = \begin{cases} e(\lambda) + e(\lambda - s_i\beta_i) + \cdots + e(w_i(\lambda)) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{if } \lambda(h_i) = -1, \\ -\Big(e(\lambda + s_i\beta_i) + e(\lambda + 2s_i\beta_i) + \cdots + e(w_i(\lambda + s_i\beta_i))\Big) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\leq -2}. \end{cases}$$

Remark 1.3.2. Let $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$. We set $\widehat{D}_w = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0])$, which does not depend on the choice of the reduced expression of $w \in W^\omega$.

1.4. **Twining characters.** Let G be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$ such that $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(T) = \mathfrak{h}$, and $\mathrm{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \mathrm{Hom}(T, GL_1)$ of T may be identified with $\mathfrak{h}_{\mathbb{Z}}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle \lambda, \alpha_i^\vee \rangle = (\mathrm{d}\lambda)(h_i)$, where $\alpha_i^\vee \in \mathrm{Hom}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. Let $\Lambda_+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \quad \text{for all } i \in I\}$ be the set of dominant weights of Λ.

There exists an automorphism of G whose differential at the identity element coincides with the diagram automorphism ω of $\mathfrak{g}$. By abuse of notation, we will denote still by ω this automorphism of G and by $\langle \omega \rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(G)$ generated by the ω. Whenever there can be ambiguity, we will write $\mathrm{d}\omega$ for the automorphism of $\mathfrak{g}$. Recall also that the Weyl group $W \subset GL(\mathfrak{h}^*)$ may be identified with $N_G(T)/T$, $N_G(T)$ the normalizer of T in G. Each $w \in W^\omega$ lifts to an element of $N_G(T)$ fixed by

$\omega \in \mathrm{Aut}(G)$, which will be denoted by $\dot{w}$. We will also denote the induced action of ω on Λ by the same letter ω, and set $\Lambda^\omega = \{\lambda \in \Lambda \mid \omega \cdot \lambda = \lambda\}$, $\Lambda_+^\omega = \Lambda^\omega \cap \Lambda_+$. Note that, under the identification $\Lambda \simeq \mathfrak{h}_\mathbb{Z}^* \subset \mathfrak{h}^*$, this action of ω on Λ coincides with the restriction of $((\mathrm{d}\omega)^{-1})^* = ((\mathrm{d}\omega)^*)^{-1}$ to $\mathfrak{h}_\mathbb{Z}^*$.

By a $G \rtimes \langle\omega\rangle$-module M, we will always mean a finite-dimensional rational G-module that admits a $\mathbb{C}$-linear $\langle\omega\rangle$-action such that

$$\omega \cdot (g\,m) = \omega(g)(\omega \cdot m) \quad \text{for all } g \in G,\ m \in M.$$

Regarding the semi-direct product $G \rtimes \langle\omega\rangle$ of G and $\langle\omega\rangle$ as a linear algebraic group, this is the same as a finite-dimensional rational $G \rtimes \langle\omega\rangle$-module. Likewise for $B \rtimes \langle\omega\rangle$- and $T \rtimes \langle\omega\rangle$-modules. Let $\mathbb{C}[\Lambda^\omega]$ be the group algebra over $\mathbb{C}$ of Λ^ω with basis $e(\lambda)$, $\lambda \in \Lambda^\omega$. Let M be a $T \rtimes \langle\omega\rangle$-module, and let

$$M = \coprod_{\lambda\in\Lambda} M_\lambda \quad \text{with} \quad M_\lambda = \{m \in M \mid t\,m = \lambda(t)m \quad \text{for all} \quad t \in T\}$$

be the weight space decomposition with respect to T. Now we define the twining character $\mathrm{ch}^\omega(M)$ of M to be

$$\mathrm{ch}^\omega(M) = \sum_{\lambda\in\Lambda^\omega} \mathrm{Tr}(\omega|_{M_\lambda})\, e(\lambda) \in \mathbb{C}[\Lambda^\omega].$$

Remark 1.4.1. It easily follows that for each $t \in T$,

$$\mathrm{Tr}((t,\omega)\ ;\ M) = \sum_{\lambda\in\Lambda^\omega} \mathrm{Tr}(\omega|_{M_\lambda})\, \lambda(t) \in \mathbb{C}$$

since $\omega \cdot M_\lambda = M_{\omega(\lambda)}$ for $\lambda \in \Lambda$.

