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# SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY

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*Abstract.* Tuneski [2] introduced a new sufficient condition for starlikeness. The object of the present paper is to improve Tuneski's result and to obtain new sufficient conditions for starlikeness, convexity, strongly starlikeness and strongly convexity of order $\alpha$.

## 1. Introduction.

Let $A$ be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)$ in $A$ is said to be starlike if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0\quad(z\in U).$$

We denote by $S^{\star}$ the subclass of $A$ consisiting of all starlike functions in $U$. A function $f(z)$ in $A$ is said to be convex if it satisfies

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0\quad(z\in U).$$

We denote by $C$ the subclass of $A$ consisiting of all convex functions in $U$. A function $f(z)$ in $A$ is said to be starlike of order $\alpha$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\quad(z\in U)$$

for some $\alpha$ $(0\leq\alpha<1)$. We denote by $S^{\star}(\alpha)$ the subclass of $A$ consisiting of all starlike functions of order $\alpha$ in $U$. A function $f(z)$ in $A$ is said to be convex of order $\alpha$ if it satisfies

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\quad(z\in U)$$

for some $\alpha$ $(0\leq\alpha<1)$. We denote by $C(\alpha)$ the subclass of $A$ consisiting of all convex functions of order $\alpha$ in $U$. A function $f(z)$ in $A$ is said to be strongly starlike of order $\alpha$ if it satisfies

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$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad (z \in U)$$

for some $\alpha$ $(0 < \alpha \leq 1)$. We denote by $SS^\star(\alpha)$ the subclass of $A$ consisiting of all strongly starlike functions of order $\alpha$ in $U$. A function $f(z)$ in $A$ is said to be strongly convex of order $\alpha$ if it satisfies

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad (z \in U)$$

for some $\alpha$ $(0 < \alpha \leq 1)$. We denote by $SC(\alpha)$ the subclass of $A$ consisiting of all strongly convex functions of order $\alpha$ in $U$.

In particular, we denote by $S^\star(0) = SS^\star(1) = S^\star$ and $C(0) = SC(1) = C$. It is shown that $f(z)$ is in $C(\alpha)$ if and only if $zf'(z)$ is in $S^\star(\alpha)$, and also $f(z)$ is in $SC(\alpha)$ if and only if $zf'(z)$ is in $SS^\star(\alpha)$.

Let $f(z)$ and $g(z)$ be analytic in $U$. Then we say that $f(z)$ is subordinate to $g(z)$ and we write $f(z) \prec g(z)$, if $g(z)$ is univalent in $U$, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Tuneski [2] introduced a new sufficient condition for starlikeness as the following.

**Theorem A**([2]). *If $f(z) \in A$, $f(z) \neq 0$ in $0 < |z| < 1$, and*

$$\frac{f(z)f''(z)}{f'(z)^2} \prec 2 - \frac{2}{(1-z)^2} = h(z) \quad (z \in U)$$

*then $f(z) \in S^\star$.*

In [2], the condition $f(z) \neq 0$ in $0 < |z| < 1$ was not assumed, but it is necessary to complete the proof, because $p(z) = zf'(z)/f(z)$ must be analytic in $U$ in the proof.

Furthermore, the condition $f'(z) \neq 0$ in $U$ is also necessary to complete the proof, but if $f(z) \neq 0$ in $0 < |z| < 1$ is assumed, then we have $f'(z) \neq 0$ in $U$. Because if $f'(z_0) = 0$ in $0 < |z_0| < 1$, then we have $f'(z) = (z - z_0)^l g(z)$, where $g(z)$ is analytic in $U$, $g(z_0) \neq 0$, and $l$ is a positive integer. From this, we have

$$\frac{f(z)f''(z)}{f'(z)^2} = \frac{lf(z)}{(z-z_0)^{l+1}g(z)} + \frac{f(z)g'(z)}{(z-z_0)^l g(z)^2}.$$

Letting $z \to z_0$ with

$$\arg(z - z_0) = \frac{\arg f(z_0) - \arg g(z_0)}{l+1},$$

then

$$\lim_{z \to z_0} \frac{f(z)f''(z)}{f'(z)^2} = +\infty \text{ (real number)}$$

because $f(z_0) \neq 0$. Thus

$$\lim_{z \to z_0} \frac{f(z)f''(z)}{f'(z)^2} \notin h(U)$$

because it is obtained $\{z \in \mathbb{C} : z = \mathrm{Re}\, z \geq 3/2\} \not\subset h(U)$ by an easy calculation. This contradicts the assumption and so we conclude $f'(z) \neq 0$ in $U$.

