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INTEGRAL MEANS FOR GENERALIZED SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. By means of coefficient inequalities, the authors introduce a certain family of normalized analytic functions in the open unit disk. Applying the concepts of extreme points, fractional calculus, and subordination between analytic functions, several integral means inequalities are obtained here for higher-order and fractional derivatives of functions belonging to this general family. Relevant connections of the results presented in this paper with those given in earlier works are also considered.

1. Introduction, Definitions, and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Denote by $\mathcal{A}(n)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \tag{1.2}$$

$$(a_k \geqq 0;\ k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

We denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ of functions which are also *univalent* in $\mathcal{U}$, and by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the subclasses of $\mathcal{T}(n)$ consisting of functions which are, respectively, *starlike* of order α $(0 \leqq \alpha < 1)$ and *convex* of order α $(0 \leqq \alpha < 1)$.

The classes $\mathcal{T}(n)$, $\mathcal{T}_\alpha(n)$, and $\mathcal{C}_\alpha(n)$, introduced by Chatterjea [1], were investigated systematically by Srivastava *et al.* [12]. In fact, the following special cases of these classes when $n = 1$:

$$\mathcal{T} := \mathcal{T}(1), \quad \mathcal{T}^*(\alpha) := \mathcal{T}_\alpha(1), \quad \text{and} \quad \mathcal{C}(\alpha) := \mathcal{C}_\alpha(1) \tag{1.3}$$

were considered earlier by Silverman [8]. And, as already remarked by Srivastava *et al.* [12, p. 117], the necessary and sufficient conditions for a function $f(z)$ of the form (1.2) to be in the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ would follow immediately from those given by Silverman [8, p. 110, Theorem 2; p. 111, Corollary 2] for the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ by merely setting

$$a_k = 0 \quad (k \in \mathbb{N} \setminus \{1\}). \tag{1.4}$$

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Next, following the work of Sekine and Owa [7], we denote by $\mathcal{A}(n,\vartheta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form [*cf.* Equation (1.2)]:

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^k \tag{1.5}$$

$$(\vartheta \in \mathbb{R};\ a_k \geqq 0;\ k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}),$$

so that, obviously,

$$\mathcal{A}(n,0) = \mathcal{A}(n) \qquad (n \in \mathbb{N}). \tag{1.6}$$

Thus, if we define the subclasses

$$\mathcal{T}(n,\vartheta), \quad \mathcal{T}^*_\alpha(n,\vartheta), \quad \text{and} \quad \mathcal{C}_\alpha(n,\vartheta)$$

of the class $\mathcal{A}(n,\vartheta)$ in the same way as we defined the subclasses

$$\mathcal{T}(n), \quad \mathcal{T}_\alpha(n), \quad \text{and} \quad \mathcal{C}_\alpha(n)$$

of the class $\mathcal{A}(n)$, it is easily observed that

$$\mathcal{T}(n,0) = \mathcal{T}(n), \quad \mathcal{T}^*_\alpha(n,0) = \mathcal{T}_\alpha(n), \quad \text{and} \quad \mathcal{C}_\alpha(n,0) = \mathcal{C}_\alpha(n) \qquad (n \in \mathbb{N}), \tag{1.7}$$

together with (*cf.*, *e.g.*, Silverman [8, p. 111, Corollary]).

$$\mathcal{T} = \mathcal{T}^*(0) \quad \text{and} \quad \mathcal{T}(n) = \mathcal{T}_0(n).$$

The following coefficient inequalities for functions $f(z)$ of the form (1.5) were proven recently by Sekine and Owa [7].

Lemma 1. *A function $f \in \mathcal{A}(n,\vartheta)$ of the form* (1.5) *is in the class $\mathcal{T}^*_\alpha(n,\vartheta)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k-\alpha)\, a_k \leqq 1-\alpha \qquad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.8}$$

Lemma 2. *A function $f \in \mathcal{A}(n,\vartheta)$ of the form* (1.5) *is in the class $\mathcal{C}_\alpha(n,\vartheta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k-\alpha)\, a_k \leqq 1-\alpha \qquad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.9}$$

We remark in passing that the coefficient inequalities (1.8) *and* (1.9) do not contain the parameter ϑ (and, therefore, coincide essentially with the corresponding coefficient inequalities considered earlier by Silverman [8], Chatterjea [1], and Srivastava *et al.* [12]). See also the aforementioned remark involving the coefficient specialization exhibited by (1.4).

