数理解析研究所講究録
1191 巻 2001 年 29-42

Some existence, monotonicity and uniqueness results for pulsating travelling fronts in periodic media and periodic domains

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1 Introduction

This study is devoted to the analysis of some front propagation phenomena for a class of advection-diffusion-reaction equations in a general class of periodic domains with underlying periodic diffusion and velocity fields. In the case where the coefficients of the equation are invariant in some given direction and the domain itself is invariant in that direction, then one can speak about travelling fronts which move in that direction with constant speed and whose profiles do not change as time runs. In *periodic* domains or media, the notion of travelling fronts has to be replaced by the notion of *pulsating* (or periodic) travelling fronts : a pulsating travelling front propagates in some direction with some unknown *effective* speed but its profile changes periodically as time runs. Pulsating travelling fronts appear in various physical models and can propagate in several classes of periodic domains such as straight or oscillating infinite cylinders, the whole space, or domains with periodic holes, etc. Various existence, uniqueness and monotonicity results are given for two types of reaction terms. For a combustion-type nonlinearity, the pulsating travelling fronts exist, their speed is unique and the fronts are increasing in the time variable and unique up to translation in time. For another class of nonlinearity arising either in combustion or biological models, the set of possible speeds is a semi-infinite interval, closed and bounded from below, and for each speed, a time-increasing pulsating travelling front exists. The results can all be stated in a same general class of periodic media and domains (*see* section 6), and, as well as more general ones, they are proved in two papers [7] and [8] written with H. Berestycki and with H. Berestycki and N. Nadirashvili.

2 Travelling fronts and *pulsating* travelling fronts in straight infinite cylinders

Let us first deal with the case of a straight infinite cylinder

$$\Omega = \{(x,y),\ x \in \mathbb{R},\ y \in \omega\}$$

where ω is a smooth bounded and connected subset of $\mathbb{R}^{N-1}$ and let us consider the classical solutions $u(t,x,y)$ of the following advection-diffusion-reaction equation

$$\frac{\partial u}{\partial t} - \Delta u + q(x,y)\cdot\nabla_{x,y}u = f(u), \quad t\in\mathbb{R},\ (x,y)\in\overline{\Omega} \tag{2.1}$$

together with Neumann boundary conditions on $\partial\Omega$

$$\partial_\nu u = 0, \quad (t,x,y)\in\mathbb{R}\times\partial\Omega \tag{2.2}$$

where $\nu = \nu(x,y) = \nu(y)$ is the outward unit normal to $\partial\Omega$ and $\partial_\nu u = \frac{\partial u}{\partial\nu}$. These Neumann boundary condition mean that there is no flux of u across the wall of the cylinder.

The underlying velocity field $q(x,y) = (q_1(x,y),\cdots,q_N(x,y))$ is given in $\overline{\Omega}$, bounded in $C^1(\overline{\Omega})$ and one assumes that

$$\left\{\begin{array}{rcl} \operatorname{div} q &=& 0 \text{ in } \overline{\Omega} \\ \forall (x,y)\in\overline{\Omega},\quad q(x+L,y) &=& q(x,y) \\ \displaystyle\int_{(0,L)\times\omega} q_1(x,y)\,dx\,dy &=& 0 \\ q\cdot\nu &=& 0 \text{ on } \partial\Omega \end{array}\right. \tag{2.3}$$

where the period L of q with respect to the variable x is some given positive number. This field q is divergence-free (which corresponds to the incompressibility assumption for the underlying medium) and may represent some turbulent fluctuations with respect to a mean field.

Such semilinear parabolic equations can arise in the modelling of thermodiffusive premixed flame propagation with a unit Lewis number and a simple chemistry, and u then represents an adimensionalized temperature (*see e.g.* [9], [52], [58]. These equations can also come from biological models of population dynamics where u stands for the relative concentration of some substance [1], [20]. One of our goals is to analyze the influence of periodic advection, and of other periodic phenomena, on the propagation of fronts (flames in combustion theory). Related questions in combustion theory have been treated in [2], [16], [57]. In dimension $N\geq 2$, equation (2.1) can then arise in turbulent combustion models to describe the propagation of a premixed flame in an array of vortical cells. Generally speaking, equation (2.1) is a transport equation for a passive quantity u in a periodic excitable medium.

