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ON THE FEKETE-SZEGÖ AND ARGUMENT INEQUALITIES FOR STRONGLY CLOSE-TO-STAR FUNCTIONS

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ABSTRACT. Let $\mathcal{CS}(\beta)$ be the class of normalized strongly close-to-star functions of order β in the open unit disk. We obtain sharp Fekete-Szegö inequalities for functions belonging to the class $\mathcal{CS}(\beta)$. Some sufficient conditions for close-to-star functions also are investigated in a sector. Furthermore, we consider the integral preserving properties for functions in $\mathcal{CS}(\beta)$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions. We also denote by $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{U}$ (see, e.g., Srivastava and Owa [18]).

For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function $w(z)$ such that $w(0) = 0, |w(z)| < 1 \quad (z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

Let

$$\mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}\ ;\ -1 \le B < A \le 1) \right\}$$

and

$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}\ ;\ -1 \le B < A \le 1) \right\}.$$

The class $\mathcal{S}^*[A,B]$ was studied by Janowski [5] and (more recently) by Silverman and Silvia [17]. Applying the Briot-Bouquet differential

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subordination [10, p. 81], we can easily see that $\mathcal{K}[A,B] \subset \mathcal{S}^*[A,B]$. We also note that $\mathcal{S}^*[1,-1] = \mathcal{S}^*$ and $\mathcal{K}[1,-1] = \mathcal{K}$. Furthermore, Silverman and Silvia [17] proved that a function f is in $\mathcal{S}^*[A,B]$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (z \in \mathcal{U}\ ;\ B \neq -1) \tag{1.2}$$

and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-A}{2} \qquad (z \in \mathcal{U}\ ;\ B = -1). \tag{1.3}$$

A classical result of Fekete and Szegö [4] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter μ, for functions belonging to $\mathcal{S}$. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [2,6-8,14]).

Denote by $\mathcal{CS}(\beta)$ the class of strongly close-to-star functions of order $\beta (\beta \geq 0)$. Thus $f \in \mathcal{CS}(\beta)$ if and only if there exists $g \in \mathcal{S}^*$ such that for $z \in \mathcal{U}$,

$$\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| \leq \frac{\pi}{2}\beta.$$

For the case $\beta = 1$, $\mathcal{CS}(\beta)$ is the class of close-to-star functions introduced by Reade [16]. The close-to-star and similar other functions have been extensively studied by Ahuja and Mogra [1], Padmanabhan and Parvatham [12], Paravatham and Srinivasan [13], Sudharsan et. al. [19] and others.

In the present paper, we prove sharp Fekete-Szegö inequalities for functions belonging to the class $\mathcal{CS}(\beta)$. Argument properties also are investigated, which give conditions for close-to-star functions. Furthermore, we consider the integral preserving properties for functions in the class $\mathcal{CS}(\beta)$.

2. Results

To prove our main results, we need the following lemmas.

Lemma 2.1 [3,15]. *Let p be analytic in $\mathcal{U}$ and satisfy* $\operatorname{Re}\{p(z)\} > 0$ *for $z \in \mathcal{U}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then*

$$|p_n| \leq 2 \quad (n \geq 1)$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.2 [11]. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\left|\arg\ \{p(z)\}\right| \quad < \quad \frac{\pi}{2}\eta \quad \textit{for } |z| < |z_0| \tag{2.1}$$

and

$$\left|\arg\ \{p(z_0)\}\right| \quad = \quad \frac{\pi}{2}\eta (0 < \eta \leq 1). \tag{2.2}$$

Then

$$\frac{z_0 p\,'(z_0)}{p(z_0)} \quad = \quad ik\eta, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg\ \{p(z_0)\} = \frac{\pi}{2}\eta, \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg\ \{p(z_0)\} = -\frac{\pi}{2}\eta, \tag{2.5}$$

and

$$\{p(z_0)\}^{\frac{1}{\eta}} = \pm ia \ \ (a > 0). \tag{2.6}$$

Lemma 2.3 [9]. *Let h be convex(univalent) function in $\mathcal{U}$ and ω be an analytic function in $\mathcal{U}$ with* Re $\{\omega(z)\} \geq 0$. *If p is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then*

$$p(z) + \omega(z) z p'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$

With the help of Lemma 2.1, we now derive

Theorem 2.1. *Let $f \in \mathcal{CS}(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 + 2(1+\beta)^2(1-2\mu) & \textit{if } \mu \leq \frac{\beta}{2(1+\beta)}, \\ 1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)} & \textit{if } \frac{\beta}{2(1+\beta)} \leq \mu \leq \frac{1}{2}, \\ 1 + 2\beta & \textit{if } \frac{1}{2} \leq \mu \leq \frac{2+\beta}{2(1+\beta)}, \\ -1 + 2(1+\beta)^2(2\mu - 1) & \textit{if } \mu \geq \frac{2+\beta}{2(1+\beta)}. \end{cases}$$