Motivated largely by the coefficient inequalities (1.8) and (1.9), we now introduce a general family $\mathcal{A}(n;\{B_k\},\vartheta)$ of functions $f \in \mathcal{A}(n,\vartheta)$ of the form (1.5), which satisfy the following inequality:

$$\sum_{k=n+1}^{\infty} B_k\, a_k \leqq 1 \tag{1.10}$$

$$(B_k > 0;\ k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N})$$

for every *positive* sequence $\{B_k\}$ of real numbers.

The class $\mathcal{A}(n;\{B_k\})$ given by

$$\mathcal{A}(n;\{B_k\}) := \mathcal{A}(n;\{B_k\},0) \tag{1.11}$$

was studied earlier by Sekine [6] (and, subsequently, by Owa *et al.* [5]). As a matter of fact, Sekine [6] presented an interesting (and useful) classification (*cf.* [6, pp. 3-4]) of the analytic functions in $\mathcal{A}(n)$ $(n \in \mathbb{N})$ by using the inequality (1.10). Indeed it is fairly easy to verify each of the following classifications:

$$\mathcal{A}(n;\{k\},\vartheta) = \mathcal{T}_0^*(n,\vartheta) =: \mathcal{T}^*(n,\vartheta) = \mathcal{T}(n,\vartheta) \tag{1.12}$$

$$\mathcal{A}\left(n;\left\{\frac{k-\alpha}{1-\alpha}\right\},\vartheta\right) = \mathcal{T}_\alpha^*(n,\vartheta) \qquad (0 \leqq \alpha < 1), \tag{1.13}$$

and

$$\mathcal{A}\left(n;\left\{\frac{k(k-\alpha)}{1-\alpha}\right\},\vartheta\right) = \mathcal{C}_\alpha(n,\vartheta) \qquad (0 \leqq \alpha < 1). \tag{1.14}$$

It follows also from (1.10) that

$$\mathcal{A}(n;\{B_k\},\vartheta) \subseteq \mathcal{A}(n;\{C_k\},\vartheta) \qquad (0 < C_k \leqq B_k), \tag{1.15}$$

which readily yields the inclusion relations:

$$\mathcal{C}_\alpha(n,\vartheta) \subset \mathcal{T}_\alpha^*(n,\vartheta) \subseteq \mathcal{T}^*(n,\vartheta)$$
$$(0 \leqq \alpha < 1;\ \vartheta \in \mathbb{R};\ n \in \mathbb{N}).$$

The main object of this paper is to apply the familiar concepts of extreme points, fractional calculus, and subordination between analytic functions with a view to obtaining several integral means inequalities for higher-order and fractional derivatives of functions in the general class $\mathcal{A}(n;\{B_k\},\vartheta)$ which we have introduced here. We also point out relevant connections of the results presented in this paper with those given in earlier works by (for example) Silverman [9], Kim and Choi [2], and others.

2. Basic Properties of the Class $\mathcal{A}(n;\{B_k\},\vartheta)$

The proof of each of the following results (Theorem 1, Theorem 2, and Corollary 1 below) is much akin to that of the corresponding result in Owa *et al.* [5], and we choose to omit the details involved.

Theorem 1. *$\mathcal{A}(n;\{B_k\},\vartheta)$ is the convex subfamily of the class $\mathcal{A}(n,\vartheta)$.*

Theorem 2. *Let*

$$f_1(z) = z \quad \text{and} \quad f_k(z) = z - \frac{e^{i(k-1)\vartheta}}{B_k} z^k \tag{2.1}$$
$$(k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}).$$

Then $f \in A(n;\{B_k\},\vartheta)$ if and only if $f(z)$ can be expressed as

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z), \tag{2.2}$$

where

$$\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1 \tag{2.3}$$
$$(\lambda_1 \geqq 0;\ \lambda_k \geqq 0;\ k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}).$$

Corollary 1. *The extreme points of the class* $\mathcal{A}(n;\{B_k\},\vartheta)$ *are the functions* $f_1(z)$ *and* $f_k(z)$ $(k \geqq n+1)$ *given by* (2.1).

By means of the relationships exhibited by (1.12), (1.13), and (1.14), we can easily deduce from Corollary 1 the extreme points of various other subclasses of the class $\mathcal{A}(n,\vartheta)$. Thus, for example, we obtain Corollary 2 and Corollary 3 below.