1.5. **An important example.** Let $\lambda \in \Lambda_+^\omega$ and $L(\lambda)$ the simple rational G-module of highest weight λ. We can make $L(\lambda)$ into a $G \rtimes \langle\omega\rangle$-module as follows. Let v_λ be a (nonzero) highest weight vector of $L(\lambda)$. If $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, there is an isomorphism of $\mathfrak{U}(\mathfrak{g})$-modules

$$\mathfrak{U}(\mathfrak{g})/\mathfrak{J}(\lambda) \simeq L(\lambda) \quad \text{via } x \mapsto x\,v_\lambda,$$

where $\mathfrak{J}(\lambda)$ is the left ideal of $\mathfrak{U}(\mathfrak{g})$ given by

$$\mathfrak{J}(\lambda) = \sum_{i\in I} \Big(\mathfrak{U}(\mathfrak{g})e_i + \mathfrak{U}(\mathfrak{g})(h_i - \langle\lambda, \alpha_i^\vee\rangle) + \mathfrak{U}(\mathfrak{g})f_i^{\langle\lambda,\alpha_i^\vee\rangle+1}\Big).$$

Since $\lambda \in \Lambda^\omega$, the left ideal $\mathfrak{J}(\lambda)$ of $\mathfrak{U}(\mathfrak{g})$ is ω-invariant, i.e., $\mathrm{d}\omega$-invariant, and hence $L(\lambda)$ admits a structure of $\langle\omega\rangle$-module such that

$$\omega \cdot (x\,v_\lambda) = \Big((\mathrm{d}\omega)(x)\Big)v_\lambda \quad \text{for all } x \in \mathfrak{U}(\mathfrak{g}).$$

Therefore, the $L(\lambda)$ admits a structure of $G \rtimes \langle\omega\rangle$-module such that $\omega \cdot v_\lambda = v_\lambda$. Note that a $G \rtimes \langle\omega\rangle$-module structure on $L(\lambda)$ such that $\omega \cdot v_\lambda = v_\lambda$ is unique since $L(\lambda)$ is a cyclic G-module generated by v_λ.

On the other hand, for each $i \in \widehat{I}$, we have $(P_\omega^*)^{-1}(\lambda)(\widehat{h}_i) = \lambda(h_i)$. Hence $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$ is dominant integral. If $\widehat{L}(\widehat{\lambda})$ is the simple $\widehat{\mathfrak{g}}$-module of highest weight $\widehat{\lambda}$, we know that

$$\mathrm{ch}^\omega(L(\lambda)) = P_\omega^*\left(\mathrm{ch}\,\widehat{L}(\widehat{\lambda})\right), \tag{1.5.1}$$

where P_ω^* on the right-hand side is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*] \tilde{\rightarrow} \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ defined by

$$P_\omega^*(e(\widehat{\mu})) = e(P_\omega^*(\widehat{\mu})) \quad \text{for } \widehat{\mu} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*.$$

Assume now that $J = I_i = \{\omega^k(i) \mid 0 \le k \le N_i - 1\} \subset I$, $i \in \widehat{I}$, and let P_J be the standard parabolic subgroup of G associated to J. Let $\nu \in \Lambda^\omega$ with $\langle \nu, \alpha_i^\vee \rangle \ge 0$ (hence $\langle \nu, \alpha_j^\vee \rangle \ge 0$ for all $j \in J$). If $L_J(\nu)$ is the simple rational P_J-module of highest weight ν, then it remains simple as a rational module over the Levi factor L_J of P_J with the unipotent radical U_J of P_J acting trivially. We can make $L_J(\nu)$ into a $P_J \rtimes \langle\omega\rangle$-module in the same way as $L(\lambda)$ above.

The following lemma is a first (but important) step towards our main result (Theorem 0.1).

LEMMA 1.5.1. *With the notation and assumption as above, we have in* $\mathbb{C}[\Lambda^\omega]$,

$$\mathrm{ch}^\omega(L_J(\nu)) = \widehat{D}_i(e(\nu)).$$

2. Proof of the main result.

Since the proof of our main result is so simple and clear modulo some algebro-geometric arguments, we give a "detailed outline" of it in this section. Fix $w \in W^\omega$ and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, i.e., the Zariski closure of the Bruhat cell $B\dot{w}B/B$ in the flag variety G/B. For a $B \rtimes \langle\omega\rangle$-module M, the ω-Euler characteristic $\chi_w^\omega(M)$ is defined to be

$$\chi_w^\omega(M) = \sum_{j \ge 0} (-1)^j\, \mathrm{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) \in \mathbb{C}[\Lambda^\omega].$$

Here recall that, since M is a $B \rtimes \langle\omega\rangle$-module, the $\mathcal{O}_{X(w)}$-module $\mathcal{L}_{X(w)}(M)$ associated to M is a $(B, \langle\omega\rangle)$-equivariant ($\doteqdot B \rtimes \langle\omega\rangle$-equivariant) sheaf, and hence the cohomology groups $H^j(X(w), \mathcal{L}_{X(w)}(M))$, $j \ge 0$, are $B \rtimes \langle\omega\rangle$-modules.