Two main types of nonlinearities f are considered here. Namely, the given function f is assumed to be Lipschitz-continuous in $[0,1]$ and to be of one of the following types : either

$$\left\{\begin{array}{l} \exists\theta\in(0,1),\ f(s)=0 \text{ for all } s\in[0,\theta],\ f(s)>0 \text{ for all } s\in(\theta,1),\ f(1)=0 \\ \exists\mu\in(0,1-\theta),\quad f \text{ is nonincreasing on } [1-\mu,1], \end{array}\right. \tag{2.4}$$

$$\text{or}\qquad \left\{\begin{array}{l} f>0 \text{ on } (0,1),\quad f(0)=f(1)=0 \\ \exists\mu>0,\quad f \text{ is nonincreasing on } [1-\mu,1] \\ \exists\delta>0,\quad f\in C^{1,\delta}([0,1]). \end{array}\right. \tag{2.5}$$

Case (2.4) is usually referred to as the combustion nonlinearity with positive ignition temperature θ [36]. Case (2.5) can be viewed as a combustion nonlinearity with ignition temperature equal to 0 [36], or can also be thought of as the production rate of a population in biological models [1], [22], [38], in which case the quantity u represents the density of a population.

One is interested in some particular solutions of (2.1-2.2), namely the pulsating travelling fronts, which propagate in a given direction, say to the left, with an unknown effective speed $c \neq 0$, in the sense that

$$u\left(t + \frac{L}{c}, x, y\right) = u(t, x + L, y) \text{ for all } t \in I\!R,\ (x, y) \in \overline{\Omega}. \tag{2.6}$$

Such fronts are assumed to have prescribed limiting conditions as $x \to \pm\infty$:

$$\forall t \in I\!R, \quad u(t, -\infty, y) = 0, \quad u(t, +\infty, y) = 1 \text{ uniformly with respect to } y. \tag{2.7}$$

Such pulsating fronts (which correspond to flames with pulsating shapes in combustion theory) are of particular interest since, in periodic media, they can describe the behavior at large time of the solutions of the related Cauchy problem with front-like initial conditions. However, the question of the stability of the pulsating solutions is not addressed here.

The first analyses of the propagation phenomena for advection-diffusion-reaction equations like (2.1) have dealt with the case of planar travelling fronts, for one-dimensional equations $u_t = u_{xx} + f(u)$ with a zero velocity field $q = 0$. Such travelling fronts $u(t, x)$ move with constant speed and their shape does not change as time runs : they satisfy (2.6) for any $L \in I\!R$ and can be written as $u(t, x) = \phi(x + ct)$. Travelling fronts are then particular types of pulsating traveling fronts. Since the pioneering paper of Kolmogorov, Petrovsky and Piskunov [38] in 1937 for nonlinearities of the type (2.5), there have been many papers on the questions of existence, uniqueness or stability properties of planar travelling fronts for various kinds of reaction terms $f(u)$, more general than (2.4) or (2.5), arising in combustion or biological models (*see e.g.* Aronson and Weinberger [1], Fife and McLeod [21], Kanel' [36]). In the case (2.4), there exists a unique speed c (which is positive) and a unique - up to translation - front u. In the case (2.5), travelling fronts with speed c exist if and only if $c \geq c^*$ for some (positive) minimal speed c^* and, for any given $c \geq c^*$, the fronts with speed c are unique up to translation. Many papers have also been devoted to the study of planar travelling fronts for systems of one-dimensional diffusion-reaction equations [5], [12], [14], [19], [42], [51].

For the one-dimensional equation $u_t = u_{xx} + f(x, u)$ with no advection and with a function f similar to (2.5), Hudson and Zinner [35] have got the existence of a semi-infinite line $[c^*, +\infty)$ of possible speeds of pulsating travelling fronts, as well as the formula (7.2) below for c^*.

These existence and uniqueness results have almost entirely been generalized in the multi-dimensional case of straight infinite cylinders $\Omega = I\!R \times \omega$ with shear flows $q = (\alpha(y), 0, \cdots, 0)$, by Berestycki, Larrouturou, Lions [10] and Berestycki, Nirenberg [13]. *In the case of shear flows*, the velocity field q is L-periodic in x for all L and the equation (2.1) is invariant by translation in the variable x. In this framework, travelling fronts are solutions of the type $u(t, x, y) = \phi(x + ct, y)$ (the problem for travelling fronts is then reduced to a semilinear elliptic equation for the function ϕ). The known results for these travelling fronts are the following : if f is of type (2.4), there exists a unique speed c and a unique travelling front $\phi(x + ct, y)$ (ϕ is increasing in $s = x + ct$ and unique up to translation in s) whereas if f is of type (2.5), there exists a speed c^* such that travelling fronts $\phi(x + ct, y)$ exist if and only if $c \geq c^*$ and, for each given $c \geq c^*$, the front ϕ is increasing and unique up to translation in s if $f'(0) > 0$. The cases of monotone shear flows along the main direction of the cylinder, and of almost parallel flows with more general reaction terms have been considered in [27], [28]. Lastly, similar existence or uniqueness results with Dirichlet conditions on $\partial\Omega$ have been obtained in [24] and [50].