Corollary 2. *The extreme points of the class* $\mathcal{T}_\alpha^*(n,\vartheta)$ *are the functions* $f_1(z)$ *and* $f_k(z)$ $(k \geqq n+1)$ *given by*

$$f_1(z) = z \quad \text{and} \quad f_k(z) = z - \left(\frac{1-\alpha}{k-\alpha}\right) e^{i(k-1)\vartheta}\, z^k \tag{2.4}$$

$$(k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}).$$

Corollary 3. *The extreme points of the class* $\mathcal{C}_\alpha(n,\vartheta)$ *are the functions* $f_1(z)$ *and* $f_k(z)$ $(k \geqq n+1)$ *given by*

$$f_1(z) = z \quad \text{and} \quad f_k(z) = z - \left(\frac{1-\alpha}{k(k-\alpha)}\right) e^{i(k-1)\vartheta}\, z^k \tag{2.5}$$

$$(k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}).$$

A *further* special case of *each* of these last results (Corollary 2 and Corollary 3 above) when

$$\vartheta = 0 \quad \text{and} \quad n = 1 \tag{2.6}$$

was given by Silverman [9, Theorem 9 (Corollary 1 and Corollary 2)] for the classes $\mathcal{T}^*(\alpha)$ *and* $\mathcal{C}(\alpha)$ investigated by him (see also [8]).

3. Fractional Calculus and Subordination Principle

We begin by recalling the following definitions of fractional calculus (that is, fractional integrals and fractional derivatives) given by Owa [4] (see also Srivastava and Owa [10] and [11]).

Definition 1. The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta \qquad (\lambda > 0), \tag{3.1}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta \qquad (0 \leqq \lambda < 1), \tag{3.2}$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1 above.

Definition 3. Under the hypotheses of Definition 2, the *fractional derivative of order* $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^\lambda f(z) \qquad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{3.3}$$

It readily follows from Definitions 1 and 2 that

$$D_z^{-\lambda} z^\kappa = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\lambda+1)} z^{\kappa+\lambda} \qquad (\lambda > 0;\ \Re(\kappa) > -1) \tag{3.4}$$

and

$$D_z^{\lambda} z^\kappa = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\lambda+1)} z^{\kappa-\lambda} \qquad (0 \leqq \lambda < 1;\ \Re(\kappa) > -1). \tag{3.5}$$

Next we recall the concept of subordination between analytic functions. Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathcal{U}$, the function $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $w(z)$, analytic in $\mathcal{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathcal{U}), \tag{3.6}$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathcal{U}). \tag{3.7}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{3.8}$$

The following subordination theorem will be required in our present investigation.

Theorem 3 (Littlewood [3]). *If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with*

$$g(z) \prec f(z),$$

then

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\mu d\theta \qquad (\mu > 0;\ 0 < r < 1). \tag{3.9}$$

4. Integral Means Inequalities Involving Higher-Order Derivatives

The familiar *Stirling numbers* $s(m,l)$ *of the first kind* are usually defined by means of the generating function:

$$\prod_{l=1}^{m} (z-l+1) = \sum_{l=0}^{m} s(m,l)\, z^l \qquad (m \in \mathbb{N}_0), \tag{4.1}$$

so that, obviously,

$$s(m,0) = \delta_{m,0}, \quad s(m,1) = (-1)^{m+1}(m-1)!, \quad \text{and} \quad s(m,m) = 1,$$

where $\delta_{m,n}$ denotes the Kronecker delta. Here (*and in what follows*) an *empty* product is interpreted (as usual) to be 1.

Upon setting $z = n+1$ $(n \in \mathbb{N})$, we immediately obtain

$$\sum_{l=0}^{m} s(m,l)(n+1)^l = \prod_{l=1}^{m} (n-l+2) \qquad (m \in \mathbb{N}_0;\ n \in \mathbb{N}). \tag{4.2}$$

Making use of the relationships (4.1) *and* (4.2), we now prove

Theorem 4. *Suppose that*

$$f \in \mathcal{A}\left(n; \{k^p B_k\}, \vartheta\right) \qquad (B_k \leqq B_{k+1};\ p = 2, 3, \ldots, n+1;\ n \in \mathbb{N}).$$

Also let the function $f_{n+1}(z)$ be defined by (2.1) with B_k replaced by $k^p B_k$. Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left| f^{(j)}(z) \right|^{\mu} d\theta \leqq \int_0^{2\pi} \left| f_{n+1}{}^{(j)}(z) \right|^{\mu} d\theta, \tag{4.3}$$

where $\mu > 0$ and j is integer such that $2 \leqq j \leqq p$ for $p = 2, 3, \ldots, n+1$.

