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A Limit Theorem for Measure-Valued Processes in A Super-Diffusive Medium†

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Abstract

We consider the Cauchy problem for nonlinear reaction-diffusion equation in a superdiffusive random medium. The non-degeneracy of L^1- norm of the positive solutions can be established as a result of longtime asymptotic behaviors. We construct locally finite non-negative measure valued stochastic processes associated with the nonlinear equations in question and show stochastic convergence of the processes in a superdiffusive medium as a probabiblistic counter-phenomenon.

I Introduction

The technical term *catalytic branching* is used in most cases for stochastic models which are introduced, for instance, based upon the following two distinct viewpoints in catalytic chemical systems or in catalytic biological systems. The first one is a microscopic view in the chemical reaction, where a molecule reveals a certain chemical reaction only in the places where exists the catalyst. The second one is just the case where, in the macroscopic view, the chemical reaction is described by reaction diffusion equations and the catalyst enters as a spatially heterogeneous rate function. In some cases there are catalysts present only in the localized regions such as networks of filaments or the surfaces of pellets.

Mathematically, such systems are modelled by the following catalytic reaction diffusion equations in $\mathbf{R}^d$

$$-\frac{\partial u}{\partial s} = \frac{1}{2}\Delta u + \rho_s \cdot R(u), \qquad 0 \leq s \leq t \tag{1}$$

with terminal condition $u|_{s=t} = \varphi$. Here R is a reaction term, and ρ_s is a spatial density of the catalyst at time s with continuous measure-valued path : $s \mapsto \rho_s \in \mathcal{M}(\mathbf{R}^d)$. Let $\hat{p}(r,b)$ denote the transition density of a standard Brownian motion in $\mathbf{R}^d$. Then the above (1) can be formulated rigorously by the following integral equation:

$$u(s,t,a) = \int p(t-s, b-a)\varphi(b)db + \int_s^t dr \int p(r-s, b-a)R(u(r,t,b))\rho_r(db). \tag{2}$$

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Our main concerns are firstly to formulate the equation (2) meaningfully for measure-valued paths ρ as generalized as possible, and secondly to investigate long-time asymptotic behaviors of the solutions, whereby we aim at studying the asymptotic behaviors of the associated stochastic processes. In this paper we will treat simply the typical case $R(u) = u^2$.

For the cases when ρ_s in (1) are *nice* measures having mass on an open set or a hypersurface, the equation (1) has been studied via analytic method by Chadan-Yin [2], Chan-Fung [3], Bramson-Neuhauser [1], and Durrett-Swindle [19]. On the other hand, the relationship between semilinear reaction diffusion equations, branching particle systems, and superprocesses (or measure-valued processes) has been investigated by Dynkin-Kuznetsov [21], Le Gall [24], and Gorostiza-Wakolbinger [23]. At the same time this implies that probabilistic research on analysis of this sort of equation like (1) may provide with a natural approach to the asymptotic problem, in connection with superprocesses associated with catalytic reaction diffusion equations. As to the works for stochastic processes with catalytic branching, there can be found interesting and exciting new results in series of papers written by Dawson-Fleischmann [5,6,7], and Fleischmann-Le Gall [22].

This paper is organized as follows. In Section II we introduce basic notations and preliminaries used in the succeeding sections through the whole paper. Section III is devoted to the construction of catalytic superdiffusion in a superdiffusive random medium. In particular, in Section III.1 we shall look at a quick review of superdiffusion in terms of Dynkin's formulation [20], which plays an essential role later as catalyst process in construction of catalytic superdiffusion. The useful tools called *branching rate functionals* (BRF) are provided in Section III.2, where we introduce several classes of BRF. Each class possesses its own peculiar feature to work effectively in the investigation of properties of the corresponding measure-valued processes, such as existence of process itself, its characterization, existence of modification with continuous sample paths, etc. Diffusive collision local time (DCLT) is constructed in Section III.5, whereby the existence of catalytic superdiffusion with DCLT as its branching rate functional is shown as well. In Section IV we study longtime asymptotic behaviors, which are the chief themes in this paper. In Section IV.1 asymptotic non-degeneracy of the L^1- norm of positive solutions to nonlinear catalytic reaction diffusion equations is proved. Section IV.2 is devoted to the asymptotic behaviors of the associated processes, namely, catalytic superdiffusions (CSD). It is shown that CSD with Lebesgue measure as its initial measure converges stochastically to the Lebesgue measure as time parameter goes to infinity.

II Notation and Preliminaries

Let p be a positive number such that $p > d$, where d is the space dimension parameter. φ_p is a reference function defined by

$$\varphi_p(x) := (1 + |x|^2)^{-p/2}, \qquad x \in \mathbf{R}^d.$$

We denote by $\mathcal{C}^p$ the space of continuous functions f on $\mathbf{R}^d$ such that $|f| \le C_f \varphi_p$ for some positive constant C_f depending on f. The norm $\|f\|$, $f \in \mathcal{C}^p$ is defined by

$$\|f\| := \|f/\varphi_p\|_\infty,$$

and $(\mathcal{C}^p, \|\cdot\|)$ becomes a Banach space. $\mathcal{C}^p_+$ is the totality of positive elements of $\mathcal{C}^p$. For a time interval I in $\mathbf{R}_+$, $\mathcal{C}^{p,I}$ denotes the space of all functions $f(s,x)$ in $\mathcal{C}(I \times \mathbf{R}^d)$ such that there exists a positive constant C_f depending on f, satisfying