Many works have been devoted to the behavior at large time, and especially to the convergence to travelling fronts, of solutions of Cauchy problems for equations like (2.1) under a large class of initial conditions. These works have been initiated by Kolmogorov, Petrovsky and Piskunov [38] in the one-dimensional case with no advection (see also [1], [15], [21], [49]) and followed by the study of the stability of travelling waves in infinite cylinders with shear flows (*see* [11], [40], [46], [47]). So far, few works have dealt with the question of the stability of pulsating travelling fronts in periodic media like the real line or the whole space [39], [53].

The above results for shear flows can be for the most part generalized for pulsating travelling fronts in straight infinite cylinders with periodic advection q :

Theorem 2.1 [7] *Let q be a velocity field satisfying (2.3).*

1) If f satisfies (2.4), there exists a unique solution (c, u) of (2.1)-(2.2) and (2.6)-(2.7), u being increasing in t and unique up to translation in t. Moreover, $0 < u < 1$ and $c > 0$.

2) If f satisfies (2.5), there exists a positive real number c^* such that : if $c < c^*$, there is no solution (c, u) of (2.1)-(2.2) and (2.6)-(2.7); if $c \geq c^*$, there exists a solution (c, u), such that $0 < u < 1$ and u is increasing in t; if $f'(0) > 0$ and $c \geq c^*$, then any solution u of (2.1)-(2.2) and (2.6)-(2.7) is increasing in t.

Remark 2.2 If f satisfies (2.5) and the additional assumption $f'(0) > 0$, one conjectures that, for each speed $c \geq c^*$, the solutions u are unique up to translation in t.

Remark 2.3 The function u may not be increasing in the variable x. This indeed can be observed in some remarquable experiments carried out by P. Ronney and collaborators [45] on some Taylor-Couette cells in the framework of autocatalytic chemical waves.

3 Cylinder type domains with periodic boundaries

The periodicity of the velocity field can actually derive directly from the periodicity of the domain. That is the case when, instead of a straight infinite cylinder, one considers an infinite cylinder Ω with a smooth and oscillating boundary :

$$\Omega = \{(x, y) \in I\!R^N,\ x \in I\!R,\ y \in \omega(x)\} \tag{3.1}$$

where the function $x \mapsto \omega(x)$ is periodic with period $L > 0$. Straight infinite cylinders correspond to the case where $\omega = constant$. Let now q be a $C^1(\overline{\Omega})$ velocity field satisfying

$$\left\{\begin{array}{rcl} \operatorname{div} q & = & 0 \text{ in } \overline{\Omega} \\ \forall (x,y) \in \overline{\Omega},\quad q(x+L, y) & = & q(x, y) \\ \displaystyle\int_{\{x\in(0,L),\ y\in\omega(x)\}} q_1(x, y)\, dx\, dy & = & 0 \\ q \cdot \nu & = & 0 \text{ on } \partial\Omega. \end{array}\right. \tag{3.2}$$

In the case where f is of the "bistable" type and where $q = 0$, some conditions for the existence or non-existence of pulsating travelling fronts have been given by Matano [41].

In the cases where f is of the types (2.4) or (2.5), the same result as Theorem 2.1 holds :

Theorem 3.1 [7] *Under the assumptions (3.1)-(3.2), parts 1) and 2) of Theorem 2.1 hold.*

4 Fronts in the whole space with periodic flows

A natural question about pulsating travelling fronts concerns the case where the domain Ω is the whole space $I\!R^N$. Let us consider the advection-diffusion-reaction equation

$$\frac{\partial u}{\partial t} - \Delta u + q(x)\cdot\nabla_x u = f(u), \quad t\in I\!R,\ x\in I\!R^N. \tag{4.1}$$

If the velocity field q in (4.1) is equal to a constant vector q_0, then planar travelling fronts of the type $u(t,x)=\phi(x\cdot e+ct)$, propagating in a given direction $-e\in S^{N-1}$, exist in both cases (2.4) or (2.5), and the set of possible speeds is equal to the set of planar speeds for the equation with $q\equiv 0$, translated with the shift $q_0\cdot e$.

Similarly, if q is a shear flow $q=\alpha(x)e$ where $e\cdot\nabla\alpha=0$ and α is periodic with respect to the variables orthogonal to e, travelling fronts of the type $u(t,x)=\phi(x\cdot e+ct, x\cdot e_2,\cdots,x\cdot e_N)$, where e has been completed into an orthonormal basis $(e,e_2,\cdots,e_N)$ of $I\!R^N$, also exist. In that last case, planar travelling fronts of the type $u(x,t)=\phi_0(x\cdot e'+c_0t)$ exist for any direction $e'\in S^{N-1}$ such that $e'\perp e$, where the couple (c_0,ϕ_0) does not depend on q and is the unique solution of $\phi_0''-c_0\phi_0'+f(\phi_0)=0$ with $\phi_0(-\infty)=0$, $\phi_0(+\infty)=1$. Furthermore, it can easily be checked in that case that, provided that $q=\alpha(x)e$ is not constant, there exists no travelling front in a direction e' other than $\pm e$ or the directions perpendicular to e. This example shows that, even for shear flows, the notion of travelling fronts is not sufficient to describe the propagation of fronts in most of the directions of S^{N-1}.