Proof. It follows from the hypothesis of Theorem 4 that

$$(n+1)^{p-m} B_{n+1} \sum_{k=n+1}^{\infty} k^m a_k \leqq \sum_{k=n+1}^{\infty} k^p B_k a_k \leqq 1 \qquad (m = 1, \ldots, p), \tag{4.4}$$

so that

$$\sum_{k=n+1}^{\infty} k^m a_k \leqq \frac{1}{(n+1)^{p-m} B_{n+1}} \qquad (m = 1, \ldots, p). \tag{4.5}$$

Also, from (1.5) and (2.1) with B_k replaced by $k^p B_k$, we readily obtain the following derivative formulas:

$$f^{(j)}(z) = -\sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^{k-j} \prod_{l=1}^{j} (k-l+1) \qquad (z \in \mathcal{U};\ 2 \leqq j \leqq p) \tag{4.6}$$

and

$$f_{n+1}{}^{(j)}(z) = -e^{in\vartheta} \left(\frac{\prod_{l=1}^{j} (n-l+2)}{(n+1)^p B_{n+1}} \right) z^{n-j+1} \qquad (z \in \mathcal{U};\ 2 \leqq j \leqq p). \tag{4.7}$$

Upon substituting from (4.6) and (4.7) into the desired inequality (4.3), if we apply Theorem 3, it would suffice to show that

$$\begin{aligned} &\sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^{k-j} \prod_{l=1}^{j} (k-l+1) \\ &\qquad \prec e^{in\vartheta} \left(\frac{\prod_{l=1}^{j} (n-l+2)}{(n+1)^p B_{n+1}} \right) z^{n-j+1} \qquad (2 \leqq j \leqq p). \end{aligned} \tag{4.8}$$

If we put

$$\begin{aligned} &\sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^{k-j} \prod_{l=1}^{j} (k-l+1) \\ &\qquad = e^{in\vartheta} \left(\frac{\prod_{l=1}^{j} (n-l+2)}{(n+1)^p B_{n+1}} \right) \{w(z)\}^{n-j+1}, \end{aligned} \tag{4.9}$$

then we have

$$\begin{aligned} \{w(z)\}^{n-j+1} := &\left(\frac{(n+1)^p B_{n+1}}{\prod_{l=1}^{j} (n-l+2)} \right) \\ &\cdot \sum_{k=n+1}^{\infty} e^{i(k-n-1)\vartheta} a_k z^{k-j} \prod_{l=1}^{j} (k-l+1), \end{aligned}$$

so that, in view of (4.1) *and* (4.2),

$$
\begin{aligned}
|w(z)|^{n-j+1} &\leqq \left(\frac{(n+1)^p B_{n+1}}{\prod_{l=1}^{j}(n-l+2)}\right) \sum_{k=n+1}^{\infty} a_k |z|^{k-j} \prod_{l=1}^{j}(k-l+1) \\
&\leqq \left(\frac{(n+1)^p B_{n+1}}{\prod_{l=1}^{j}(n-l+2)}\right) |z| \sum_{k=n+1}^{\infty} a_k \sum_{l=0}^{j} s(j,l)\, k^l \\
&\leqq \left(\frac{(n+1)^p B_{n+1}}{\prod_{l=1}^{j}(n-l+2)}\right) |z| \sum_{l=0}^{j} s(j,l) \sum_{k=n+1}^{\infty} k^l a_k \\
&\leqq \left(\frac{(n+1)^p B_{n+1}}{\prod_{l=1}^{j}(n-l+2)}\right) |z| \sum_{l=0}^{j} s(j,l) \frac{1}{(n+1)^{p-l} B_{n+1}} \\
&= \left(\frac{|z|}{\prod_{l=1}^{j}(n-l+2)}\right) \sum_{l=0}^{j} s(j,l)(n+1)^l \\
&= |z| < 1 \qquad (z \in \mathcal{U}). \qquad (4.10)
\end{aligned}
$$

Thus we have shown that the function $w(z)$, occurring in (4.9), satisfies each of the conditions in (3.6). Hence the subordination in (4.8) holds true, and this evidently completes the proof of Theorem 4.