In the present paper, we improve Theorem A and we obtain new sufficient conditions for starlikeness, convexity, strongly starlikeness and strongly convexity of order $\alpha$.

We need the following lemma due to Miller and Mocanu [1].

**Lemma**([1]). *Let $q(z)$ be univalent in $U$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

**(i):** *$Q(z)$ is starlike in $U$,*

**(ii):** $\mathrm{Re}\,\{zh'(z)/Q(z)\} = \mathrm{Re}\,\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0$, $z \in U$.

*If $p(z)$ is analytic in $U$, with $p(0) = q(0)$, $p(U) \subseteq D$ and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z)$$

*then $p(z) \prec q(z)$.*

## 2. New sufficient conditions for starlikeness and convexity of order $\alpha$.

We begin with the statement and the proof of the following result.

**Theorem 1.** *If $f(z) \in A$, $f(z) \neq 0$ in $0 < |z| < 1$, $-1 < \alpha \leq 1$, and*

$$\frac{f(z)f''(z)}{f'(z)^2} \prec h_\alpha(z) = \begin{cases} \dfrac{\alpha+1}{\alpha}\left(1 - \dfrac{1}{(1-\alpha z)^2}\right) & (\alpha \neq 0), \\ -2z & (\alpha = 0) \end{cases}$$

$$(z \in U)$$

*then $f(z) \in S^*\left(\dfrac{1-\alpha}{2}\right)$.*

**Proof.** We choose $p(z) = zf'(z)/f(z)$, $q(z) = (1-\alpha z)/(1+z)$, $\theta(w) = 1 - (1/w)$, $\phi(w) = 1/w^2$. Then $q(z)$ is univalent in $U$, $\theta(w)$ and $\phi(w)$ are analytic in the domain $D = \mathbb{C} \setminus \{0\}$ which contains $q(U) = \{z \in \mathbb{C} : \mathrm{Re}\, z > (1-\alpha)/2\}$ and $\phi(w) \neq 0$ when $w \in q(U)$.

Further,

$$Q(z) = zq'(z)\phi(q(z)) = -\frac{(1+\alpha)z}{(1-\alpha z)^2}$$

is starlike in $U$, and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \frac{(\alpha+1)(\alpha z^2 - 2z)}{(1-\alpha z)^2} \\ &= \begin{cases} \dfrac{\alpha+1}{\alpha}\left(1 - \dfrac{1}{(1-\alpha z)^2}\right) & (\alpha \neq 0), \\ -2z & (\alpha = 0), \end{cases} \end{aligned}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} = \begin{cases} \dfrac{2}{1-\alpha z} & (\alpha \neq 0), \\ 2 & (\alpha = 0). \end{cases}$$

Thus,

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in U).$$

Further, $p(z)$ is analytic in $U$ because $f(z) \neq 0$ in $0 < |z| < 1$, $p(0) = q(0) = 1$ and $0 \notin p(U)$ because it is obtained $f'(z) \neq 0$ in $U$ applying the assumptions $f(z) \neq 0$ in $0 < |z| < 1$ and $f(z)f''(z)/f'(z)^2 \prec h_\alpha(z)$ by the preceding argument of Theorem A. Thus, $p(U) \subset D$. Therefore, the conditions of Lemma are satisfied and so we obtain that if

$$\begin{aligned} \theta(p(z)) + zp'(z)\phi(p(z)) &= \frac{f(z)f''(z)}{f'(z)^2} \\ &\prec \begin{cases} \dfrac{\alpha+1}{\alpha}\left(1 - \dfrac{1}{(1-\alpha z)^2}\right) & (\alpha \neq 0) \\ -2z & (\alpha = 0) \end{cases} \\ &= h(z) \quad (z \in U) \end{aligned}$$

then

$$\frac{zf'(z)}{f(z)} = p(z) \prec q(z) = \frac{1-\alpha z}{1+z}.$$

Thus, $f(z) \in S^\star\left(\dfrac{1-\alpha}{2}\right)$. $\square$

**Remark 1.** Theorem A is obtained from Theorem 1 ($\alpha = 1$). Therefore, Theorem 1 is an expansion of Theorem A.

**Remark 2.** We have that $f(z)f''(z)/f'(z)^2$ and $h_\alpha(z)$ defined in Theorem 1 are analytic in $U$, $f(0)f''(0)/f'(0)^2 = h_\alpha(0) = 0$ and $h_\alpha(z)$

is univalent in $U$. So, we have that the condition of Theorem 1 is equivalent with

$$\frac{f(z)f''(z)}{f'(z)^2} \in h_\alpha(U) \quad (z \in U).$$

**Example.** The function $f(z) = a(1 - e^{-z/a})$, $a > 1/\log 3$, is in $A$, $f(z) \neq 0$ in $0 < |z| < 1$, and

$$\left|\frac{f(z)f''(z)}{f'(z)^2}\right| = |1 - e^{z/a}| < |1 - e^{\log 3}| = 2 \quad (z \in U).$$

Therefore, $f(z) \in S^\star(1/2)$.