Let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$. Note that we have $\ell(w) = \ell(w_{i_1}) + \cdots + \ell(w_{i_n})$. We want to show that

$$\chi_w^\omega(M) = \widehat{D}_{i_1} \cdots \widehat{D}_{i_n}(\mathrm{ch}^\omega(M)),$$

where $\widehat{D}_j$ for $j = i_1, \ldots, i_n$ is the ω-Demazure operator defined in §1.3. In particular, we will obtain a twining character formula of the Demazure module $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$ for $\lambda \in \Lambda_+^\omega$, where $\mathbb{C}_\lambda$ is the one-dimensional $B \rtimes \langle\omega\rangle$-module on which B acts by the weight λ through the quotient $B \to T$ and $\langle\omega\rangle$ trivially.

2.1. **Formula for the ω-Euler characteristics.** Set $\widehat{D}_w = \widehat{D}_{i_1} \cdots \widehat{D}_{i_n}$. Then we are to show

$$\chi_w^\omega(M) = \widehat{D}_w(\mathrm{ch}^\omega(M)). \tag{2.1.1}$$

Let us first make some reductions. Since both sides of (2.1.1) are additive in M, we may assume that M is one-dimensional of weight $\mu \in \Lambda^\omega$ on which ω is acting by a scalar ζ^k for a primitive N-th root of unity ζ in $\mathbb{C}$ and $k \in \mathbb{Z}$. We will denote such M by $\mathbb{C}_{\mu,k}$. Thus we are reduced to showing that

$$\chi_w^\omega(\mathbb{C}_{\mu,k}) = \widehat{D}_w(\mathrm{ch}^\omega(\mathbb{C}_{\mu,k})),$$

where $\mathrm{ch}^\omega(\mathbb{C}_{\mu,k}) = \zeta^k e(\mu)$.

Put for simplicity $z_j = w_{i_j}$, $1 \le j \le n$. We have an isomorphism of $B \rtimes \langle\omega\rangle$-modules

$$H^\bullet(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_{\mu,k})) \simeq H^\bullet(X(z_1, \ldots, z_n), \mathcal{L}_{X(z_1,\ldots,z_n)}(\mathbb{C}_{\mu,k})), \tag{2.1.2}$$

and for each s with $1 \le s \le n-1$, a $B \rtimes \langle\omega\rangle$-equivariant spectral sequence

$$H^i(X(z_s), \mathcal{L}(H^j(X(z_{s+1}, \ldots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k})))) \Rightarrow H^{i+j}(X(z_s, \ldots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k})). \tag{2.1.3}$$

Here $X(z_s, \ldots, z_t)$ for $1 \le s \le t \le n$ is the so-called Bott-Samelson variety, and $\mathcal{L}_{X(z_s,\ldots,z_t)}(\mathbb{C}_{\mu,k})$ is the sheaf of $\mathcal{O}_{X(z_s,\ldots,z_t)}$-modules associated to the $B \rtimes \langle\omega\rangle$-module $\mathbb{C}_{\mu,k}$. Note that, since $z_s, \ldots, z_t \in W^\omega$ and their right coset representatives $\dot{z}_s, \ldots, \dot{z}_t \in N_G(T)$ are fixed by $\omega \in \mathrm{Aut}(G)$, the Bott-Samelson variety $X(z_s, \ldots, z_t)$ is an $\langle\omega\rangle$-invariant subvariety of $(G/B)^{t-s+1}$. (The proofs of (2.1.2) and (2.1.3) are not so difficult, but rather long. For details, see our preprint [KN].)