$$|f(s,\cdot)| \le C_f \cdot \varphi_p \quad \text{for} \quad s \in I.$$

Let $\mathcal{B} \equiv \mathcal{B}(\mathbf{R}^d)$ denote the space of all Borel measurable functions on $\mathbf{R}^d$. We say that $f \in \mathcal{B}$ if $f : \mathbf{R}^d \to \mathbf{R}$ is $\mathcal{B}$- measurable. Let $\mathcal{B}^p$ denote the set of all those $f \in \mathcal{B}$ satisfying $|f| \le C_f \varphi_p$ for some constant C_f. Moreover, $f \in b\mathcal{B}^p$ means that f is a bounded element of $\mathcal{B}^p$. As is easily imagined, the symbols $\mathcal{B}^p_+$, $\mathcal{B}^{p,I}$, etc. denote those measurable counterparts of $\mathcal{C}^p_+$, $\mathcal{C}^{p,I}$, etc. respectively. Let $\mathcal{M}_p \equiv \mathcal{M}_p(\mathbf{R}^d)$ denote the set of all locally finite non-negative measures μ on $\mathbf{R}^d$, such that

$$\|\mu\|_p := \langle \mu, \varphi_p \rangle = \int_{R^d} \varphi_p(y)\mu(dy) < \infty.$$

$\mathcal{M}_p$ is also called the set of tempered measures on $\mathbf{R}^d$, equipped with the p- vague topology. While, $\mathcal{M}_F = \mathcal{M}_F(\mathbf{R}^d)$ is the set of all finite measures on $\mathbf{R}^d$.

L is the second order differential operator defined by

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(r,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(r,x)\frac{\partial}{\partial x_i} \tag{3}$$

for $(r,x) \in Q := \mathbf{R}_+ \times \mathbf{R}^d$. We assume

(A.1) (a) $A = (a_{ij})$ is non-negative definite and symmetric.
(b) L is uniformly elliptic, i.e., there is a positive constant C_0 such that

$$\sum_{i,j} a_{ij}u_i u_j \ge C_0 \sum_i |u_i|^2 \quad \text{for} \quad \forall (r,x) \in S \quad \text{and} \quad u_1, \cdots, u_d \in \mathbf{R}.$$

(c) $a_{ij}, b_i \in b\mathcal{C}(S)$ satisfying Hölder conditions: there exists a positive constant A, $0 < \alpha \le 1$ such that for any $(r,s,x,y) \in \mathbf{R}^2_+ \times \mathbf{R}^d \times \mathbf{R}^d$,

$$\begin{aligned} |a_{ij}(r,x) - a_{ij}(s,y)| &\le A\{\,|r-s|^{\alpha/2} + |x-y|^\alpha\,\}, \\ |b_i(r,x) - b_i(r,y)| &\le A|x-y|^\alpha. \end{aligned}$$

Under the assumption (A.1) there exists a fundamental solution $p(r,x;t,y)$ of the differential equation $\partial_t u + Lu = 0$ such that for any $f \in b\mathcal{C}(\mathbf{R}^d)$,

$$u(r,x) = \int_{R^d} p(r,x;t,y) f(y)dy \tag{4}$$

satisfies the problem

$$\begin{cases} \dfrac{\partial}{\partial t}u + Lu = 0 & \text{in} \quad Q_{<t} := [0,t) \times \mathbf{R}^d \\ u(r,x) \to f(x) & \text{as} \quad r \nearrow t. \end{cases} \tag{5}$$

We denote by $(\xi_t, \Pi_{r,x})$ an L- diffusion, which is a Markov process in $\mathbf{R}^d$ with continuous paths with transition function

$$P(r,x,t,dy) = p(r,x;t,y)dy. \tag{6}$$

In addition, $S = (S_t)_{t\geq 0}$ denotes the L- diffusion semigroup.

III Superdiffusion in a Superdiffusive Medium

III.1 Superdiffusion as Catalyst Process

We begin with definition of superdiffusion (SD), which is based on the martingale problem formulation. Let Ω be the path space $\mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$, and K_0 be a special branching rate functional (BRF) given by

$$K_0(dr) := \gamma dr, \qquad \gamma > 0.$$

We consider the superdiffusion $X^{K_0} \equiv X^\gamma$ with BRF K_0. For each $\mu \in \mathcal{M}_p$ (as initial measure), there exists a probability measure $\mathbf{P}^\gamma_\mu$ on $(\Omega, \mathcal{F})$ such that $X^\gamma_0 = \mu$, $\mathbf{P}^\gamma_\mu$- a.s., and

$$M_t(\psi) := \langle X^\gamma_t, \psi\rangle - \langle \mu, \psi\rangle - \int_0^t \langle X^\gamma_s, L\psi\rangle ds, \quad (\forall t > 0, \psi \in \mathrm{Dom}(L))$$

is a continuous $\mathcal{F}_t$- martingale under $\mathbf{P}^\gamma_\mu$, where the quadratic variation process $\langle M.(\psi)\rangle_t$ is given by

$$\langle M.(\psi)\rangle_t = 2\gamma \int_0^t \int \psi(\eta)^2 X^\gamma_s(d\eta) ds, \qquad \mathbf{P}^\gamma_\mu - a.s.$$

for $\forall t > 0$. We adopt this superdiffusion X^γ as catalyst to construct a measure-valued process in catalytic random medium in the succeeding sections. We would rather use the symbol ρ as catalyst process instead of X^γ.