Since [*cf.* Equation (1.14)]

$$
\begin{aligned}
\mathcal{C}_\alpha(n,\vartheta) &:= \mathcal{A}\left(n; \left\{\frac{k(k-\alpha)}{1-\alpha}\right\}, \vartheta\right) \\
&= \mathcal{A}\left(n; \left\{k^2 \cdot \frac{k-\alpha}{k(1-\alpha)}\right\}, \vartheta\right) \qquad (4.11)
\end{aligned}
$$

and since the sequence

$$\{B_k\} \qquad \left(B_k := \frac{k-\alpha}{k(1-\alpha)}\right)$$

is an *increasing* sequence, Theorem 4 immediately yields

Corollary 4. *Suppose that $f \in \mathcal{C}_\alpha(n,\vartheta)$ and let $f_{n+1}(z)$ be defined by* (2.5). *Then, for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi} |f''(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |f_{n+1}{}''(z)|^\mu \, d\theta \qquad (\mu > 0). \tag{4.12}$$

5. Integral Means Inequalities Involving Fractional Calculus Operators

Our first integral means inequality involving fractional integrals is given by

Theorem 5. *Suppose that*

$$f \in \mathcal{A}(n; \{B_k\}, \vartheta) \qquad (B_k \leqq B_{k+1})$$

and let the function $f_{n+1}(z)$ be defined by (2.1). *Then, for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi} |D_z^{-\lambda} f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |D_z^{-\lambda} f_{n+1}(z)|^\mu \, d\theta \qquad (\lambda > 0;\ \mu > 0). \tag{5.1}$$

Proof. By means of the fractional integral formula (3.4), we find from (1.5) that

$$D_z^{-\lambda} f(z) = \frac{z^{\lambda+1}}{\Gamma(\lambda+2)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Theta(k)\, a_k\, z^{k-1} \right) \qquad (\lambda > 0) \tag{5.2}$$

or

$$\frac{\Gamma(\lambda+2)}{z^{\lambda+1}} D_z^{-\lambda} f(z) = 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Theta(k)\, a_k\, z^{k-1} \qquad (\lambda > 0),$$

where

$$\Theta(k) := \frac{\Gamma(\lambda+2)\,\Gamma(k+1)}{\Gamma(\lambda+k+1)} > 0 \qquad (\lambda > 0;\ k \geqq n+1;\ n \in \mathbb{N}) \tag{5.3}$$

is a *decreasing* function of k so that

$$0 < \Theta(k) \leqq \Theta(n+1) = \frac{\Gamma(\lambda+2)\,\Gamma(n+2)}{\Gamma(\lambda+n+2)} \tag{5.4}$$
$$(\lambda > 0;\ k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}).$$

Similarly, (2.1) and (3.4) yield

$$D_z^{-\lambda} f_{n+1}(z) = \frac{z^{\lambda+1}}{\Gamma(\lambda+2)} \left(1 - \frac{e^{in\vartheta}}{B_{n+1}} \Theta(n+1)\, z^n \right) \qquad (\lambda > 0) \tag{5.5}$$

or

$$\frac{\Gamma(\lambda+2)}{z^{\lambda+1}} D_z^{-\lambda} f_{n+1}(z) = 1 - \frac{e^{in\vartheta}}{B_{n+1}} \Theta(n+1)\ z^n \qquad (\lambda > 0),$$

where $\Theta(k)$ is given by (5.3).

Upon substituting from (5.2) and (5.5) into the desired inequality (5.1), if we apply Theorem 3, it would suffice to show that

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Theta(k)\, a_k z^{k-1} \prec 1 - \frac{e^{in\vartheta}}{B_{n+1}} \Theta(n+1)\, z^n. \tag{5.6}$$

Indeed, by setting

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Theta(k)\, a_k z^{k-1} = 1 - \frac{e^{in\vartheta}}{B_{n+1}} \Theta(n+1) \{w(z)\}^n,$$

we find that

$$\{w(z)\}^n := \frac{B_{n+1}}{\Theta(n+1)} \sum_{k=n+1}^{\infty} e^{i(k-n-1)\vartheta} \Theta(k)\, a_k\ z^{k-1},$$

so that, by virtue of the inequality (5.4), we have

$$
\begin{aligned}
|w(z)|^n &\leqq \frac{B_{n+1}}{\Theta(n+1)} \sum_{k=n+1}^{\infty} \Theta(k)\, a_k\, z^{k-1} \\
&\leqq \frac{B_{n+1}}{\Theta(n+1)}\, \Theta(n+1)\, |z| \sum_{k=n+1}^{\infty} a_k \\
&\leqq |z|\, B_{n+1} \sum_{k=n+1}^{\infty} a_k \\
&\leqq |z| \sum_{k=n+1}^{\infty} B_k\, a_k \\
&\leqq |z| < 1 \qquad (z \in \mathcal{U}),
\end{aligned}
\tag{5.7}
$$

since (by hypothesis)

$$B_k \leqq B_{k+1} \qquad (k = n+1, n+2, n+3, \ldots\,;\ n \in \mathbb{N}).$$

In light of the inequality (5.7), we have the subordination (5.6), which proves Theorem 5.