Remark 2.1.1. There are several equivalent (or inequivalent) definitions of a Bott-Samelson variety, but in the talk, we stick to that of [Ja]:

$$X(y_1, \ldots, y_n) = \{(g_1B, \ldots, g_nB) \in (G/B)^n \mid g_{i-1}^{-1}g_i \in \overline{B\dot{y}_iB} \quad \text{for all } i\}$$

for $y_1, \ldots, y_n \in W$. If J_i for $1 \le i \le n$ is a subset of I and z_{J_i} is the longest element of the subgroup $W_{J_i} = \langle r_k \mid k \in J_i\rangle$ of the Weyl group W, then the Bott-Samelson variety

$X(z_{J_1},\dots,z_{J_n})$ is smooth. Moreover, if we assume that $\ell(z_{J_1}\cdots z_{J_n})=\ell(z_{J_1})+\cdots+\ell(z_{J_n})$, then the restriction ϕ of the n-th projection $\pi_n\colon (G/B)^n\to G/B$ to the Bott-Samelson variety $X(z_{J_1},\dots,z_{J_n})\subset (G/B)^n$ gives a Demazure-Hansen desingularization

$$\phi\colon X(z_{J_1},\dots,z_{J_n})\to X(z_{J_1}\cdots z_{J_n})$$

of the Schubert variety $X(z_{J_1}\cdots z_{J_n})$. Note that the ϕ induces an isomorphism of suitable open and dense subvarieties.

Now it follows that

$$\begin{aligned}
\chi_w^\omega(\mathbb{C}_{\mu,k}) &= \sum_{j\ge 0}(-1)^j \operatorname{ch}^\omega(H^j(X(z_1,\dots,z_n),\mathcal{L}_{X(z_1,\dots,z_n)}(\mathbb{C}_{\mu,k}))) \quad \text{by (2.1.2)}\\
&= \sum_{j\ge 0}(-1)^j\left(\sum_{i\ge 0}(-1)^i \operatorname{ch}^\omega(H^i(X(z_1),\mathcal{L}(H^j(X(z_2,\dots,z_n),\mathcal{L}(\mathbb{C}_{\mu,k})))))\right) \quad \text{by (2.1.3)}\\
&= \sum_{j\ge 0}(-1)^j \chi_{z_1}^\omega(H^j(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(\mathbb{C}_{\mu,k}))).
\end{aligned}$$

By induction on n, we may assume that $w=w_i$ for some $i\in\widehat{I}$ in proving (2.1.1). So put $J=I_i$ and let $P=P_J$ be the standard parabolic subgroup of G associated to J. We are to show

$$\chi_{w_i}^\omega(\mathbb{C}_{\mu,k})=\widehat{D}_i(\zeta^k e(\mu)). \tag{2.1.4}$$

Assume first that $\langle\mu,\alpha_i^\vee\rangle\ge 0$ (and hence that $\langle\mu,\alpha_k^\vee\rangle\ge 0$ for all $k\in J$). Let $L_J(\mu)$ be the simple rational P_J-module of highest weight μ admitting an $\langle\omega\rangle$-action as in §1.5, and let ζ^k be the one-dimensional trivial P_J-module with ω acting by the scalar ζ^k.

Lemma 2.1.2. *Let the notation and assumption be as above. Then we have the following isomorphism of $P_J\rtimes\langle\omega\rangle$-modules.*

$$H^0(P_J/B,\mathcal{L}_{P_J/B}(\mathbb{C}_{\mu,k}))\simeq L_J(\mu)\otimes_{\mathbb{C}}\zeta^k.$$

(This lemma is, in a sense, crucial to the proof of our main result. Although no one doubts the truth of this lemma, its complete proof would be rather long.)

Now we deduce that

$$
\begin{aligned}
\chi^{\omega}_{w_i}(\mathbb{C}_{\mu,k}) &= \mathrm{ch}^{\omega}(H^0(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k}))) \quad \text{by Kempf's vanishing theorem} \\
&= \mathrm{ch}^{\omega}(L_J(\mu) \otimes_{\mathbb{C}} \zeta^k) \quad \text{by Lemma 2.1.2} \\
&= \zeta^k \mathrm{ch}^{\omega}(L_J(\mu)) \\
&= \zeta^k \widehat{D}_i(e(\mu)) \quad \text{by Lemma 1.5.1} \\
&= \widehat{D}_i(\zeta^k e(\mu)).
\end{aligned}
$$

If $\langle \mu, \alpha_i^\vee \rangle = -1$ (and hence $\langle \mu, \alpha_k^\vee \rangle = -1$ for all $k \in J$), then both sides of (2.1.4) vanish.