Let now q be a divergence-free velocity field q, of class $C^1(I\!R^N)$, L-periodic with respect to the space variables, in the sense that there exists an N-uple $(L_i)\in(I\!R_+^*)^N$ such that

$$\left\{\begin{array}{rcl} \text{div}\, q &=& 0 \ \text{ in } I\!R^N\\ \forall k\in\prod_{i=1}^N L_i\mathbb{Z},\quad \forall x\in I\!R^N,\quad q(x+k) &=& q(x)\\ \int_{\prod_{i=1}^N(0,L_i)} q(x)\,dx &=& 0.\end{array}\right. \tag{4.2}$$

Under the above assumptions, pulsating travelling fronts for (4.1) are the solutions $u(t,x)$ which propagate in a given direction, say $-e\in S^{N-1}$, with an effective speed $c\neq 0$:

$$\left\{\begin{array}{l} \forall k\in\prod_{i=1}^N L_i\mathbb{Z},\quad \forall x\in I\!R^N,\quad u\left(t+\dfrac{k\cdot e}{c},x\right) = u(t,x+k)\\ \forall t\in I\!R,\quad u(t,x)\underset{x\cdot e\to-\infty}{\longrightarrow}0,\quad u(t,x)\underset{x\cdot e\to+\infty}{\longrightarrow}1,\end{array}\right. \tag{4.3}$$

where the above limits hold locally in t and uniformly in the variables orthogonal to e.

The questions of the existence and uniqueness of pulsating travelling fronts have been solved by Xin [54], [56] in the case of a combustion nonlinearity f satisfying (2.4), under the additional assumption $f'(1)<0$: for each given $e\in S^{N-1}$, there exists a unique solution $u(t,x)$ of (4.1) and (4.3), and u is increasing and unique up to translation in t. This result, which actually holds for more general equations involving space-dependent diffusion terms (*see* also section 6) has been proved through a continuation method based on some invertibility properties of linearized operators. This method does not seem to easily extend to the case of a nonlinearity

f satisfying (2.5), whereas the method used in [7] allows for the following Theorem 4.1, similar to Theorems 2.1 and 3.1. Before stating this result, let us mention that the homogenization limit with velocity fields or diffusion matrices involving very small scales has been carried out by Freidlin [23], Heinze [31] and Xin [56]. Lastly, the question of front propagation in random media has been considered in [23] and [56].

Let us now turn to the statement of the following existence, monotonicity and uniqueness result of pulsating travelling fronts for the equation (4.1) :

Theorem 4.1 [7] *Let q be a C^1 velocity field satisfying (4.2) and let $e \in S^{N-1}$ be a unit vector. If f is of the type (2.4), there exists a unique solution $(c,u) = (c(e), u(e))$ of (4.1) and (4.3), the function u being increasing and unique up to translation in t. If f is of the type (2.5), there exists $c^* = c^*(e) > 0$ such that no solution (c,u) exists if $c < c^*$, and, for each $c \geq c^*$, a time-increasing solution u exists and all solutions u are increasing in t if $f'(0) > 0$.*

5 Periodic media with holes

Another class of periodic domains and media is the case where the domains have periodic holes. For instance, consider first the case of the whole space with periodic holes; namely, let Ω be a domain with a smooth boundary and such that

$$\exists\ (L_i)_{1\leq i\leq N} \in (\mathbb{R}_+^*)^N, \quad \forall\ k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \qquad \Omega + k = \Omega. \tag{5.1}$$

Let $\nu = \nu(x)$ be the outward unit normal to Ω. Let q be a $C^1(\overline{\Omega})$ velocity field such that

$$\left\{\begin{array}{rcl} \text{div } q &=& 0 \text{ in } \overline{\Omega} \\ \forall k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \quad \forall x \in \overline{\Omega}, \quad q(x+k) &=& q(x) \\ \displaystyle\int_{\prod_{i=1}^{N}(0,L_i)\ \cap\ \Omega} q(x)\ dx &=& 0 \\ q\cdot\nu &=& 0 \text{ on } \partial\Omega. \end{array}\right. \tag{5.2}$$

A pulsating travelling front in a direction $-e \in S^{N-1}$ is a solution (c,u) (with $c \neq 0$) of

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} - \Delta u + q(x)\cdot\nabla_x u &=& f(u), \quad t\in\mathbb{R},\ x\in\overline{\Omega} \\ \partial_\nu u &=& 0, \qquad t\in\mathbb{R},\ x\in\partial\Omega \\ \forall k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \quad \forall x\in\overline{\Omega}, \quad u\left(t+\dfrac{k\cdot e}{c}, x\right) &=& u(t, x+k) \\ \forall t\in\mathbb{R}, \quad u(t,x) \underset{x\cdot e\to-\infty}{\longrightarrow} 0, \quad u(t,x) \underset{x\cdot e\to+\infty}{\longrightarrow} & & 1, \end{array}\right. \tag{5.3}$$

where the above limits hold locally in t and uniformly in the variables orthogonal to e.