In precisely the same manner as detailed above, by making use of the fractional derivative formula (3.5) in place of the fractional integral formula (3.4), we can prove

Theorem 6. *Suppose that*

$$f \in \mathcal{A}(n; \{kB_k\}, \vartheta) \qquad (B_k \leqq B_{k+1})$$

and let the function $f_{n+1}(z)$ *be defined by* (2.1) *with* B_k *replaced by* kB_k. *Then, for* $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{\lambda} f_{n+1}(z)\right|^{\mu} d\theta \qquad (0 \leqq \lambda < 1;\ \mu > 0). \tag{5.8}$$

Next we prove

Theorem 7. *Suppose that*

$$f \in \mathcal{A}\left(n; \left\{k^2 B_k\right\}, \vartheta\right) \qquad (B_k \leqq B_{k+1})$$

and let the function $f_{n+1}(z)$ *be defined (as in Theorem* 6) *by* (2.1) *with* B_k *replaced by* kB_k. *Then, for* $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{1+\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{1+\lambda} f_{n+1}(z)\right|^{\mu} d\theta \qquad (0 \leqq \lambda < 1;\ \mu > 0). \tag{5.9}$$

Proof. In view of Definition 3 *and* the fractional derivative formula (3.5), we find from (1.5) that

$$D_z^{1+\lambda} f(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k\,(k-1)\, \Phi(k)\, a_k\, z^{k-1}\right) \qquad (0 \leqq \lambda < 1) \tag{5.10}$$

or

$$\frac{\Gamma(1-\lambda)}{z^{-\lambda}} D_z^{1+\lambda} f(z) = 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k\,(k-1)\, \Phi(k)\, a_k\, z^{k-1} \qquad (0 \leqq \lambda < 1),$$

where

$$\Phi(k) := \frac{\Gamma(1-\lambda)\,\Gamma(k-1)}{\Gamma(k-\lambda)} > 0 \qquad (0 \leqq \lambda < 1;\ k \geqq n+1;\ n \in \mathbb{N}) \tag{5.11}$$

is a *decreasing* function of k so that

$$0 < \Phi(k) \leqq \Phi(n+1) = \frac{\Gamma(1-\lambda)\,\Gamma(n)}{\Gamma(n-\lambda+1)} \tag{5.12}$$

$$(0 \leqq \lambda < 1;\ k = n+1, n+2, n+3, \dots;\ n \in \mathbb{N}).$$

Similarly, (2.1) (*with*, of course, B_k replaced by kB_k), (3.4), and Definition 3 would yield

$$D_z^{1+\lambda} f_{n+1}(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)}\left(1 - \frac{e^{in\vartheta}}{B_{n+1}}\, n\Phi(n+1)\, z^n\right) \qquad (0 \leqq \lambda < 1) \tag{5.13}$$

or

$$\frac{\Gamma(1-\lambda)}{z^{-\lambda}} D_z^{1+\lambda} f_{n+1}(z) = 1 - \frac{e^{in\vartheta}}{B_{n+1}}\, n\Phi(n+1)\, z^n \qquad (0 \leqq \lambda < 1),$$

where $\Phi(k)$ is given by (5.11).

Upon substituting from (5.10) and (5.13) into the desired inequality (5.9), it would suffice to show that

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k(k-1)\,\Phi(k)\, a_k\, z^{k-1}$$

$$\prec 1 - \frac{e^{in\vartheta}}{B_{n+1}}\, n\Phi(n+1)\, z^n. \tag{5.14}$$

Indeed, by letting

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k(k-1)\,\Phi(k)\, a_k\, z^{k-1}$$

$$= 1 - \frac{e^{in\vartheta}}{B_{n+1}}\, n\Phi(n+1)\, \{w(z)\}^n,$$

we find that

$$\{w(z)\}^n := \frac{B_{n+1}}{n\Phi(n+1)} \sum_{k=n+1}^{\infty} e^{i(k-n-1)\vartheta} k(k-1)\,\Phi(k)\, a_k\, z^{k-1},$$

so that, by applying the inequality (5.12), we have

$$|w(z)|^n \leqq \frac{B_{n+1}}{n\Phi(n+1)} \sum_{k=n+1}^{\infty} k(k-1)\,\Phi(k)\, a_k\, |z|^{k-1}$$

$$\leqq \frac{B_{n+1}}{n\Phi(n+1)}\, \Phi(n+1)\, |z| \sum_{k=n+1}^{\infty} k(k-1)\, a_k$$