Next we shall present a characterization of SD ρ. Actually,

$$\rho = [\, X^\gamma_t = X^{K_0}_t, \mathbf{P}^\gamma_\mu, t > 0, \mu \in \mathcal{M}_p\,] \quad \text{with} \quad p > d, \ \gamma > 0$$

is an $\mathcal{M}_p$- valued Markov process whose Laplace transition functional is given by

$$\mathbf{P}^\gamma_{s,\mu} \exp\langle X^\gamma_t, -\varphi\rangle = \exp\langle \mu, -v^{[\varphi]}(s,t,\cdot)\rangle, \qquad \varphi \in \mathcal{C}_{+,K} \tag{7}$$

where the solution $v(t) \equiv v^{[\varphi]}(t)$ of the log-Laplace equation

$$v(s,t,x) + \Pi_{s,x} \int_s^t \gamma v^2(r,t,\xi_r) dr = \Pi_{s,x}\varphi(\xi_t) \tag{8}$$

solves uniquely the nonlinear parabolic equation

$$-\frac{\partial v}{\partial s} = Lv - \gamma v^2 \qquad \text{with} \qquad v|_{s=t} = \varphi. \tag{9}$$

Note that

$$\Pi_{s,x}\varphi(\xi_t) = \int p(s,x;t,y)\varphi(y)dy.$$

III.2 Classes of Branching Rate Functionals

The additive functional (AF) $K = K(\xi)$ of diffusion process $\xi = (\xi_t)$ is a random measure $K = K(\omega, dt)$ on $(0, \infty)$ such that for any $r \le t$, $K(\cdot, (r,t))$ is measurable with respect to the completion of $\mathcal{F}(r,t)$ relative to $\Pi_{r,\mu}$, where $\Pi_{r,\mu}$ is defined by

$$\int \Pi_{r,x}\mu(dx)$$

for any $\mu \in M_F$. Let $\mathcal{K}$ be the set of all branching rate functionals. We say that $K \in \mathcal{K}$ if an AF $K = K(\xi)$ satisfies the following two conditions:
(a) (Continuity) $K(dr)$ does not carry mass at any single point set.
(b) (Local Admissibility) For $u \ge 0$,

$$\sup_{a \in R^d} \Pi_{s,a} \int_s^t \varphi_p(\xi_r)K(dr) \to 0 \quad \text{as} \quad s,t \to u.$$

DEFINITION 1. Let $K \in \mathcal{K}$. We say that $K \in \mathcal{K}^*$ if for each finite interval $I = [L,T] \subset \mathbf{R}_+$, there is a positive constant $C(I)$ such that

$$\sup_{s \in I} \Pi_{s,a} \int_s^T \varphi_p^2(\xi_r)K(dr) \le C(I) \cdot \varphi_p(a), \qquad a \in \mathbf{R}^d.$$

DEFINITION 2. We say that $K \in \mathcal{K}^\beta$ ($\beta > 0$) if for each $N > 0$, there is a positive constant $C(N)$ such that

$$\Pi_{s,a} \int_s^t \varphi_p^2(\xi_r)K(dr) \le C(N)|t-s|^\beta \cdot \varphi_p(a) \quad \text{for} \quad 0 \le s \le t \le N, \ a \in \mathbf{R}^d.$$

Notice that we have a natural inclusion $\mathcal{K}^\beta \subset \mathcal{K}^*$.

III.3 Superdiffusion with Continuous Paths

Let $K \in \mathcal{K}^\beta$ for some $\beta > 0$. Then it is easy to show that there exists a probability measure $P_{s,\mu} \in \mathcal{M}_1(\mathcal{C}(\mathbf{R}_+, \mathcal{M}_p))$ such that for $\varphi \in \mathcal{C}_{+,K}$

$$P_{s,\mu} \exp\langle X_t^K, -\varphi\rangle = \exp\langle \mu, -v(s,t)\rangle \tag{10}$$

and $v \equiv v^{[\varphi]}$ is a solution of the log-Laplace equation

$$v(s,t,a) + \Pi_{s,a} \int_s^t v^2(r,t,\xi_r)K(dr) = \Pi_{s,a}\varphi(\xi_t). \tag{11}$$

Define the centered process

$$Z_t := P_{s,\mu} X_t^K - X_t^K \qquad \text{for} \qquad t \geq s. \tag{12}$$

Since $K \in \mathcal{K}^\beta$ for some $\beta > 0$, we can assert Hölder continuity of Z_t. As a matter of fact, we obtain

LEMMA 1. *For $N > 0$, $\mu \in \mathcal{M}_p$, $k \geq 1$ and $\varepsilon \in (0, \beta/2)$, there exists a modification $\tilde{Z}$ of Z such that*

$$\sup_{0 \leq s \leq N} P_{s,\mu} [\sup_{s \leq t \leq t+h \leq N} |\langle \tilde{Z}_{t+h} - \tilde{Z}_t, \varphi \rangle| / h^\varepsilon]^k < +\infty \qquad \textit{for} \quad \varphi \in \mathcal{D}_0 \tag{13}$$

where $\mathcal{D}_0 = \{\varphi_1, \varphi_2, \cdots\}$ is a countable subset of $\mathrm{Dom}(L)$.

Proof. By applying the recursive scheme for moments (3.2.4) [6] and higher moment formula (Lemma 2.6.2, [6]) we can easily obtain

$$|P_{s,\mu} \langle S_{t-s}\mu - X_t^K, \varphi \rangle^k| \leq C_0 \Big\{ \|\varphi\|^k \|\mu\|_p (t-s)^{(k-1)\beta} .$$

$$\times \sum_{2 \leq j \leq k-2} \|\mu\|_p \|\varphi\|^{k-j} (t-s)^{(k-j-1)\beta} \|\varphi\|^j (t-s)^{j\beta/2} \sum_{i=1}^{j-1} \|\mu\|_p^i \Big\} . \tag{14}$$

