数理解析研究所講究録
1191 巻 2001 年 90-98

Traffic Models and a Solvable Difference-Differential Equation

名古屋大学理学部　中西健一 (Nakanishi, Kenichi)
Department of Physics, Nagoya Univ.

Traffic flow on no-passing freeways has been extensively studied using the car-following models. Here, let us investigate two car-following models. One is Newell-Whitham type model (NWM)[1, 2]. NWM is one of the traditional car-following model. It has been studied in the field of traffic engineering for a long time. The other is Optimal Velocity model (OVM)[3]. OVM is a new mode of the car-following model. It was proposed in 1994 by a group of particle theorists. The fundamental concept of the car-following model is the assumption that the driving behavior of car is affected only the preceding car. Both NWM and OVM have common structure that the effect of the preceding car is expressed only by an velocity function $V(h)$, depending the headway. It's sometimes called the optimal velocity. $V(h)$ is the most desirable velocity for the driver at given headway. Generally speaking, when the headway is longer, the velocity is larger. Driver tries to adjust its velocity to be the optimal one.

NWM is given by a first-order difference-differential equation:

$$\dot{x}_n(t+\tau)=V(x_{n-1}(t)-x_n(t)) \quad (n=1,2,...N).$$

x_n is the position of the n-th car. The time lag τ represents a delay of the velocity adjustment. On the other hand, OVM is described by a second order differential equation:

$$\ddot{x}_n(t) = a\left[V(x_{n-1}(t) - x_n(t)) - \dot{x}_n(t)\right] \quad (n = 1, 2, ...N).$$

The acceleration of car is proportional to the difference between the present velocity and the optimal one. A constant a is the sensitivity of driver. For skillful drivers, a become large.

By truncating the Taylor expansion of $\dot{x}_n(t+\tau)$ as

$$\dot{x}_n(t+\tau) \simeq \dot{x}_n(t) + \tau\ \ddot{x}_n(t),$$

NWM is formally reduced to OVM. Then we know a essentially corresponds to inverse of τ. Actually, NWM and OVM are quite similar in their qualitative behavior, especially with regard to the generation of density waves. Numerical simulations with a hyperbolic tangent optimal velocity function,

$$V(h) = \xi + \eta \tanh\left(\frac{h-\rho}{2\sigma}\right),$$

show that these models give rise to the spontaneous generation of traffic congestion.

For N cars on a circuit of length L, the uniform flow with headway $h(= L/N)$ and velocity $V(h)$,

$$x_n^{(0)}(t) = V(h)\ t - h\ n,$$

is a trivial solution of both models. It describes an equal spacing row of cars moving with same velocity on a circuit. If τ is small enough, the uniform flow is linearly stable for any headway. However, when τ exceeds a critical value $\tau_c = \sigma/\eta$, the uniform flow becomes linearly unstable on a intermediate range of headway;

$$|h-\rho| < 2\sigma \mathrm{Arccosh} \sqrt{\frac{\tau}{\tau_c} \frac{\sin \pi/N}{\pi/N}}.$$

In this case, we can observe the spontaneous generation of density wave in the simulation. For OVM, τ is substituted by inverse of the sensitivity.

Then we assume that NWM has the traveling wave solution of following form[4]:

$$x_n(t) = Ct - nh + A \ln \frac{\vartheta_0\,(u-\beta+\delta,\ q)}{\vartheta_0\,(u-\beta-\delta,\ q)}.$$

Here, q is the modulus parameter of the theta function and $u = \nu t - 2\beta n$. The headway $\Delta x_n(t) \equiv x_{n-1}(t) - x_n(t)$ is given as

$$\Delta x_n(t) = h + A\ \ln \frac{\vartheta_0^2(u)\ \vartheta_0^2(\delta+\beta) - \vartheta_1^2(u)\ \vartheta_1^2(\delta+\beta)}{\vartheta_0^2(u)\ \vartheta_0^2(\delta-\beta) - \vartheta_1^2(u)\ \vartheta_1^2(\delta-\beta)}.$$

It is logarithm of a rational expression of $\vartheta_0^2(u)$ and $\vartheta_1^2(u)$. If and only if the Whitham condition $v\,\tau = \beta$ [2] is satisfied, the velocity at $t+\tau$ is also written in a rational expression of $\vartheta_0^2(u)$ and $\vartheta_1^2(u)$ as

$$\dot{x}_n(t+\tau) = C + A\nu \frac{\vartheta_0^2(u)\ \big(\vartheta_0^2(\delta)\big)' - \vartheta_1^2(u)\ \big(\vartheta_1^2(\delta)\big)'}{\vartheta_0^2(u)\ \vartheta_0^2(\delta) - \vartheta_1^2(u)\ \vartheta_1^2(\delta)}.$$