Assume finally that $\langle \mu, \alpha_i^\vee \rangle \leq -2$ (and hence that $\langle \mu, \alpha_k^\vee \rangle \leq -2$ for all $k \in J$). Set $\rho_J = \frac{1}{2} \sum_{\alpha \in \Delta_J^+} \alpha$ with $\Delta_J^+ = \Delta_+ \cap \sum_{k \in J} \mathbb{Z}\alpha_k$ the positive root system of P_J. By direct checking, using the $T \rtimes \langle \omega \rangle$-module isomorphism $(\mathrm{Lie}(P)/\mathrm{Lie}(B))^* \simeq \bigoplus_{\alpha \in \Delta_J^+} \mathbb{C} f_\alpha$, we see that as $B \rtimes \langle \omega \rangle$-modules,

$$
\textstyle\bigwedge_{\mathbb{C}}^{\ell(w_i)} (\mathrm{Lie}(P)/\mathrm{Lie}(B))^* \simeq \mathbb{C}_{-2\rho_J,0} \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1},
$$

where $\ell(w_i) = \dim_{\mathbb{C}}(P/B)$ and $(-1)^{\ell(w_i)-1}$ is the one-dimensional $B \rtimes \langle \omega \rangle$-module with B acting trivially and ω by the scalar $(-1)^{\ell(w_i)-1}$. Then the $B \rtimes \langle \omega \rangle$-equivariant Serre duality reads

(2.1.5)

$$
\begin{aligned}
H^j(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k}))^* &\simeq H^{\ell(w_i)-j}(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{-\mu-2\rho_J,-k} \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1})) \\
&\simeq \begin{cases} H^0(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{-\mu-2\rho_J,-k})) \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1} & \text{if } j = \ell(w_i), \\ 0 & \text{otherwise (by Kempf).} \end{cases}
\end{aligned}
$$

(The use above of the $B \rtimes \langle \omega \rangle$-equivariant Serre duality is the most essential part of the proof of our main result.)

Remark 2.1.3. Put $X = P/B$ and $m = \dim_{\mathbb{C}} X$. Let $\mathcal{M}$ be a $(B, \langle \omega \rangle)$-equivariant $\mathcal{O}_X$-module that is locally free of finite rank over $\mathcal{O}_X$. The $B \rtimes \langle \omega \rangle$-equivariant Serre duality (see our preprint on Naito's home page) asserts that, as $B \rtimes \langle \omega \rangle$-modules,

$$
H^i(X, \mathcal{M}^\vee \otimes_X \Omega_X^m) \simeq H^{m-i}(X, \mathcal{M})^* \quad \text{for all } 0 \leq i \leq m,
$$

where $\mathcal{M}^\vee = \mathcal{H}om_X(\mathcal{M}, \mathcal{O}_X)$ is the dual sheaf of $\mathcal{M}$, $\Omega_X^m = \bigwedge_X^m \Omega_X^1$ is the canonical sheaf on X, and $H^{m-i}(X, \mathcal{M})^*$ is the dual $B \rtimes \langle \omega \rangle$-module of $H^{m-i}(X, \mathcal{M})$. This Serre duality will be a consequence of the triviality of the $B \rtimes \langle \omega \rangle$-action on the one-dimensional vector space $H^m(X, \Omega_X^m)$. Since the triviality of the B-action on it is known, it remains to show the triviality of the $\langle \omega \rangle$-action. There are many ways to show it, but the way we take here is (I think) purely algebro-geometric and elementary: first take a $P \rtimes \langle\omega\rangle$-equivariant closed immersion $\imath\colon X \to \mathbb{P} = \mathbb{P}(L(\lambda))$ for sufficiently dominant $\lambda \in \Lambda_+^\omega$; then use the fact that the full automorphism group $PGL(L(\lambda))$ of $\mathbb{P}(L(\lambda))$ acts trivially on the one-dimensional vector space $H^l(\mathbb{P}, \Omega_{\mathbb{P}}^l)$ with $l = \dim_{\mathbb{C}} \mathbb{P}$, where (though not so trivial)

$$H^l(\mathbb{P}, \Omega_{\mathbb{P}}^l) \simeq H^m(\mathbb{P}, \imath_*\Omega_X^m) \simeq H^m(X, \Omega_X^m)$$

as $P \rtimes \langle\omega\rangle$-modules.

The proof of the following lemma is easy.