A simple computation with (14) leads to the higher moment estimate of X^K, i.e.,

$$|P_{s,\mu} \langle S_{t-s}\mu - X_t^K, \varphi \rangle^k| \leq C_k (t-s)^{k\beta/2} \|\varphi\|^k \sum_{i=1}^{k-1} \|\mu\|_p^i \tag{15}$$

for $0 \leq s \leq t \leq N$, $\mu \in \mathcal{M}_p$, and $\varphi \in \mathcal{B}^p$. Hence it is not difficult to derive the following estimate of centered moments

$$P_{s,\mu} \langle Z_{t+u} - Z_t, \varphi \rangle^{2k} \leq C_1 (\|S_u\varphi - \varphi\|^{2k} + u^{k\beta} \|\varphi\|^{2k}) \sum_{i=1}^{2k-1} \|\mu\|_p^i \tag{16}$$

for $0 \leq s \leq t \leq t+u \leq N$, $\mu \in \mathcal{M}_p$, and $\varphi \in \mathcal{B}^p$, if we apply Lemma 3.2.2 [6] for (15) by employing the similar techniques discussed in the proof of Lemma 3.2.2 [6]. Consequently the above (16) implies that

$$P_{s,\mu} |\langle Z_{t+h} - Z_t, \varphi \rangle|^{2k} \leq C_2 h^{k\beta} \sum_{i=1}^{2k-1} \|\mu\|_p^i$$

for some positive constant C_2. Therefore, we may resort to a general version of the Kolmogorov criterion in order to conclude the assertion (13), because the class $\mathcal{D}_0$ is rich enough to determine in the category of $\mathcal{M}_p$. Q.E.D.

For $\varphi_k \in \mathcal{D}_0$, we can define a metric d_p in $\mathcal{M}_p$ as

$$d_p(\mu, \nu) := \sum_{m=1}^{\infty} \frac{1}{2^m} (1 \wedge |\langle \mu, \varphi_m \rangle - \langle \nu, \varphi_m \rangle|) \quad \text{for} \quad \mu, \nu \in \mathcal{M}_p. \tag{17}$$

Note that $(\mathcal{M}_p, d_p)$ becomes a metric space. In particular, $\tilde{Z}$ has $P_{s,\mu}$- a.s. locally Hölder continuous paths of order ε in the metric d_p. As a result, we obtain

PROPOSITION 2. *If $K \in \mathcal{K}^\beta$ for some $\beta > 0$, then there exists a modification $\tilde{X}$ of superdiffusion X^K with continuous paths, that is, $\tilde{X} \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$.*

Proof. From the expectation formula for the superdiffusion X^K, we have

$$P_{s,\mu}X_t^K = S_{t-s}\mu \quad \text{for} \quad \mu \in \mathcal{M}_p.$$

For the one-point compactification $\mathbf{R}_*^d$ of $\mathbf{R}^d$, we denote by $\mathcal{C}_*^p$ the subspace of all elements $f \in \mathcal{C}^p$ such that the mapping

$$F : x \to F(x) := f(x)/\varphi_p(x)$$

can be extended to a function in $\mathcal{C}(\mathbf{R}_*^d)$. Note that $\mathcal{C}_*^p$ becomes a separable Banach space. Since $t \mapsto S_t\varphi$ is a continuous curve in $\mathcal{C}_*^p$, the map $t \mapsto S_t\mu \in \mathcal{M}_p$ can be regarded as a continuous mapping. From (12) and (13), we get

$$S_{t-s}\mu - \tilde{Z}_t = P_{s,\mu}X_t^K - \tilde{Z}_t = X_t^K,$$

implying that there can be found a continuous $\mathcal{M}_p$- valued process if we retake the modification of X^K. Q.E.D.

III.4 Absolute Continuous State of Occupation Time Process

For the catalyst process $\rho = X^\gamma$, $\gamma > 0$, we define the occupation time,

$$Y_{[u,t]}^\gamma := \int_u^t X_r^\gamma \, dr, \qquad 0 \le s \le u \le t \tag{18}$$

where $Y_{[u,t]}^\gamma$ is a measure on $\mathbf{R}^d$ and is distributed according to the law $\mathbf{P}_\mu$, ($\mu \in \mathcal{M}_p$ given). Next we shall define the potential kernel,

$$q(u,t,a,b) := \int_u^t p(0,a;r,b)dr \quad \text{for} \quad 0 \le u \le t, \quad a,b \in \mathbf{R}^d \tag{19}$$

and

$$\mu * q(u,t,b) := \int q(u,t,a,b)\mu(da) \quad \text{for} \quad \mu \in \mathcal{M}_p, \;\; 0 \le u \le t, \;\; b \in \mathbf{R}^d. \tag{20}$$

In addition we can show that, for $\mu \in \mathcal{M}_p$, $r > 0$, $x \in \mathbf{R}^d$

$$\varphi_p(x)\mu * p(0,r,z) \le C_0(\mu)\sqrt{\left(1 + \frac{1}{r}\right)^p r^{p-d}}. \tag{21}$$

We have the following lemma.

LEMMA 3. *If $K \in \mathcal{K}^*$, $\mu \in \mathcal{M}_p$ and $z \in \mathbf{R}^d$, then for $r' = (s'-r)_+$ or $r' = (t-r)$, we have the convergence*

$$\Pi_{s,\mu} \int_s^t q^2(\varepsilon + r', \delta + r', \xi_r, z)\, K(dr) \to 0 \quad as \quad 0 < \varepsilon \leq \delta \searrow 0.$$

Define

$$y^{\varepsilon}_{[u,t]}(z) := \langle Y^{\gamma}_{[u,t]}, p(0,\cdot;\varepsilon,z)\rangle$$

for a small parameter $\varepsilon > 0$. By virtue of Lemma 3, we can show the existence of the $L^2(\mathbf{P}_\mu)$- limit of a family $\{y^{\varepsilon}_{[u,t]}(z)\}_\varepsilon$ as ε tends to zero, and we write its limit as $y^{\gamma}_{[u,t]}(z)$. Hence it follows that the random measure $Y^{\gamma}_{[u,t]}$ on $\mathbf{R}^d$ is absolutely continuous with respect to the Lebesgue measure dz, and we can show that $y^{\gamma}_{[u,t]}$ becomes a density field. Moreover, applying Sugitani's result [25] on jointly continuous property, we can deduce the existence of continuous density field.