$$\leqq \frac{|z|}{n} B_{n+1} \sum_{k=n+1}^{\infty} k^2 a_k$$

$$\leqq \frac{|z|}{n} \sum_{k=n+1}^{\infty} k^2 B_k\, a_k$$

$$\leqq \frac{|z|}{n} < 1 \qquad (z \in \mathcal{U}), \tag{5.15}$$

since (by hypothesis) $f \in \mathcal{A}\left(n; \left\{k^2 B_k\right\}, \vartheta\right)$ *and*

$$B_k \leqq B_{k+1} \qquad (k = n+1, n+2, n+3, \ldots ;\ n \in \mathbb{N}).$$

In view of the inequality (5.15), we arrive immediately at the subordination (5.14), which evidently completes the proof of Theorem 7.

Similarly, we can prove

Theorem 8. *Suppose that*

$$f \in \mathcal{A}\left(n; \left\{k^2 B_k\right\}, \vartheta\right) \qquad (B_k \leqq B_{k+1})$$

and let the function $f_{n+1}(z)$ *be defined by* (2.1) *with* B_k *replaced by* $k^2 B_k$. *Then, for* $z = re^{i\theta}$ *and* $0 < r < 1$,

$$(5.16) \qquad \int_0^{2\pi} \left|D_z^{1+\lambda} f(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|D_z^{1+\lambda} f_{n+1}(z)\right|^\mu d\theta \qquad \left(0 \leqq \lambda < \frac{n+1}{n+2};\ \mu > 0\right).$$

Finally, we prove the following interesting extension of the integral means inequality (4.3) asserted by Theorem 4.

Theorem 9. *Suppose that*

$$f \in \mathcal{A}\left(n; \left\{k^p B_k\right\}, \vartheta\right) \qquad (B_k \leqq B_{k+1};\ p = 2, 3, \ldots, n).$$

Also let the function $f_{n+1}(z)$ *be defined (as in Theorem* 4) *by* (2.1) *with* B_k *replaced by* $k^p B_k$. *Then, for* $z = re^{i\theta}$ *and* $0 < r < 1$,

$$(5.17) \qquad \int_0^{2\pi} \left|D_z^\lambda f^{(j)}(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|D_z^\lambda f_{n+1}{}^{(j)}(z)\right|^\mu d\theta,$$

where $\mu > 0$, $0 \leqq \lambda < 1$ *and* j *is integer such that* $2 \leqq j \leqq p$ *for* $p = 2, 3, \ldots, n$.

Proof. First of all, operating upon both sides of (4.6) by D_z^λ and applying the fractional derivative formula (3.5), we get

$$(5.18) \qquad D_z^\lambda f^{(j)}(z) = -\sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Psi(k)\, a_k z^{k-j-\lambda} \prod_{l=1}^{j+1} (k - l + 1)$$
$$(0 \leqq \lambda < 1;\ 2 \leqq j \leqq p;\ p = 2, 3, \ldots, n),$$

where

$$(5.19) \qquad \Psi(k) := \frac{\Gamma(k-j)}{\Gamma(k-j-\lambda+1)} > 0 \qquad (0 \leqq \lambda < 1;\ k \geqq n+1;\ 2 \leqq j \leqq p)$$

is a *decreasing* function of k so that

$$(5.20) \qquad 0 < \Psi(k) \leqq \Psi(n+1) = \frac{\Gamma(n-j+1)}{\Gamma(n-j-\lambda+2)}$$
$$(0 \leqq \lambda < 1;\ k = n+1, n+2, n+3, \ldots ;\ 2 \leqq j \leqq p).$$

Similarly, we find from (4.7) and (3.5) that

$$(5.21) \qquad D_z^\lambda f_{n+1}{}^{(j)}(z) = -e^{in\vartheta} \left(\frac{\prod_{l=1}^{j+1} (n - l + 2)}{(n+1)^p B_{n+1}}\right) \Psi(n+1)\, z^{n-j-\lambda+1}$$
$$(0 \leqq \lambda < 1;\ 2 \leqq j \leqq p),$$

where $\Psi(k)$ is given by (5.19). Thus, by virtue of Theorem 3, it would suffice to show that

$$\sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Psi(k)\,a_k\;z^{k-j-\lambda}\prod_{l=1}^{j+1}(k-l+1)$$

$$\prec e^{in\vartheta}\left(\frac{\prod_{l=1}^{j+1}(n-l+2)}{(n+1)^p B_{n+1}}\right)\Psi(n+1)\,z^{n-j-\lambda+1} \qquad (2 \leqq j \leqq p). \tag{5.22}$$