LEMMA 2.1.4. *Let J be an ω-invariant subset of I, w_J the longest element of the Weyl group W_J of P_J, and let $\nu \in \Lambda^\omega$ be such that $\langle \nu, \alpha_k^\vee\rangle \geq 0$ for all $k \in J$. Then we have the following isomorphism of $P_J \rtimes \langle\omega\rangle$-modules.*

$$L_J(\nu)^* \simeq L_J(-w_J(\nu)).$$

The isomorphism (2.1.5) together with Lemmas 2.1.2 and 2.1.4 implies that, as $B \rtimes \langle\omega\rangle$-modules,

$$\text{(2.1.6)} \quad \begin{aligned} H^{\ell(w_i)}(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k})) &\simeq \left(L_J(-\mu - 2\rho_J)^* \otimes_{\mathbb{C}} \zeta^k\right) \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1} \\ &\simeq L_J(w_i(\mu + 2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}. \end{aligned}$$

Then, setting $\widehat{\mu} = (P_\omega^*)^{-1}(\mu)$,

$$\begin{aligned} \chi_{w_i}^\omega(\mathbb{C}_{\mu,k}) &= (-1)^{\ell(w_i)} \operatorname{ch}^\omega(L_J(w_i(\mu + 2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}) \quad \text{by (2.1.6)} \\ &= -\zeta^k \operatorname{ch}^\omega(L_J(w_i(\mu + 2\rho_J))) \\ &= -\zeta^k \widehat{D}_i(e(w_i(\mu + 2\rho_J))) \quad \text{by Lemma 1.5.1} \\ &= -\zeta^k \left(P_\omega^* \circ D_{\widehat{r}_i} \circ (P_\omega^*)^{-1}\right)(e(w_i(\mu + 2\rho_J))) \\ &= -\zeta^k \left(P_\omega^* \circ D_{\widehat{r}_i}\right)(e(\widehat{r}_i(\widehat{\mu} + \widehat{\alpha}_i))) \quad \text{since } (P_\omega^*)^{-1}(2\rho_J) = \widehat{\alpha}_i \\ &= -\zeta^k P_\omega^*(-D_{\widehat{r}_i}(e(\widehat{\mu}))) \\ &= \zeta^k \left(\widehat{D}_i \circ P_\omega^*\right)(e(\widehat{\mu})) \\ &= \zeta^k \widehat{D}_i(e(\mu)) \\ &= \widehat{D}_i(\zeta^k e(\mu)). \end{aligned}$$

Thus in all cases (2.1.4) holds, and we are done.

If $\lambda \in \Lambda_+^\omega$, then for any Schubert variety $X(w)$,

$$H^j(X(w), \mathcal{L}_{X(w)}(\lambda)) = 0 \quad \text{for all } j \geq 1$$

by the Demazure vanishing theorem of Andersen et al. Hence we have proved

THEOREM 2.1.5. *Let M be a finite-dimensional rational $B \rtimes \langle\omega\rangle$-module and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda_\omega]$,*

$$\chi_w^\omega(M) = \sum_{j \geq 0} (-1)^j \operatorname{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) = \widehat{D}_w(\operatorname{ch}^\omega(M)),$$

where $\widehat{D}_w = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n}$ for any reduced expression $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. In particular, for $\lambda \in \Lambda_+^\omega$, we have

$$\operatorname{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) = \widehat{D}_w(e(\lambda)),$$

where $\mathbb{C}_\lambda$ is the one-dimensional $B \rtimes \langle\omega\rangle$-module on which B acts by the weight λ through the quotient $B \to T$ and ω trivially.

Theorem 2.1.5 above reveals that there exists a striking relation between the ω-Euler characteristic $\chi_w^\omega(\mathbb{C}_\lambda) \in \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ for $\mathfrak{g}$ and the ordinary Euler characteristic for the orbit Lie algebra $\widehat{\mathfrak{g}}$. To state the relation, we need some notation. Recall that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the dual complex semi-simple Lie algebra of the fixed point subalgebra $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$ of $\mathfrak{g}$. Let $\widehat{G}$ be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus $\widehat{T}$ and Borel subgroup $\widehat{B} \supset \widehat{T}$ such that $\operatorname{Lie}(\widehat{G}) = \widehat{\mathfrak{g}}$, $\operatorname{Lie}(\widehat{T}) = \widehat{\mathfrak{h}}$, and $\operatorname{Lie}(\widehat{B}) = \widehat{\mathfrak{b}}$. For $\widehat{w} \in \widehat{W} \simeq N_{\widehat{G}}(\widehat{T})/\widehat{T}$, we take a right coset representative $\dot{\widehat{w}} \in N_{\widehat{G}}(\widehat{T})$ of $\widehat{w}$, and define the Schubert variety $\widehat{X}(\widehat{w})$ over $\mathbb{C}$ by

$$\widehat{X}(\widehat{w}) = \overline{\widehat{B}\dot{\widehat{w}}\widehat{B}/\widehat{B}} = \overline{\widehat{B}\dot{\widehat{w}}\widehat{B}}/\widehat{B} \subset \widehat{G}/\widehat{B}.$$

For each $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, we denote by $\mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})$ the (locally free) $\widehat{B}$-equivariant sheaf of $\mathcal{O}_{\widehat{X}(\widehat{w})}$-modules associated to the one-dimensional $\widehat{B}$-module $\mathbb{C}_{\widehat{\lambda}}$ on which $\widehat{B}$ acts by the weight $\widehat{\lambda}$ through the quotient $\widehat{B} \to \widehat{T}$.