PROPOSITION 4. *For $d \leq 3$, $\delta > 0$, $\mu \in \mathcal{M}_p$, and $\mathbf{P}_\mu \sim X^\gamma$ fixed, there exists a jointly continuous field $\tilde{y}^{\gamma}_{\delta} := \{\tilde{y}_{[\delta,\delta+t]}(z);\ t \geq 0,\ z \in \mathbf{R}^d\}$ such that*

$$\mathbf{P}_\mu(Y^{\gamma}_{[\delta,\delta+t]}(dz) = \tilde{y}_{[\delta,\delta+t]}(z)dz, \quad \forall t \geq 0\,) = 1.$$

Proof. This is greatly due to Sugitani's work [25]. First of all, it is interesting to note that for $s > 0$, $\mu * q(s, s+r, z)$ is locally Lipschitz continuous in $(r,z) \in \mathbf{R}_+ \times \mathbf{R}^d$ by virtue of the condition (21). While, we have

$$\mathbf{P}_\mu y_{[s,s+t]}(z) = \mu * q(s, s+t, z)$$

by the expectation formula for density field. So that, we can define the centered field as

$$\Xi_r(s,z) = \mu * q(s, s+t, z) - y_{[s,s+t]}(z).$$

Hence the proposition can be attributed to the assertion that $\Xi_r(s,z)$ has a jointly continuous modification. So we need to derive the moment estimates of Ξ_r. Indeed, a direct computation with the argument of Lemma 3.2.1 [6] provides with the following moment estimate

$$\mathbf{P}_\mu|\Xi_{t+h}(s,z) - \Xi_t(s,z)|^{2k} \leq C_k \gamma^k h^{k/2} \sum_{i=1}^{2k-1} \{\, \mu * q(s+t, s+t+2h, z)\,\}^i. \tag{22}$$

On the other hand, set

$$\Xi^{\varepsilon}_t(s,z) := \mu * q(\varepsilon + s, \varepsilon + s + t, z) - y^{\varepsilon}_{[s,s+t]}(z).$$

By applying the recursive scheme (in Lemma 4, [7]) we can prove the moment estimate

$$\mathbf{P}_\mu|\Xi^{\varepsilon}_t(s,z) - \Xi^{\varepsilon}_t(s,\zeta)|^k \leq C_k \gamma^{k/2} |z-\zeta|^{k\alpha}.$$

$$\times \sum_{i=1}^{k-1} \{ \mu * q(s, 2(\varepsilon + s - t), z) + \mu * q(s, 2(\varepsilon + s - t), \zeta) \}^i \tag{23}$$

for a small parameter $\alpha \in (0,1)$, and all ε, γ, $|z|$, $|\zeta|$, t, $s+t \in (0, N]$, where we employed the relation

$$|u^{(k)}(s,a)| \leq k! C_k \gamma^{k/2} |z-\zeta|^{k\alpha}.$$

$$\times \{q((\delta - s)_+, 2(\varepsilon + \delta + t - s), a, z) + q((\delta - s)_+, 2(\varepsilon + \delta + t - s), a, \zeta)\}$$

for the sequence $\{u^{(n)}(s,a)\}$ of

$$u(s,a) = \Pi_{s,a} \int_s^T v^2(r, \xi_r) K_0(dr)$$

(cf. Lemma 11, [7]). On this account, by the above-mentioned local Lipschitz continuity and the L^2- convergence of $\{y^{\varepsilon}_{[u,t]}(z)\}$ together with (23) it is easy to show that

$$\mathbf{P}_\mu |\Xi_t(s,z) - \Xi_t(s,\zeta)|^{2k} \leq C_k \gamma^k |z-\zeta|^{2k\alpha}.$$

$$\times \sum_{i=1}^{2k-1} \{ \mu * q(s, 2(s+t), z) + \mu * q(s, 2(s+t), \zeta) \}^i. \tag{24}$$

Thus, by choosing k sufficiently large, we may apply the Kolmogorov moment criterion to deduce from (22), (24) that there exists a jointly continuous modification of $\Xi_\cdot(s,\cdot)$, if we pay attention to the fact that each sum remains finite by continuity. Q.E.D.

In addition, since we have that

$$\text{the mapping} : [r,z] \longrightarrow \mu * q(0,r,z)$$

is finite and continuous on $\mathbf{R}_+ \times \mathbf{R}^d$, we can extend the above joint continuity result to the case where $\delta = 0$ for $\tilde{y}^\gamma_\delta$. Furthermore, we have

LEMMA 5. *Let $d \leq 3$. For $\alpha \in (0, \xi_0)$, $0 < \xi_0 < 1$, $\delta \geq 0$, and $\mathbf{P}$ fixed, there exists a modification $\tilde{y}_\delta$ of y^γ_δ such that*

$$\sup \frac{|\tilde{y}_{[\delta,\delta+t]}(z)\varphi_p(z) - \tilde{y}_{[\delta,\delta+s]}(\zeta)\varphi_p(\zeta)|}{|[t,z]-[s,\zeta]|^\alpha} < \infty, \quad \mathbf{P} - a.s.$$

holds, where the supremum is taken over the region : $0 \leq t, s \leq N$, $z, \zeta \in \mathbf{R}^d$, and $[t,z] \neq [s,\zeta]$.