In this case, we can eliminate $\vartheta_0^2(u)$ and $\vartheta_1^2(u)$ from the above two expressions and $\dot{x}_n(t+\tau)$ is expressed by a single-valued function of $\Delta x_n(t)$. We find that the resulting optimal velocity function is given by a hyperbolic tangent. Thus, we find the relations between the ansatz parameters and the coefficients of the model:

$$
\begin{aligned}
\xi &= C + \frac{\sigma\beta}{2\tau}\frac{\partial}{\partial\beta}\ln\frac{\vartheta_1(2\delta+\beta,\ q)}{\vartheta_1(2\delta-\beta,\ q)}, \\
\eta &= \frac{\sigma\beta}{2\tau}\frac{\partial}{\partial\beta}\ln\frac{\vartheta_1^2(\beta,\ q)}{\vartheta_1(2\delta+\beta,\ q)\vartheta_1(2\delta-\beta,\ q)}, \\
\rho &= h - \sigma\ln\frac{\vartheta_1(2\delta-\beta,\ q)}{\vartheta_1(2\delta+\beta,\ q)}, \\
\sigma &= A.
\end{aligned}
$$

The original OVM, which has a constant sensitivity and a single-valued optimal velocity function, can not solved by the theta function ansatz, because $\vartheta_0(u)\vartheta_1(u)\vartheta_2(u)\vartheta_3(u)$-terms appear both in $\dot{x}_n$ and in $\ddot{x}_n$. However, if we allow the sensitivity a to be headway-dependent, these terms can cancel out. Thus, we find that OVM

$$\ddot{x}_n = a(\Delta x_n)\left[V(\Delta x_n) - \dot{x}_n\right]$$

with the headway-dependent sensitivity

$$a(h) = \alpha\cosh\left(\frac{h-\rho}{2\sigma}\right)$$

is solved by the the theta function ansatz. The headway-dependent sensitivity $a(h)$ becomes small in the intermediate headway. Since the car bunching occurs first in this range, this model is easier to generate congestion than the original OVM.

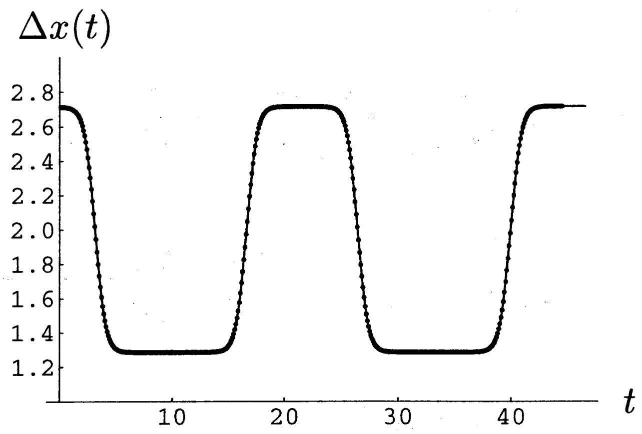

Figure 1:

Now, let us investigate spectrum of the solution of NWM. Under the cyclic boundary condition

$$x_{n+N}(t) = x_n(t) - L,$$

the mean headway of the solution must be given by $h = L/N$, and β must be one of the discrete values:

$$\beta = \frac{1}{2N}, \frac{2}{2N}, \ldots, \frac{n_\mathrm{b}}{2N}, \ldots.$$

For given mean headway h, we can obtain the spectrum for the possible n_b and q. Once n_b and q is determined, we can calculate all other parameter of the ansatz and concretely construct the multibunch solution[5]. In Figure 1, we show a result of a numerical simulation with $N = 20$, $\tau = 0.58228$ and $V(h) = \tanh(h-2) + \tanh 2$ after a sufficient relaxation time, $t \simeq 6 \times 10^4$. The curve plotted by the dots represents the results of the numerical simulation, while the solid line represents the analytic one-bunch solution with $q = 0.70792140328755$.

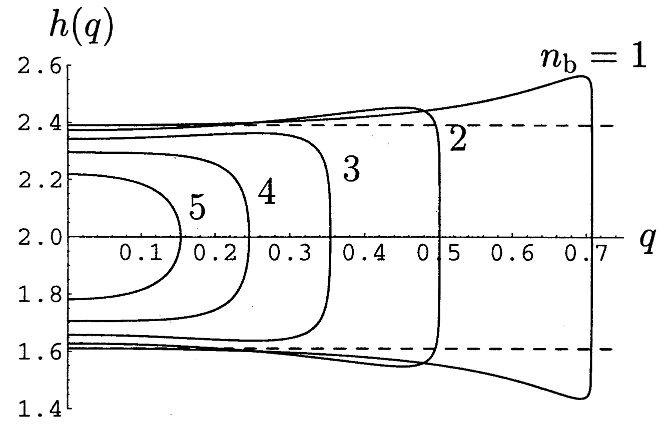

Figure 2:

During the simulation, we observe a cascade of transitions of multibunch states. Until $t \simeq 400$, three-bunch configuration was formed. It gradually became distorted, with one of the bunches moving closer to a neighboring bunch. At $t \simeq 4680$, the fusion of these bunches occurred. The two-bunch configuration existed about ten times longer than the three-bunch one. However, eventually, one of the bunches began to shrink. This bunch was finally absorbed by the other at $t \simeq 51560$. The simulation was continued until $t = 2 \times 10^5$, with the one-bunch solution continuing to survive, unchanged in form.