Now we are ready to state the following

COROLLARY 2.1.6. *Let $\lambda \in (\mathfrak{h}_\mathbb{Z}^*)^0$ and $w \in W^\omega$. We set $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$. Then we have in the algebra $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$,*

$$\begin{aligned} \chi_w^\omega(\mathbb{C}_\lambda) &= \sum_{j \geq 0} (-1)^j \operatorname{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) \\ &= P_\omega^* \left(\sum_{j \geq 0} (-1)^j \operatorname{ch} H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})) \right), \end{aligned}$$

where $\operatorname{ch} H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})) \in \mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ for $j \in \mathbb{Z}_{\geq 0}$ is the ordinary character of the j-th cohomology group $H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}}))$ of $\widehat{X}(\widehat{w})$.

(This immediately follows from Theorem 2.1.5 and the ordinary Demazure character formula for the orbit Lie algebra $\widehat{\mathfrak{g}}$.)

2.2. **Joseph's modules.** Let us finally return to Joseph's module $J_w(\lambda)$, with $w \in W^\omega$ and $\lambda \in \Lambda_+^\omega$. Thus let v_λ^* be a (nonzero) lowest weight vector of the dual module $L(\lambda)^*$ (which is the dual element of a (nonzero) highest weight vector v_λ of $L(\lambda)$), and let $\dot{w} \in N_G(T)^\omega$ representing $w \in W^\omega$. Since v_λ^* is fixed by ω, so is $\dot{w}\, v_\lambda^*$. Joseph's module $J_w(\lambda)$ of highest weight $-w(\lambda)$ in $L(\lambda)^*$ is defined to be

$$J_w(\lambda) = \mathfrak{U}(\mathfrak{b})\,(\dot{w}\, v_\lambda^*) \subset L(\lambda)^*,$$

where $\mathfrak{U}(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{b} = \mathrm{Lie}(B)$. Note that, since $\omega \cdot (\dot{w}\, v_\lambda^*) = \dot{w}\, v_\lambda^*$, Joseph's module $J_w(\lambda)$ is a $B \rtimes \langle\omega\rangle$-submodule of $L(\lambda)^*$. Moreover, since $\dot{w}_0\, v_\lambda^*$ is a (nonzero) highest weight vector of $L(\lambda)^*$ fixed by ω, there is an isomorphism of $G \rtimes \langle\omega\rangle$-modules

$$L(\lambda)^* \simeq L(-w_0(\lambda)), \tag{2.2.1}$$

which enables us to regard $J_w(\lambda)$ as a $B \rtimes \langle\omega\rangle$-submodule of $L(-w_0(\lambda))$. Then we obtain a short exact sequence of $B \rtimes \langle\omega\rangle$-modules

$$0 \leftarrow J_w(\lambda)^* \leftarrow L(-w_0(\lambda))^* \leftarrow J_w(\lambda)^\perp \leftarrow 0,$$

with $J_w(\lambda)^\perp = \{\phi \in L(-w_0(\lambda))^* \mid \phi(J_w(\lambda)) = 0\}$. On the other hand, Lemma 2.1.2 for the case $J = I$ combined with (2.2.1) yields an isomorphism of $G \rtimes \langle\omega\rangle$-modules

$$H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) \simeq L(-w_0(\lambda))^*.$$

Since the restriction map

$$H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) \to H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$$

is known to be a ($B \rtimes \langle\omega\rangle$-equivariant) surjection, we obtain an isomorphism of $B \rtimes \langle\omega\rangle$-modules

$$J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)),$$

or equivalently

$$J_w(\lambda) \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*. \tag{2.2.2}$$

We now define a $\mathbb{C}$-linear conjugation $\bar{\ } : \mathbb{C}[\Lambda^\omega] \to \mathbb{C}[\Lambda^\omega]$ by

$$\overline{\sum_{\mu \in \Lambda^\omega} a_\mu\, e(\mu)} = \sum_{\mu \in \Lambda^\omega} a_\mu\, e(-\mu) \quad \text{with } a_\mu \in \mathbb{C} \text{ for } \mu \in \Lambda^\omega.$$

Then we obtain the following theorem from the $B \rtimes \langle\omega\rangle$-module isomorphism (2.2.2).