For each μ, there is a function in $\mathcal{CS}(\beta)$ such that equality holds in all cases.

Proof. Let $f \in \mathcal{CS}(\beta)$. Then it follows from the definition that we may write

$$\frac{f(z)}{g(z)} = p^{\beta}(z),$$

where g is starlike and p has positive real part. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, and let p be given as in Lemma 2.1. Then by equating coefficients, we obtain

$$a_2 = b_2 + \beta p_1$$

and

$$a_3 = b_3 + \beta p_1 b_2 + \frac{\beta(\beta-1)}{2} p_1^2 + \beta p_2.$$

So, with $x = 1 - 2\mu$, we have

$$(a_3 - \mu a_2^2) = b_3 + \frac{1}{2}(x-1)b_2^2 + \beta\left(p_2 + \frac{1}{2}(\beta x - 1)p_1^2\right) + \beta x p_1 b_2. \quad (2.7)$$

Since rotations of f also belong to $\mathcal{CS}(\beta)$, we may assume, without loss of generality, that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\mathrm{Re}(a_3 - \mu a_2^2)$.

For some functions $h(z) = 1 + k_1 z + k_2 z^2 + \cdots$ $(z \in \mathcal{U})$ with positive real part, we have $zg'(z) = g(z)h(z)$. Hence, by equating coefficients, $b_2 = k_1$ and $b_3 = (k_2 + k_1^2)/2$. So by Lemma 2.1,

$$\begin{aligned}\mathrm{Re}\left(b_3 + \frac{1}{2}(x-1)b_2^2\right) &= \frac{1}{2}\mathrm{Re}\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1+2x}{4}\mathrm{Re}k_1^2 \\ &\le 1 - \rho^2 + (1+2x)\rho^2\cos 2\phi, \qquad (2.8)\end{aligned}$$

where $b_2 = k_1 = 2\rho e^{i\theta\phi}$ for some ρ in [0,1]. We also have

$$\begin{aligned}\mathrm{Re}\left(p_2 + \frac{1}{2}(\beta x - 1)p_1^2\right) &= \mathrm{Re}\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{2}\beta x \mathrm{Re}p_1^2 \\ &\le 2(1-r^2) + 2\beta x r^2\cos 2\theta, \qquad (2.9)\end{aligned}$$

where $p_1 = 2re^{i\theta}$ for some r in [0,1]. From (2.7-9), we obtain

$$\begin{aligned}\mathrm{Re}(a_3 - \mu a_2^2) \le 1 - \rho^2 &+ (1+2x)\rho^2\cos 2\phi + 2\beta((1-r^2) \\ &+ \beta x r^2 \cos 2\theta + 2xr\rho\cos(\theta+\phi)), \qquad (2.10)\end{aligned}$$

and we now proceed to maximize the right-hand side of (2.10). This function will be denote ψ whenever all parameters except x are held constant.

Assume that $\beta/(2(1+\beta)) \le \mu \le 1/2$, so that $0 \le x \le 1/(1+\beta)$. Since the expression $-t^2 + t^2\beta x\cos 2\theta + 2xt$ is the largest when $t = x/(1 - \beta x \cos 2\theta)$, we have

$$-t^2 + t^2\beta x \cos 2\theta + 2xt \le \frac{x^2}{1 - \beta x \cos 2\theta} \le \frac{x^2}{1 - \beta x}.$$