For a nonlinearity f satisfying (2.4), the existence of pulsating travelling fronts has been proved by Heinze [32] in the limit of asymptotically small holes, by using a perturbation technique around the homogenized equation.

With the method used in [7], the same result as for the whole space holds :

Theorem 5.1 [7] *If (5.1) and (5.2) are satisfied, Theorem 4.1 holds for the solutions of (5.3).*

6 General periodic domains

The results presented above can all be written in a more general framework which we describe now. Let Ω be a connected unbounded open set, with a smooth boundary, and such that

$$\begin{cases} \exists 1 \leq d \leq N,\ \exists L_1, \cdots, L_d > 0,\ \forall k = (k_i)_{1 \leq i \leq d} \in \prod_{i=1}^{d} L_i \mathbb{Z}, \qquad \Omega + \sum_{i=1}^{d} k_i e_i = \Omega \\ \text{and } \Omega \text{ is bounded with respect to the variables } x_{d+1}, \cdots, x_N, \end{cases} \tag{6.1}$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of $\mathbb{R}^N$. Let us denote by $x = (x_1, \cdots, x_d)$ the first d coordinates and by $y = (x_{d+1}, \cdots, x_N)$ the last $N - d$ ones. Let $\nu = \nu(x, y)$ be the outward unit normal to Ω. Let C be the periodicity cell defined by

$$C = \{(x, y) \in \Omega,\ x \in (0, L_1) \times \cdots \times (0, L_d)\}.$$

We say that a field $v(x, y)$ defined in Ω is L-periodic with respect to the variable x if $v(x+k, y) = v(x, y)$ for all $k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}$ and for all $(x, y) \in \overline{\Omega}$.

Note that that class of domains includes all domains described above : the infinite cylinders with straight or oscillating boundaries, the whole space with or without periodic holes. Domains of the class (6.1) also include infinite cylinders or slabs with periodic holes.

Let $q = (q_1, \cdots, q_N)$ denote a globally C^1 vector field defined in $\overline{\Omega}$ and such that

$$\begin{cases} \operatorname{div} q &= 0 \text{ in } \overline{\Omega} \\ q \text{ is } L\text{-periodic w.r.t. } x & \\ \forall 1 \leq i \leq d,\ \int_C q_i\, dx\, dy &= 0 \\ q \cdot \nu &= 0 \text{ on } \partial\Omega. \end{cases} \tag{6.2}$$

Furthermore, let $A(x, y) = (A_{ij}(x, y))_{1 \leq i,j \leq N}$ be a globally $C^1(\overline{\Omega})$ matrix field such that

$$\begin{cases} \exists 0 < c_1 \leq c_2, \quad \forall \xi \in \mathbb{R}^N, \quad \forall (x, y) \in \overline{\Omega}, \quad c_1|\xi|^2 \leq \sum_{1 \leq i,j \leq N} A_{ij}(x, y)\xi_i\xi_j \leq c_2|\xi|^2 \\ A \text{ is symmetric and } L\text{-periodic w.r.t. } x. \end{cases} \tag{6.3}$$

In the sequel, if z and z' are two vectors in $\mathbb{R}^N$ and B is an $N \times N$-matrix, then zBz' denotes the number $zBz' := \sum_{1 \leq i,j \leq N} z_i B_{ij} z'_j$.

Let e be any given unit vector in $\mathbb{R}^d$ and let f be of the type (2.4) or (2.5). Let us now study the questions of the existence and of the qualitative properties of pulsating travelling fronts $u(t, x, y)$, moving in direction $-e$ with an effective speed $c \neq 0$, and solving

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) + q \cdot \nabla u &= f(u), \quad t \in \mathbb{R},\ (x, y) \in \overline{\Omega} \\ \nu A \nabla u &= 0, \quad t \in \mathbb{R},\ (x, y) \in \partial\Omega \\ \forall k \in \prod_{i=1}^{d} L_i\mathbb{Z}, \quad u\left(t + \dfrac{k \cdot e}{c}, x, y\right) &= u(t, x + k, y) \text{ for all } (t, x, y) \in \mathbb{R} \times \overline{\Omega} \\ u(t, x, y) \underset{x \cdot e \to -\infty}{\longrightarrow} 0, \quad u(t, x, y) &\underset{x \cdot e \to +\infty}{\longrightarrow} 1 \text{ for each } (t, y), \end{cases} \tag{6.4}$$

where the above limits hold locally in t and uniformly in y and in the directions of $I\!R^d$ orthogonal to e.