In order to prove the subordination (5.22), we set

$$\sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Psi(k)\,a_k\;z^{k-j-\lambda}\prod_{l=1}^{j+1}(k-l+1)$$
$$= e^{in\vartheta}\left(\frac{\prod_{l=1}^{j+1}(n-l+2)}{(n+1)^p B_{n+1}}\right)\Psi(n+1)\,\{w(z)\}^{n-j-\lambda+1}$$

and observe that

$$|w(z)|^{n-j-\lambda+1} \leqq \frac{(n+1)^p B_{n+1}}{\Psi(n+1)\prod_{l=1}^{j+1}(n-l+2)} \cdot \sum_{k=n+1}^{\infty}\Psi(k)\,a_k\,|z|^{k-j-\lambda}\prod_{l=1}^{j+1}(k-l+1)$$

$$\leqq \frac{(n+1)^p B_{n+1}}{\Psi(n+1)\prod_{l=1}^{j+1}(n-l+2)} \cdot \Psi(n+1)\sum_{k=n+1}^{\infty} a_k\,|z|^{k-j-\lambda}\prod_{l=1}^{j+1}(k-l+1)$$

$$\leqq \frac{(n+1)^p B_{n+1}}{\prod_{l=1}^{j+1}(n-l+2)}\sum_{k=n+1}^{\infty} a_k\,|z|^{k-j-\lambda}\prod_{l=1}^{j+1}(k-l+1), \tag{5.23}$$

which, in view of (4.1) *and* (4.2), would lead us to the inequality:

$$|w(z)| < 1 \qquad (z \in \mathcal{U}) \tag{5.24}$$

just as in (4.10). This evidently completes the proof of Theorem 9.

Each of our integral means inequalities (given by Theorems 4 to 9 above) can be suitably specialized in order to derive the corresponding results for numerous simpler classes of analytic and univalent functions. For example, in its special cases when

(i) $n = 1, \quad \vartheta = 0, \quad$ and $\quad B_k = 1,$

(ii) $n = 1, \quad \vartheta = 0, \quad$ and $\quad B_k = \dfrac{k-\alpha}{k(1-\alpha)} \quad (0 \leqq \alpha < 1),$

and

(iii) $n = 1, \quad \vartheta = 0, \quad$ and $\quad B_k = \dfrac{k-\alpha}{1-\alpha} \quad (0 \leqq \alpha < 1),$

Theorem 6 would immediately yield the integral means inequalities proven earlier by Kim and Choi [2, p. 49, Theorem 1 (i); p. 51, Theorem 3 (i) and (ii)], who also gave several obvious special cases of Theorem 7 to hold true for such familiar function classes as $\mathcal{T}$, $\mathcal{T}^*(\alpha)$, and $\mathcal{C}(\alpha)$ involved in the relationship (1.3).

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REFERENCES

[1] S.K. Chatterjea, On starlike functions, *J. Pure Math.* **1** (1981), 23-26.

[2] Y.C. Kim and J.H. Choi, Integral means of the fractional derivative of univalent functions with negative coefficients, *Mathematica Japonica*, **51**(2000), 453-457.

[3] J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* (2) **23** (1925), 481-519.

[4] S. Owa, On the distortion theorems. I., *Kyungpook Math. J.* **18** (1978), 53-59.

[5] S. Owa, T. Sekine, T. Yaguchi, M. Nunokawa, and D. Pashkouleva, Notes on certain class of analytic functions with negative coefficients, *Internat. J. Math. and Math. Sci.* **12** (1989), 199-204.

[6] T. Sekine, On new generalized classes of analytic functions with negative coefficients, *Rep. Res. Inst. Sci. Tech. Nihon Univ.* **35** (1987), 1-26.

[7] T. Sekine and S. Owa, Coefficient inequalities for certain univalent functions, *Math. Inequal. Appl.* **2** (1999), 535-544.

[8] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109-116.

[9] H. Silverman, Integral means for univalent functions with negative coefficients, *Houston J. Math.* **23** (1997), 169-174.

[10] H.M. Srivastava and S. Owa, An application of the fractional derivative, *Math. Japon.* **29** (1984), 383-389.

[11] H.M. Srivastava and S. Owa (Editors), *Current Topics in Univalent Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.

[12] H.M. Srivastava, S. Owa, and S.K. Chatterjea, A note on certain classes of starlike functions, *Rend. Sem. Mat. Univ. Padova* **75** (1987), 115-124.

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