Thus

$$\psi(x) \le 1 + 2x + 2\beta\left(1 + \frac{x^2}{1 - \beta x}\right) = 1 + 2\beta + \frac{2(1-2\mu)}{1 - \beta(1-2\mu)}$$

and with (2.10) this estiablishes the second inequality in the theorem. Equality occurs only if

$$p_1 = \frac{2(1-2\mu)}{1-\beta(1-2\mu)},\ p_2 = b_2 = 2,\ b_3 = 3,$$

and the corresponding function f is defined by

$$f(z) = \frac{z}{(1-z)^2}\left(\lambda\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}\right)^\beta,\quad f(0) = 0,$$

where

$$\lambda = \frac{1 + (1-2\beta)(1-2\mu)}{2(1-\beta(1-2\mu)).}$$

We now prove the first inequlity. Let $\mu \le \beta/(2(1+\beta))$, so that $x \ge 1/(1+\beta)$. With $x_0 = 1/(1+\beta)$, we have

$$\begin{aligned}\psi(x) &= \psi(x_0) + 2(x - x_0)(\rho^2\cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\rho\beta r\cos(\theta+\phi)\\ &\le \psi(x_0) + 2(x-x_0)(1+\beta)^2\\ &\le 1 + 2(1+\beta)^2(1-2\mu),\end{aligned}$$

as required. Equality occurs only if $p_1 = p_2 = b_2 = 2,\ b_3 = 3$, and the corresponding function f is defined by

$$f(z) = \frac{z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^\beta,\quad f(0) = 0.$$

Let $x_1 = -1/(1+\beta)$. We shall find that $\psi(x_1) = 1 + 2\beta$, and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

$$\begin{aligned}\psi(x) &\le \psi(x_1) + 2|x - x_1|(1+\beta)^2\\ &\le -1 + 2(1+\beta)^2(2\mu - 1),\end{aligned}$$

if $x \le x_1$, that is, $\mu \ge (2+\beta)/(2(1+\beta))$. Equality occurs only if $p_1 = 2i,\ p_2 = -2,\ b_2 = 2i,\ b_3 = -3$, and the corresponding function f is defined by

$$f(z) = \frac{z}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^\beta,\quad f(0) = 0.$$

Also, for $0 \le \lambda \le 1$,

$$\psi(\lambda x_1) = \lambda\psi(x_1) + (1-\lambda)\psi(0) \le \lambda(1+2\beta) + (1-\lambda)(1+2\beta) = 1+2\beta,$$

so, we obtain $\psi(x) \le 1+2\beta$ for $x_1 \le x \le 0$, i.e., $1/2 \le \mu \le (2+\beta)/2(1+\beta)$. Equality occurs only if $p_1 = b_2 = 0,\ p_2 = 2,\ b_3 = 1$, and the corresponding function f is defined by

$$f(z) = \frac{z(1+z^2)^\beta}{(1-z^2)^{1+\beta}}, \quad f(0) = 0.$$

We now show that $\psi(x_1) \le 1+2\beta$. We have

$$-t^2 + t^2\beta x \cos 2\theta + 2xt\rho\cos(\theta+\phi) \le \frac{x^2\rho^2\cos^2(\theta+\phi)}{1-\beta x\cos 2\theta}$$

for real t, and so

$$\psi(x) - 1 - 2\beta \le \rho^2\left(-1 + (1+2x)\cos 2\phi + \frac{\beta x^2(1+\cos 2(\theta+\phi))}{1-\beta x\cos 2\theta}\right).$$

Thus we consider the inequality

$$\beta x^2(1+\cos 2(\theta+\phi)) + (1-\beta x\cos 2\theta)(-1+(1+2x)\cos 2\phi) \le 0$$

with $x = x_1$. After some simplifications, this becames

$$2\beta^2\sin^2\phi\cos^2\phi + 2\beta\cos\theta\sin\theta\sin\phi + \cos^2\phi \ge 0. \tag{2.11}$$

Now, for all real t, we note that

$$2t^2 + 2t\sin\theta\cos\phi + \cos^2\phi \ge 0,$$

so, by taking $t = \beta\sin\phi\cos\theta$, we obtain (2.11). Therefore we complete the proof of Theorem 2.1.