That framework for the propagation of pulsating travelling fronts contains all situations described in the previous sections. Note that the Laplace operator has been replaced with a general heterogeneous diffusion operator $\mathrm{div}(A\nabla u)$. Such operators have also been considered in the onedimensional case or in the case of the whole space (*see* [44], [53], [54], [55], [56]).

In that general framework, the following Theorem, generalizing Theorems 2.1, 3.1, 4.1 and 5.1, holds :

Theorem 6.1 [7] *Let Ω be a domain satisfying (6.1). Let e be a unit vector in $I\!R^d$. Let q and A be two globally $C^1(\overline{\Omega})$ vector and matrix fields satisfying (6.2) and (6.3).*

1) Let f be a nonlinearity of the ignition temperature type (2.4). There exists a unique solution $(c,u)=(c(e),u(e))$ of (6.4), the function u being increasing and unique up to translation in t. Moreover, $0<u<1$ and $c(e)>0$.

2) Let f be a nonlinearity of the type (2.5). There exists $c^*(e)>0$ such that problem (6.4) has no solution (c,u) if $c<c^*(e)$ while, for each $c\geq c^*(e)$, it has a solution (c,u) such that u is increasing in t. Moreover, if $f'(0)>0$, then any solution u of (6.4) is increasing in t.

Remark 6.2 Theorems 2.1, 3.1, 4.1 and 5.1 hold in the general case where the Laplace operator is replaced with a divergence type operator $\mathrm{div}(A\nabla u)$ together with Neumann type boundary conditions $\nu A\nabla u=0$ on $\partial\Omega$.

Remark 6.3 All above theorems work in the case where the nonlinearity $f(u)$ is replaced with $h(x,y)f(u)$ if h is a continuous, positive function which is L-periodic w.r.t. x (*see* [7]).

At this stage, the question of the uniqueness of the pulsating travelling fronts for each speed $c\geq c^*$, in the case where f satisfies (2.5), remains open, even under the assumption $f'(0)>0$.

Another related open problem concerns the case where the function f is of the bistable type, namely, there exists $\theta\in(0,1)$ such that $f(0)=f(\theta)=f(1)$, $f<0$ on $(0,\theta)$, $f>0$ on $(\theta,1)$ and f is nonincreasing in a right neighborhood of 0 and in a left neighborhood of 1. Some conditions for the existence or nonexistence of pulsating travelling fronts in infinite cylinders with periodic boundary have been given by Matano [41]. Other existence, nonexistence or stability results have been obtained by Xin [53], [55] and Papanicolaou and Xin [44] in the case of the whole space with almost uniform diffusion and advection coefficients, and by Nakamura [43] for the one-dimensional case with periodic diffusion coefficient.

Lastly, let us mention here that the methods used in [7] to prove the uniqueness and monotonicity properties of the pulsating travelling fronts in the case of a nonlinearity f with positive ignition temperature (2.4) actually work and lead to the same uniqueness and monotonicity results in the case of a bistable nonlinearity f.

7 Further results : formulas for the speeds

One of the most important questions related to the front propagation phenomena is the determination of the speed of propagation of the travelling fronts, or of the pulsating travelling fronts in the periodic framework. In the theory of combustion for instance, the determination of the burning velocity of a deflagration flame is a fundamental question.

Many works have been devoted to finding some formulas for the speeds of propagation of travelling waves for advection-diffusion-reaction equations more general than those arising in combustion models. The first formula comes back to the paper of Kolmogorov, Petrovsky and Piskunov [38] and concerns the minimal speed $c^* = 2\sqrt{f'(0)}$ of planar travelling fronts for the equation $u_t = u_{xx} + f(u)$ with nonlinearities of the "Fisher-KPP" ([22], [38]) type

$$\begin{array}{l} f(0) = f(1) = 0,\ f > 0,\ f(s) \le f'(0)s \text{ on }]0,1[\\ \text{and } \exists \mu > 0,\ f \text{ is nonincreasing on } [1-\mu, 1]. \end{array} \tag{7.1}$$