If $zf'(z)$ is in $S^\star(\alpha)$, then $f(z)$ is in $C(\alpha)$. Therefore, we have the following corollary.

**Corollary 1.** *If $f(z) \in A$, $f'(z) \neq 0$ in $0 < |z| < 1$, $-1 < \alpha \leq 1$ and*

$$\frac{zf'(z)\{2f''(z) + zf'''(z)\}}{\{f'(z) + zf''(z)\}^2} \prec \begin{cases} \dfrac{\alpha+1}{\alpha}\left(1 - \dfrac{1}{(1-\alpha z)^2}\right) & (\alpha \neq 0), \\ -2z & (\alpha = 0) \end{cases}$$
$$(z \in U)$$

*then* $f(z) \in C\left(\dfrac{1-\alpha}{2}\right)$.

## 3. New sufficient conditions for strongly starlikeness and strongly convexity of order $\alpha$.

We have the following theorem from similar argument of Theorem 1.

**Theorem 2.** *If $f(z) \in A$, $f(z) \neq 0$ in $0 < |z| < 1$, $0 < \alpha \leq 1$, and*

$$\frac{f(z)f''(z)}{f'(z)^2} \prec k_\alpha(z) = 1 - \left(\frac{1+z}{1-z}\right)^\alpha - \frac{2\alpha z(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}} \quad (z \in U)$$

*then* $f(z) \in SS^\star(\alpha)$.

**Proof.** We choose $p(z) = zf'(z)/f(z)$, $q(z) = \{(1-z)/(1+z)\}^\alpha$, $\theta(w) = 1 - (1/w)$, $\phi(w) = 1/w^2$. Then $q(z)$ is univalent in $U$, $\theta(w)$ and $\phi(w)$ are analytic in the domain $D = \mathbb{C} \setminus \{0\}$ which contains $q(U) = \{z \in \mathbb{C} : |\arg z| < \pi\alpha/2\}$ and $\phi(w) \neq 0$ when $w \in q(U)$. Further,

$$Q(z) = zq'(z)\phi(q(z)) = -\frac{2\alpha z(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}}$$

is starlike in $U$, and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= 1 - \left(\frac{1+z}{1-z}\right)^{\alpha} - \frac{2\alpha z(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}} \end{aligned}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} = \frac{2(1+\alpha z)}{1-z^2}$$

Thus,

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 1 > 0 \quad (z \in U).$$

Further, $p(z)$ is analytic in $U$ because $f(z) \neq 0$ in $0 < |z| < 1$, $p(0) = q(0) = 1$ and $0 \notin p(U)$ because it is obtained $f'(z) \neq 0$ in $U$ applying the assumptions $f(z) \neq 0$ in $0 < |z| < 1$ and $f(z)f''(z)/f'(z)^2 \prec k_\alpha(z)$ by the preceding argument of Theorem A. Thus, $p(U) \subset D$. Therefore, the conditions of Lemma are satisfied and so we obtain that if

$$\begin{aligned} \theta(p(z)) + zp'(z)\phi(p(z)) &= \frac{f(z)f''(z)}{f'(z)^2} \\ &\prec 1 - \left(\frac{1+z}{1-z}\right)^{\alpha} - \frac{2\alpha z(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}} \\ &= h(z) \quad (z \in U) \end{aligned}$$

then

$$\frac{zf'(z)}{f(z)} = p(z) \prec q(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}.$$

Thus, $f(z) \in SS^\star(\alpha)$. □

If $zf'(z)$ is in $SS^\star(\alpha)$, then $f(z)$ is in $SC(\alpha)$. Therefore, we have the following corollary.

**Corollary 2.** *If $f(z) \in A$, $f'(z) \neq 0$ in $0 < |z| < 1$, $0 < \alpha \leq 1$ and*

$$\frac{zf'(z)\{2f''(z) + zf'''(z)\}}{\{f'(z) + zf''(z)\}^2} \prec 1 - \left(\frac{1+z}{1-z}\right)^{\alpha} - \frac{2\alpha z(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}} \quad (z \in U)$$

*then $f(z) \in SC(\alpha)$.*

## References

[1] S.S.Miller and P.T.Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J. **32**(1985), 185-195.

[2] N.Tuneski, *On certain sufficient conditions for starlikeness*, Internat. J. Math. Sci. **23**(2000), 521-527.

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