Next, we prove

Theorem 2.2. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left\{\left(\frac{f'(z)}{g'(z)}\right)^\alpha\left(\frac{f(z)}{g(z)}\right)^\beta\right\}\right| < \frac{\pi}{2}\delta \quad (\alpha > 0;\ \beta \in \mathbb{R};\ 0 < \delta \le 1)$$

for some $g \in \mathcal{K}[A,B]$, then

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\pi}{2}\eta,$$

where η $(0 < \eta \le 1)$ is the solution of the equation :

$$\delta = \begin{cases} (\alpha+\beta)\eta + \frac{2}{\pi}\alpha \tan^{-1}\left(\frac{\eta \sin[\frac{\pi}{2}\{1-t(A,B)\}]}{\frac{1+A}{1+B}+\eta\cos[\frac{\pi}{2}\{1-t(A,B)\}]}\right) & (B \neq -1) \\ (\alpha+\beta)\eta & (B = -1) \end{cases} \tag{2.12}$$

and

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB}\right). \tag{2.13}$$

Proof. Let

$$p(z) = \frac{f(z)}{g(z)} \quad \text{and} \quad q(z) = \frac{zg'(z)}{g(z)}.$$

Then, by a simple calculation, we have

$$\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f(z)}{g(z)}\right)^{\beta} = (p(z))^{\alpha+\beta}\left(1+\frac{1}{q(z)}\frac{zp'(z)}{p(z)}\right)^{\alpha}.$$

Since $g \in \mathcal{K}[A,B]$, $g \in \mathcal{S}^*[A,B]$. If we let

$$q(z) = \rho e^{i\frac{\pi}{2}\phi} \quad (z \in \mathcal{U}),$$

then it follows from (1.2) and (1.3) that

$$\begin{cases} \frac{1-A}{1-B} < \rho < \frac{1+A}{1+B} \\ -t(A,B) < \phi < t(A,B) \quad (B \neq -1) \end{cases}$$

and

$$\begin{cases} \frac{1-A}{2} < \rho < \infty \\ -1 < \phi < 1 \quad (B = -1), \end{cases}$$

where $t(A,B)$ is defined by (2.13).

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.2) we obtain (2.3) under the restrictions (2.4-6).

At first, we suppose that

$$\{p(z_0)\}^{\frac{1}{\eta}} = ia \quad (a > 0).$$

For the case $B \neq -1$, we then obtain

$$\begin{aligned}
&\arg\left\{\left(\frac{f'(z_0)}{g'(z_0)}\right)^{\alpha}\left(\frac{f(z_0)}{g(z_0)}\right)^{\beta}\right\}\\
&= \arg\left\{(p(z_0))^{\alpha+\beta}\left(1+\frac{1}{q(z_0)}\frac{z_0p'(z_0)}{p(z_0)}\right)^{\alpha}\right\}\\
&= \arg\left\{(p(z_0))^{\alpha+\beta}\right\}+\arg\left\{\left(1+i\eta k(\rho e^{i\frac{\pi}{2}\phi})^{-1}\right)^{\alpha}\right\}\\
&= (\alpha+\beta)\frac{\pi}{2}\eta+\alpha\tan^{-1}\left(\frac{\eta k\sin[\frac{\pi}{2}(1-\phi)]}{\rho+\eta k\cos[\frac{\pi}{2}(1-\phi)]}\right)\\
&\geq (\alpha+\beta)\frac{\pi}{2}\eta+\alpha\tan^{-1}\left(\frac{\eta\sin[\frac{\pi}{2}\{1-t(A,B)\}]}{\frac{1+A}{1+B}+\eta\cos[\frac{\pi}{2}\{1-t(A,B)\}]}\right)\\
&= \frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively. Similarly, for the case $B=-1$, we have

$$\arg\left\{\left(\frac{f'(z_0)}{g'(z_0)}\right)^{\alpha}\left(\frac{f(z_0)}{g(z_0)}\right)^{\beta}\right\} \geq (\alpha+\beta)\frac{\pi}{2}\eta=\frac{\pi}{2}\delta.$$

These evidently contradict the assumption of the theorem.

Next, in the case $p(z_0)^{\frac{1}{\eta}} = -ia \quad (a>0)$, applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

By setting $\alpha=1,\ \beta=0,\ \delta=1,\ A=1$ and $B=-1$ in Theorem 2.2, we have

Corollary 2.1. *Every close-to-convex function is close-to-star in $\mathcal{U}$.*

If we put $g(z)=z$ in Theorem 2.2, then, by letting $B\to A$ $(A<1)$, we obtain

Corollary 2.2. *If $f\in\mathcal{A}$ and*

$$\left|\arg\left\{(f'(z))^{\alpha}\left(\frac{f(z)}{z}\right)^{\beta}\right\}\right| < \frac{\pi}{2}\delta \ (\alpha>0;\ \beta\in\mathbb{R};\ 0<\delta\leq 1),$$

then

$$|\arg\{f'(z)\}| < \frac{\pi}{2}\eta,$$

where η $(0<\eta\leq 1)$ is the solution of the equation :

$$\delta=(\alpha+\beta)\eta+\frac{2}{\pi}\alpha\tan^{-1}(\eta).$$

For a function f belonging to the class $\mathcal{A}$, we define the integral operator F_c as follows :

$$F_c(f) := F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt \quad (c \geq 0\ ;\ z \in \mathcal{U}). \tag{2.14}$$

For various interesting developments involving the operator (2.14), the reader may be referred (for example) to the recent works of Miller and Mocanu [10] and Srivastava and Owa [18].