Other formulas of the variational type have been derived for such one-dimensional equations. Let us for instance mention the formula

$$c^* = \min_{\rho:[0,1]\to I\!R,\ \rho(0)=0,\ \rho'(0)>0,\ \rho>0 \text{ in } (0,1]} \ \sup_{u\in(0,1]} \left(\rho'(u) + \frac{f(u)}{\rho(u)}\right)$$

of Hadeler and Rothe [26] for nonlinearities of the type (2.5). The latter implies $2\sqrt{f'(0)} \le c^* \le 2\sqrt{\sup_{(0,1]} f(u)/u}$ and gives $c^* = 2\sqrt{f'(0)}$ in the case (7.1). Integral formulations have been given by Benguria and Depassier [4]. Other variational formulas have been obtained for systems of one-dimensional equations [42], [48], [51], or for equations with discrete diffusion [30]. Some formulas have been generalized by Hamel [29], Heinze, Papanicolaou and Stevens [33] in the multidimensional case with shear flows, and by Hudson and Zinner [34] in the discrete case. For instance, in the case (2.4), the unique speed c of travelling fronts $\phi(x+ct,y)$ solving (2.1) in a cylinder $\Omega = I\!R \times \omega$ with a shear flow $q = (\alpha(y), 0, \cdots, 0)$, is given by

$$c = \min_{w\in\mathcal{E}} \sup_{(x_1,y)\in\overline{\Omega}} \left(\frac{\Delta w + f(w)}{\partial_x w} - \alpha(y)\right) = \max_{w\in\mathcal{E}} \inf_{(x_1,y)\in\overline{\Omega}} \left(\frac{\Delta w + f(w)}{\partial_x w} - \alpha(y)\right)$$

where $\mathcal{E} = \{w \in W^{2,p}_{loc}(\Omega),\ \Delta w \in C(\overline{\Omega}),\ 0 < w < 1,\ \partial_x w > 0 \text{ in } \overline{\Omega},\ \partial_\nu w = 0 \text{ on } \partial\Omega,\ w(-\infty,\cdot) = 0,\ w(+\infty,\cdot) = 1\}$ and $p > N$ (*see* [29]). In the case (2.5) with $f'(0) > 0$, the minimal speed c^* for travelling fronts is equal to

$$c^* = \min_{w\in\mathcal{E}} \sup_{(x_1,y)\in\overline{\Omega}} \left(\frac{\Delta w + f(w)}{\partial_x w} - \alpha(y)\right).$$

Explicit formulas for the speeds of propagation of travelling waves have been obtained in some asymptotic cases, like in the limit of high activation energies (*see* [12] in the one-dimensional case, and [6] in the multi-dimensional case). Formal asymptotics in the case of shear flows with large amplitude have been derived by Audoly, Berestycki and Pomeau in [3].

We also refer to [17], [18] and [37] for some *a priori* bounds of the speeds of propagation of the solutions of the Cauchy problem associated to (2.1) with front-like initial conditions. Namely, Constantin, Kiselev, Oberman and Ryzhik have defined the notion of bulk burning rate $V(t) = |\omega|^{-1}\int_{I\!R\times\omega} u_t(t,x,y)dxdy$ ($|\omega|$ is the Lebesgue-measure of ω), and, from a subtle decomposition of the velocity field q into positive and negative parts, they have obtained some lower bounds for $V(t)$ (or for the time-average of $V(t)$) if u is a solution of the corresponding Cauchy problem with front-like initial conditions [17], [37]. These bounds have been obtained both for shear-like percolating or cellular flows and especially lead to some lower bounds for the effective speed c of any pulsating travelling front solving (2.1-2.2) and (2.6-2.7), since, for such a solution u, one has $T^{-1}\int_{t_0}^{t_0+T}V(t)dt = c$ with $T = L/c$, for any $t_0 \in I\!R$.

For pulsating travelling fronts in periodic media, the only formula, derived by Hudson and Zinner [35], concerns the minimal speed of propagation in the one-dimensional case $u_t = u_{xx} + f(x,u)$, where f is 1-periodic in x, $f(x,u) > 0$ for $u \in]0,\overline{u}(x)[$, $f(x,0) = f(x,\overline{u}(x)) = 0$ and $\mu(x) = f'_u(x,0) = \sup_{u\in]0,\overline{u}(x)[}\ f(x,u)/x$. Namely, Hudson and Zinner have obtained the following formula for the minimal speed :

$$c^* = \min_{r>0}\ \min_{\{\psi=\psi(x)\in C^2(I\!R),\ \psi>0,\ \psi\ 1-\text{periodic}\}}\ \max_{x\in[0,1]} \frac{\psi'' + 2r\psi' + (r^2+\mu(x))\psi}{r\psi}. \tag{7.2}$$

In the paper [8], the question of the determination of the minimal speed of pulsating travelling waves solving (6.4) in a domain of the class (6.1) is considered under the additional assumption that the function f satisfies (7.1).

From Theorem 6.1, under the assumptions (6.1)-(6.3) and for each given unit direction e of $I\!R^d$, there exists a minimal speed $c^*(e)$ of the pulsating travelling fronts. Our goal in [8] has been to find an explicit formula for the minimal speed $c^*(e)$.