III.5 Regular Paths and Catalytic Superdiffusion

First of all we shall introduce the concept of regular paths. Let $N > 0$, $0 < \varepsilon \leq 1$ be fixed, and take $\eta \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$. We define

$$R^\varepsilon_N(\eta) := \sup_{\substack{0 \leq s \leq N \\ a \in R^d}} \int_s^{s+\varepsilon} \langle \eta_r, \varphi_p \cdot p(s,a;r,\cdot) \rangle \, dr. \tag{25}$$

Suggested by Dawson-Fleischmann [7], we shall give below the definition of regular paths. If the path is regular, then the existence of the corresponding catalytic BRF is able to be guaranteed.

DEFINITION 3. We say that η is a regular path if $R_N^\varepsilon(\eta) \to 0$ holds for any $N > 0$ as ε tends to zero. Then we write $\eta \in \mathcal{R}$.

For the catalyst process $\rho = X^\gamma = X^{K_0}$ with $K_0(dr) = \gamma dr$, $\gamma > 0$, we know that $\rho \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$ with probability one and moreover, $\rho \in \mathcal{R}$, namely, we observe that the catalyst X^γ has a regular path in the sense of Definition 3. Indeed we prove

LEMMA 6. *The realization $\rho_{\delta+(\cdot)}$ is a regular path with* **P**- *probability one.*

Proof. Fix $N > 0$, $\varphi \in \mathcal{B}_+^p$. Let us consider the integral

$$I := \int_s^{s+\varepsilon} \langle \rho_{\delta+r}, \varphi \cdot p(s,a;r,\cdot) \rangle \, dr. \tag{26}$$

Since the catalyst ρ has a continuous density field $y_\delta = \{y_{[\delta,\delta+t]}(z)\}$ from Sugitani's jointly continuous criterion argument, a locally finite random measure

$$\lambda_\delta^z(dt) \approx y^\gamma_{[\delta+,\delta+(\cdot)]}(z) \cdot dt$$

on $\mathbf{R}_+$ is naturally determined. Then we have

$$\begin{aligned}
I &= \int_s^{s+\varepsilon} dr \int \varphi(b) p(s,a;r,b) X^\gamma_{\delta+r}(db) \\
&= \int \varphi(b) db \int_s^{s+\varepsilon} p(s,a;r,b) \lambda_\delta^b(dr) \\
&= \int \varphi(b) db \int_s^{s+\varepsilon} \left(p(0,a;\varepsilon,b) + \int_r^{r+\varepsilon} \left| \frac{\partial}{\partial\sigma} p(s,a;\sigma,b) \right| d\sigma \right) \lambda_\delta^b(dr) \\
&\le \int \varphi(b) db \int_s^{s+\varepsilon} \left(p(0,a;\varepsilon,b) + \int_r^{r+\varepsilon} \frac{C_0}{(\sigma-s)^{d/2+1}} \cdot \right. \\
&\quad \left. \times \exp\left\{ C_1(\sigma-s) - \frac{|b-a|^2}{C_2(\sigma-s)} \right\} d\sigma \right) \lambda_\delta^b(dr)
\end{aligned}$$

where we made use of elementary properties of fundamental solutions for parabolic equations. Moreover, a simple estimation gives

$$\begin{aligned}
&\le \int \varphi(b) db \int_s^{s+\varepsilon} p(0,a;\varepsilon,b) \lambda_\delta^b(dr) \\
&\quad + \int \varphi(b) db \int_s^{s+\varepsilon} \left(\int_r^{r+\varepsilon} \frac{C_4}{\sigma-s} \hat{p}(\sigma-s, b-a) d\sigma \right) \lambda_\delta^b(dr) \\
&\le \int \varphi(b) p(0,a;\varepsilon,b) \lambda_\delta^b([s,s+\varepsilon]) db \\
&\quad + C_4 \int \varphi(b) db \int_s^{s+\varepsilon} (\sigma-s)^{-1} \hat{p}(\sigma-s, b-a) \lambda_\delta^b([s,\sigma]) d\sigma
\end{aligned}$$

$$\leq \int C_\varphi \cdot \varphi_p(b) p(0,a;\varepsilon,b) \tilde{y}_{[\delta+s,\delta+s+\varepsilon]}(b) db$$
$$+C_4 \int C_\varphi \cdot \varphi_p(b) db \int_s^{s+\varepsilon} (\sigma - s)^{-1} \hat{p}(\sigma - s, b-a) \tilde{y}_{[\delta+s,\delta+\sigma]}(b) d\sigma$$

because we interchanged the integral order in the above second line. Since we have

$$\sup_b \tilde{y}_{[s,t]}(b)\varphi_p(b) \leq C'(t-s)^\alpha \quad \text{for} \quad 0 \leq s \leq t \leq N, \ \exists \alpha \in (0,1)$$

from the Hölder modulus estimate in Lemma 5, we continue

$$\begin{aligned}
&\leq C_\varphi \int p(0,a;\varepsilon,b) \left\{ \sup_b \tilde{y}_{[\delta+s,\delta+s+\varepsilon]}(b)\varphi_p(b) \right\} db \\
&\quad +C_5 \int db \int_s^{s+\varepsilon} (\sigma - s)^{-1} \hat{p}(\sigma - s, b-a) \left\{ \sup_b \tilde{y}_{[\delta+s,\delta+\sigma]}(b)\varphi_p(b) \right\} d\sigma \\
&\leq C_\varphi C' \varepsilon^\alpha \int p(0,a;\varepsilon,b) db + C_5 \int db \int_s^{s+\varepsilon} (\sigma - s)^{-1} \hat{p}(\sigma - s, b-a) \cdot C''(\sigma - s)^\alpha d\sigma \\
&\leq C_6 \varepsilon^\alpha + C_7 \int_s^{s+\varepsilon} (\sigma - s)^{\alpha - 1} \left(\int \hat{p}(\sigma - s, b-a) db \right) d\sigma \leq C_8 \varepsilon^\alpha.
\end{aligned}$$

Therefore, taking Definition 3 into account, from (26) and the above estimate we can conclude immediately that $\rho \in \mathcal{R}$ with **P**- probability one. Q.E.D.