Finally, we prove

Theorem 2.3. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\frac{f(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \ (0 < \gamma \leq 1;\ 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}^*[A,B]$, then

$$\left|\arg\left(\frac{F_c(f))}{F_c(g)} - \gamma\right)\right| < \frac{\pi}{2}\eta,$$

where the operator F_c is given by (2.14) and $\eta(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{3\sin\frac{\pi}{2}(1-t(A,B,c))}{(\frac{1+A}{1+B}+c)+\eta\cos\frac{\pi}{2}(1-t(A,B,c))}\right) & \text{for } B \neq -1, \\ \eta & \text{for } B = -1, \end{cases}$$

when

$$t(A,B,c) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB+c(1-B^2)}\right) \tag{2.15}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{F_c(f)}{F_c(g)} - \gamma\right) \text{ and } q(z) = \frac{zF_c'(g)}{F_c(g)}.$$

From the assumption for g and an application of Briot-Bouquet differential equation [10, p. 81], we see that $F_c(g) \in \mathcal{S}^*[A,B]$. Using the equation

$$zF_c'(f)(z) + cF_c(f)(z) \;=\; (1+c)f(z)$$

and simplying, we obtain

$$\frac{1}{1-\gamma}\left(\frac{f(z)}{g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)+c}.$$

Then, by applying (1.2) and (1.3), we have

$$q(z) + c = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$\begin{cases} \frac{1-A}{1-B}+c \ < \ \rho \ < \ \frac{1+A}{1+B}+c \\ -t(A,B,c) \ < \ \phi \ < \ t(A,B,c) \text{ for } B \neq -1, \end{cases}$$

when $t(A,B,c)$ is given by (2.16), and

$$\begin{cases} \frac{1-A}{2}+c \ < \ \rho \ < \ \infty \\ -1 \ < \ \phi \ < \ 1 \text{ for } B = -1. \end{cases}$$

Here, we note that p is analytic in U with $p(0) = 1$ and Re $p(z) > 0$ in $\mathcal{U}$ by applying the assumption and Lemma 2.3 with $\omega(z) = 1/(q(z)+c)$. Hence $p(z) \neq 0$ in U. The remaining part of the proof of Theorem 2.3 is similar to that of Theorem 2.2, and so we omit it.

Remark. From Theorem 2.3, we see easily that every function in $\mathcal{CS}(\delta)$ $(0 < \delta \leq 1)$ preserves the angles under the integral operator defined by (2.14).

By letting $A = 1-2\beta(0 \leq \beta \leq 1),\ B = -1,\ \delta = 1$ in Theorem 2.3, we obtain

Corollary 2.3. *If $f \in \mathcal{A}$ and*

$$\text{Re}\left\{\frac{f(z)}{g(z)}\right\} > \gamma \quad (0 \leq \gamma < 1;\ z \in \mathcal{U}),$$

for some g such that

$$\text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \beta \quad (0 \leq \beta < 1;\ z \in \mathcal{U}),$$

then

$$\text{Re}\left\{\frac{F_c(f)}{F_c(g)}\right\} > \gamma \quad (0 \leq \gamma < 1;\ z \in \mathcal{U}),$$

where F_c is given by (2.14).

If we take $g(z) = z$ in Theorem 2.3, then, by letting $B \to A$ $(A < 1)$, we have

Corollary 2.4. *If $f \in \mathcal{A}$ and*

$$\left|\arg\left(\frac{f(z)}{z} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < 1;\ 0 < \delta \leq 1),$$

then

$$\left|\arg\left(\frac{F_c(f)}{z} - \gamma\right)\right| < \frac{\pi}{2}\eta,$$

where F_c is given by (2.14) and $\eta(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left(\frac{\eta}{1+c} \right).$$

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