THEOREM 2.2.1. *Let $\lambda \in \Lambda_+^\omega$ and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda^\omega]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = \overline{\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)))}.$$

By combining Theorems 2.1.5 and 2.2.1, we obtain the following

COROLLARY 2.2.2. *Let $\lambda \in \Lambda_+^\omega$ and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda^\omega]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = \overline{\widehat{D}_w(e(\lambda))}.$$

Finally, by combining Corollary 2.1.6 and Theorem 2.2.1, we obtain a remarkable relation between the twining character $\mathrm{ch}^\omega(J_w(\lambda))$ of Joseph's module $J_w(\lambda)$ for $\mathfrak{g}$ and the ordinary character of Joseph's module for the orbit Lie algebra $\widehat{\mathfrak{g}}$, which is the dual complex semi-simple Lie algebra of $\mathfrak{g}^0$. For each $\widehat{w} \in \widehat{W}$, let

$$\widehat{J}_{\widehat{w}}(\widehat{\lambda}) = \mathfrak{U}(\widehat{\mathfrak{b}})\,(\widehat{w}\,\widehat{v}_{\widehat{\lambda}}^*) \subset \widehat{L}(\widehat{\lambda})^*$$

be Joseph's module of highest weight $-\widehat{w}(\widehat{\lambda})$, with $\widehat{v}_{\widehat{\lambda}}^* \in \widehat{L}(\widehat{\lambda})^*$ a lowest weight vector of $\widehat{L}(\widehat{\lambda})^*$.

COROLLARY 2.2.3. *Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$ be dominant and $w \in W^\omega$. We set $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*$. Then we have in $\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\left(\mathrm{ch}\,\widehat{J}_{\widehat{w}(\widehat{\lambda})}(\widehat{\lambda})\right).$$

REFERENCES

[AK] A. Altman and S. Kleiman, *Introduction to Grothendieck Duality Theory*, Lecture Notes in Math. Vol. 146, Springer-Verlag, Berlin, 1970.

[An] H. H. Andersen, *Schubert varieties and Demazure's character formula*, Invent. Math. **79** (1985), 611–618.

[D1] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. **7** (1974), 53–88.

[D2] ———, *Une nouvelle formule des caractères*, Bull. Sci. Math. **98** (1974), 163–172.

[DG] M. Demazure and P. Gabriel, *Groupes algébriques* I, Masson & Cie and North-Holland, Paris and Amsterdam, 1970.

[FRS] J. Fuchs, U. Ray, and C. Schweigert, *Some automorphisms of generalized Kac-Moody algebras*, J. Algebra **191** (1997), 518–540.

[FSS] J. Fuchs, B. Schellekens, and C. Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Commun. Math. Phys. **180** (1996), 39–97.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. Vol. 52, Springer-Verlag, Berlin, 1977.

[Ja] J. C. Jantzen, *Representations of Algebraic Groups*, Pure and applied mathematics Vol. 131, Academic Press, Boston, 1987.

[Jo] A. Joseph, *On the Demazure character formula*, Ann. Sci. École Norm. Sup. **18** (1985), 389–419.

[Kan] M. Kaneda, *The Frobenius morphism of Schubert schemes*, J. Algebra **174** (1995), 473–488.

[KN] M. Kaneda and S. Naito, *A twining character formula for Demazure modules*, preprint.

[Kas] M. Kashiwara, *The crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993), 839–858.

[Ku] S. Kumar, *Demazure character formula in arbitrary Kac-Moody setting*, Invent. Math. **89** (1987), 395–423.

[Ma] O. Mathieu, *Formules de caractères pour les algèbres de Kac-Moody générales*, Astérisque **159–160** (1988).

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory* (3rd enlarged edition), Springer-Verlag, Berlin, 1994.

[MR] V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. **122** (1985), 27–40.

[N1] S. Naito, *Embedding into Kac-Moody algebras and construction of folding subalgebras for generalized Kac-Moody algebras*, Japan. J. Math. (New Series) **18** (1992), 155–171.

[N2] ———, *Twining character formula of Kac-Wakimoto type for affine Lie algebras*, preprint.

[N3] ———, *Twining characters, Kostant's homology formula, and the Bernstein-Gelfand-Gelfand resolution*, preprint.

[Ra] A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Invent. Math. **80** (1985), 283–294.

[RR] S. Ramanan and A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*, Invent. Math. **79** (1985), 217–224.

Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

E-mail address: `naito@math.tsukuba.ac.jp`