Then for $0 < \varepsilon \leq 1$ we define a continuous additive functional (CAF) of L- diffusion ξ by

$$L^\varepsilon(\rho) \equiv L^\varepsilon(\xi,\rho)(dr) := \langle \rho_r, p(0,\xi_r;\varepsilon,\cdot)\rangle \, dr. \tag{27}$$

Hence a general theory for additive functionals deduces the existence of the limit $L(\rho)$ of $\{L^\varepsilon(\rho)\}$.

PROPOSITION 7. *There exists an AF $L(\rho) \equiv L(\xi,\rho)$ of L- diffusion ξ such that for any $\psi \in C_+^{p,I}$ with $I = [0,N]$, $N > 0$,*

$$\sup_{\substack{0 \leq s \leq N \\ a \in R^d}} \Pi_{s,a} \sup_{s \leq t \leq N} \left| \int_s^t \psi(r,\xi_r) L^\varepsilon(\rho)(dr) - \int_s^t \psi(r,\xi_r) L(\rho)(dr) \right|^2 \to 0 \quad (\varepsilon \searrow 0). \tag{28}$$

Proof. Take an element $\psi \in C_+^{p,[0,N]}$. Then

$$\text{the map} : t \mapsto \psi(t,x)\rho_t(dx)$$

is a continuous $\mathcal{M}_F$- valued path on $[0,N]$. Define a continuous AF $A^\varepsilon = A^\varepsilon(\xi,\psi\rho)$ as

$$A^\varepsilon(\xi,\psi\rho)(dr) := \langle \psi(r,\xi_r)\rho_r, p(0,\xi_r;\varepsilon,\cdot)\rangle dr$$

in line with (27). Recall that ρ belongs to the class $\mathcal{R}$, so that, we observe that

$$\sup_{\substack{0\le s\le N\\ a\in R^d}} \int_s^{s+\varepsilon} dr \int \psi(r,b)p(s,a;r,b)X_r^{\gamma}(db)$$

vanishes as ε tends to zero for each $N > 0$. An application of Theorem 4.1 (p.144, [28]) with a slight modification changed into the restriction of

$$\int \psi dX_t^{\gamma} \quad \text{on} \quad t \in [0, N],$$

deduces that there exists a continuous AF $A(\xi, \psi\rho)$ of L- diffusion ξ such that

$$\sup_{s,a} \Pi_{s,a}\left(\sup_{0\le t\le N} |A^{\varepsilon}(\xi,\psi\rho)(s,t) - A(\xi,\psi\rho)(s,t)|^2\right) \to 0 \quad (\varepsilon \downarrow 0) \tag{29}$$

for $N > 0$. On this account, we have only to set

$$L(\rho) \equiv L(\xi,\rho)(dr) := \psi(r,\xi_r)^{-1} \cdot A(\xi,\psi\rho)(dr).$$

Thus we can assert the existence of AF $L(\rho)$ of L- diffusion ξ. Q.E.D.

Furthermore, it is possible to state a stronger result on the above convergence (29). Let h be a function : $[0,1] \to \mathbf{R}_+$ such that $h(u) \searrow 0$ as $u \to 0$. For $M \in \mathrm{N}$, $\psi \in C_+^{p,I}$, define the set $\Phi(h, M)$ as

$$\left\{\eta \in \mathcal{R}: \int_0^N \eta_s(1)ds \le M, \ \sup_{s,a} \int_0^u dr \int p(s,a;r,b)\psi(r,b)\eta_r(db) \le h(u), \ \forall u \le 1\right\}.$$

PROPOSITION 8. *The convergence* (29) *in the above Proposition 7 is uniform on* $\Phi(h, M)$.

Proof. Take a sequence $\{s(k)\}$ such that $s(k) \nearrow N$ as $k \to \infty$. Set

$$M_t^{\varepsilon} := \Pi_{s,a}[A^{\varepsilon}(\xi,\psi\eta)(s,s(\infty)) \mid \xi_u, u \le t]$$

By Markov property we can rewrite it as

$$M_t^{\varepsilon} = A^{\varepsilon}(\xi,\psi\eta)(s,t) + \Pi_{t,\xi_t} A^{\varepsilon}(\xi,\psi\eta)(t,s(\infty)). \tag{30}$$

Then notice that M_t^{ε} is a nonnegative $L^2(\Pi_{s,a})$- martingale such that

$$\lim_{t\to N} M_t^{\varepsilon} = A^{\varepsilon}(\xi,\psi\eta)(s,N), \Pi_{s,a} - a.s.$$

Therefore, we may apply the Doob maximal L^2 inequality to get

$$\begin{aligned}
&\Pi_{s,a}(\sup_t |M_t^\varepsilon - M_t^\delta|^2) \\
\leq\ & C \cdot \Pi_{s,a}|A^\varepsilon(\xi, \psi\eta)(s, s(\infty)) - A^\delta(\xi, \psi\eta)(s, s(\infty))|^2 \\
\leq\ & 2C \cdot \Pi_{s,a} \int_s^{s(\infty)} \left(\int \{p(0, \xi_u; \varepsilon, b) - p(0, \xi_u; \delta, b)\} \psi(u, \xi_u) \eta_u(db) \right) \cdot \\
& \times \Pi_{u,\xi_u} \int_s^{s(\infty)} \left(\int \{p(0, \xi_r; \varepsilon, b) - p(0, \xi_r; \delta, b)\} \psi(r, \xi_r) \eta_r(db) \right) dr du
\end{aligned}$$

$$\leq 4C \, |||\Pi_{\cdot,\cdot} A(\xi, \psi\eta)(0, N)|||_\infty \cdot \left|\left|\left| \Pi_{\cdot,\cdot} \int (\psi\eta_r) * p(\varepsilon) dr - \Pi_{\cdot,\cdot} \int (\psi\eta_r) * p(\delta) dr \right|\right|\right|_\infty. \tag{31}$$