We have obtained the following equivalent variational formulas for $c^*(e)$:

$$c^*(e) = \min\ \{c,\ \exists\lambda > 0,\ \mu_c(\lambda) = f'(0)\} \tag{7.3}$$

where $\mu_c(\lambda)$ is the principal eigenvalue of the elliptic operator $-L_{c,\lambda}\psi = -\text{div}(A\nabla\psi) - \lambda(\text{div}(A\tilde{e}\ \psi) + \tilde{e}A\nabla\psi) + q\cdot\nabla\psi + (\lambda q\cdot\tilde{e} + \lambda c - \lambda^2\tilde{e}A\tilde{e})\psi$ on the set E of L-periodic with respect to x functions $\psi(x,y)$ such that $\nu A(\tilde{e}\lambda\psi + \nabla\psi) = 0$ on $\partial\Omega$. Here, $\tilde{e}$ denotes the vector $\tilde{e} = (e_1,\cdots,e_d,0,\cdots,0)$. Thus, under the KPP assumption (7.1), the minimal speed $c^*(e)$ can be explicitely given in terms of e, the domain Ω, the coefficients q and A and of $f'(0)$. In the general case where f satisfies (2.5) and $f'(0) > 0$, the minimal speed $c^*(e)$ is always greater than or equal to the right hand side of (7.3). Note also that the formula (7.3) is similar to that of Berestycki and Nirenberg [13] for travelling waves in infinite cylinders with shear flows.

As observed in [56], the above formula (7.3) is equivalent to the following one :

$$c^*(e) = \min_{\lambda>0}\frac{-k(\lambda)}{\lambda} \tag{7.4}$$

where $k(\lambda)$ is the principal eigenvalue of the operator $-L_\lambda\psi = -\text{div}(A\nabla\psi) - \lambda(\text{div}(A\tilde{e}\ \psi) + \tilde{e}A\nabla\psi) + q\cdot\nabla\psi + (\lambda q\cdot\tilde{e} - \lambda^2\tilde{e}A\tilde{e} - f'(0))\psi$ on the same set E of functions ψ as above. Note that the formula (7.4) is similar to that of Gärtner and Freidlin [25] for the asymptotic speed of propagation of solutions of Cauchy problem in $I\!R^N$ with compactly supported initial conditions and periodic diffusion coefficients (*see* [8] for a further study of the asymptotic speeds of propagation). Note also that when $\Omega = I\!R^N$, $A = I$ and $q = 0$, this formula (7.4) gives the KPP formula $c^*(e) = 2\sqrt{f'(0)}$ for the minimal speed of planar fronts.

Lastly, the following formula also holds

$$c^*(e) = \min_{\lambda>0} \min_{\psi\in F} \max_{(x,y)\in\overline{\Omega}} \frac{L_\lambda \psi}{\lambda\psi} \tag{7.5}$$

where $F = \{\psi \in E, \psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \overline{\Omega}\}$. This formula is obtained from (7.4) and from some characterizations of principal eigenvalues of elliptic operators. This formula (7.5) for the minimal speed of multidimensional pulsating fronts generalizes the formula (7.2) of Hudson and Zinner [35] for the minimal speed of pulsating travelling fronts in the case of one-dimensional equations of the type $u_t = u_{xx} + f(x,u)$.

8 Short sketch of the proofs

The monotonicity and uniqueness results stated in part 1) of Theorem 6.1, in the case where the function f satisfies (2.4), are based on a sliding method in another set of variables $(s,x,y) = (x\cdot\tilde{e}+ct,x,y)$, for which the equation is *elliptic degenerate*, and on the parabolic maximum principle in the original variables (t,x,y) (remember that for travelling fronts with constant speed c, the equation of the profile of the front is *elliptic* in some variables, say $(x+ct,y)$ in the case of an infinite straight cylinder). The existence of a solution (c,u) in part 1) of Theorem 6.1 is obtained as a limit of solutions of regularized elliptic equations in approximated bounded domains. The main difficulty is to deal with the degeneracy of the equations and to prove that the solution obtained at the limit is not trivial. One especially proves some Bernstein-type gradient estimates and one uses some exponentially decaying upper solutions in some semi-infinite domains.

In the case where the function f satisfies (2.5), the existence of a solution for the minimal speed $c^*(e)$ is obtained as a limit of solutions for nonlinearities f_θ of the type (2.4) and approximating f (with small ignition temperatures θ). The existence of solutions for any speed $c \geq c^*(e)$ is obtained through a method using sub- and super-solutions, and the non-existence of solutions with speeds $c < c^*(e)$ follows from a sliding method and from a comparison with suitable sub-solutions.

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