In Figure 2, we show the spectrum of the multibunch solution for $N = 20$, $\tau = 0.58228$ and $V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2$. In this case, maximally five bunch solution can exists. The dashed line indicates the boundary of linear stability. We can see that one- and two-bunch solution can stay in the linear stable region. It means that there are coexistence phases

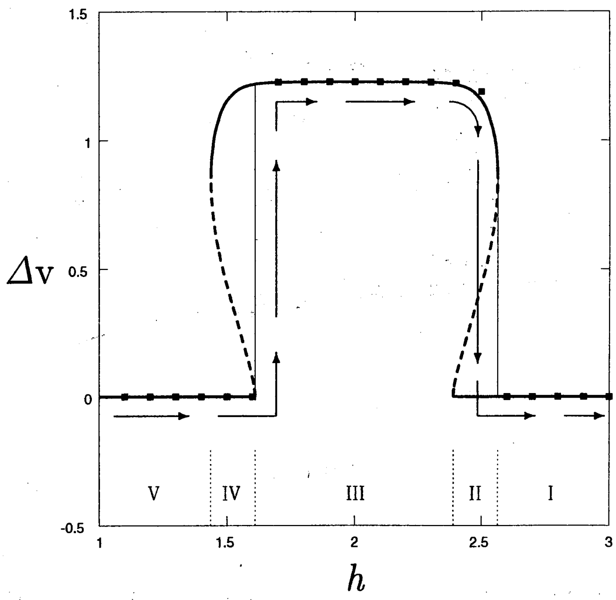

Figure 3:

of uniform and congested flow. Thus, we find that there are three different phase. One is the uniform flow phase, where a stable uniform flow only exists. Next is the congested flow phase, where a stable congested flow and an unstable uniform flow exist. The third is the coexistence phase, where a stable congested flow and a stable uniform flow exist. In this phase, an unstable congested flow also exists. If we prepare an unstable congested flow in the coexistence phase as an initial configuration of numerical simulation, it transforms into a stable congested flow or a uniform flow.

In the numerical simulation, we observe a hysteresis phenomena on the phase transition between the uniform flow phase and the congested flow phase[6] (Figure 3). It exhibits the feature of the "subcritical Hopf bifurcation". Taking the velocity amplitude Δv as an order parameter, we find that the numerical result agrees quite well with the values expected from the analytical solution. We prepare a stable uniform configuration in the uniform flow phase (V). When we increase h, the uniform flow survives in the coexistence phase (IV). A jump from a uniform flow to a congested flow occurs at the boundary between the coexistence phase and the congested flow phase (III). The congested flow continues to realize even in the coexistence phase (II). It jumps into a uniform flow at the boundary between the coexistence phase and the uniform flow phase (I).

Now I summarize the results. First, We find that NWM and OVM with headway-depending sensitivity have a set of solutions described by elliptic theta functions. They represents the multibunch traveling wave. By the numerical simulation, we observe the cascade of transitions of multibunch states. The series of transitions from a multi-bunch configuration to the next observed here, suggests that each multi-bunch solution corresponds to a heteroclinic point of the system. However, no flow out of the one-bunch solution was observed, which suggests that it is an attractor. It is also possible that each of the multi-bunch solutions is a kind of Milnor attractor[7], which is unstable with respect to any small perturbation, but globally attracts orbits. To fully understand the situation we must investigate the stability and the attracting domain of the multi-bunch solutions more precisely. Second, There are coexistence phase of uniform flow and congested flow, which are separated by an unstable congested solution. We also find an hysteresis phenomenon on the phase transition between uniform flow and congested flow. It exhibits the feature of the subcritical Hopf bifurcation.

References

[1] L. A. Pipes, J. Appl. Phys. **24**, 274 (1953); G. F. Newell, Oper. Res. **9**, 209 (1961); D. C. Gazis, R. Herman and R. W. Rothery, Oper. Res. **9**, 545 (1961).

[2] G.B. Whitham, Proc. R. Soc. London A **428**, 49 (1990)

[3] M. Bando, K. Hasebe, A. Nakayama, A. Shibata and Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995); Jpn. J. Ind. Appl. Math. **11**, 203 (1994).

[4] Y. Igarashi, K. Itoh and K. Nakanishi, J. Phys. Soc. Japan **68**, 791 (1999)

[5] K. Nakanishi, Phys. Rev. E **62**, 3349 (2000)

[6] Y. Igarashi, K. Itoh, K. Nakanishi, K. Ogura and K. Yokokawa, patt-sol/9908002

[7] J. Milnor, Comm. Math. Phys. **99** 177 (1985); **102** 517 (1985).