Combining (31) with (30) we get

$$\begin{aligned}
&\sup_{s,a} \Pi_{s,a} \left(\sup_{0 \leq t \leq N} |A^\varepsilon(\xi, \psi\eta)(s, t) - A^\delta(\xi, \psi\eta)(s, t)|^2 \right) \\
\leq\ & C' \left|\left|\left| \Pi_{\cdot,\cdot} \int \int (p(\varepsilon) - p(\delta)) \psi\eta_r(db) dr \right|\right|\right|_\infty^2
\end{aligned}$$

$$+C'' \, |||\Pi_{\cdot,\cdot} A(\xi, \psi\eta)(0, N)|||_\infty \times \left|\left|\left| \Pi_{\cdot,\cdot} \int \int (p(\varepsilon) - p(\delta)) \psi\eta_r(db) dr \right|\right|\right|_\infty. \tag{32}$$

Hence it is obvious from the fact

$$\lim_{\varepsilon \downarrow 0} \left|\left|\left| \Pi_{\cdot,\cdot} A(\xi, \psi\eta)(0, s(\infty)) - \Pi_{\cdot,\cdot} \int (\psi\eta_r) * p(\varepsilon) dr \right|\right|\right|_\infty = 0$$

uniformly in $\eta \in \Phi(h, M)$ that the term $|||\Pi . A(\xi, \psi\eta)(0, N)|||_\infty$ is uniformly bounded with respect to $\eta \in \Phi(h, M)$, because

$$\left|\left|\left| \Pi_{\cdot,\cdot} \int (\psi\eta_r) * p(\varepsilon) dr \right|\right|\right|_\infty \leq C(\varepsilon) \cdot \sup_{s,a} \int_0^u dr \int p(s, a; r, b) \psi(r, b) \eta_r(db)$$

holds. Thus we attain that (32) converges to zero as $\varepsilon, \delta \to 0$ uniformly relative to $\eta \in \Phi(h, M)$. Q.E.D.

We call $L(\rho)$ a catalytic BRF. On this account, we can construct the corresponding catalytic SD $X^{L(\rho)}$ with branching rate functional $L(\rho)$. Actually, $L(\rho)$ is nothing but a diffusive collision local time (DCLT) in the sense of Barlow-Evans-Perkins [26]. Our $L(\rho)$ is a generalization of Brownian collision local time $L_{[W,\rho]}$ defined by Dawson-Fleischmann [7].

PROPOSITION 9. *For any $\mu \in \mathcal{M}_p$, for $\mathbf{P}_\mu$- a.a. realization $\rho(w)$, there exists a catalytic BRF $L(\rho) = L(\xi, \rho) \in \mathcal{K}^\beta$ for some $\beta > 0$.*

Proof. From Definition 2 in Section III.2 it suffices to show that for each $N > 0$,

$$\Pi_{s,a} \int_s^t \varphi_p(\xi_r)^2 L(\rho)(dr) \leq C_N \cdot |t - s|^\beta \varphi_p(a) \tag{33}$$

holds for $0 \le s \le t \le N$, $a \in \mathbf{R}^d$ for some $\beta > 0$. By virtue of (b) of Proposition 6, [7] if ρ lives in $\mathcal{R}$, then Diffusive CLT $L(\rho)$ allows to have the expectation formula

$$\Pi_{s,a} \int_s^t \psi(r, \xi_r) L(\xi, \rho)(dr) = \int_s^t dr \int \psi(r, b) p(s, a; r, b) \rho_r(db) \tag{34}$$

for any $\psi \in C_+^{p,I}$. By using (34) we can rewrite the left-hand side of (33) into

$$J := \int_s^t dr \int \varphi_p(b)^2 p(s, a; r, b) \rho_r(db). \tag{35}$$

By the similar argument in the proof of Lemma 6, we proceed to estimate (35), that is,

$$\begin{aligned}
J &= \int \varphi_p(b)^2 db \int_s^t p(s, a; r, b) \lambda_0^b(dr) \\
&\le \int \varphi_p(b)^2 db \int_s^t \left(p(0, a; t-s, b) + \int_r^{r+t-s} \left| \frac{\partial}{\partial \sigma} p(s, a; \sigma, b) \right| d\sigma \right) \lambda_0^b(dr) \\
&\le \int \varphi_p(b)^2 p(s, a; t, b) \lambda_0^b([s, t]) db \\
&\quad + C_1 \int \varphi_p(b)^2 db \int_s^t \left(\int_r^{r+t-s} (\sigma - s)^{-1} \hat{p}(\sigma - s, b - a) d\sigma \right) \lambda_0^b(dr) \\
&\le C_2 \int \varphi_p(b) p(s, a; t, b) |t - s|^\beta db \\
&\quad + C_3 \int \varphi_p(b) db \int_s^t (\sigma - s)^{-1} \cdot \hat{p}(s, a; \sigma, b) C'(\sigma - s)^\beta d\sigma \\
&\le C_2' \varphi_p(a) |t - s|^\beta \int p(s, a; t, b) db + C_4 \int \varphi_p(b) db \int_s^t (\sigma - s)^{\beta - 1} \hat{p}(s, a; \sigma, b) d\sigma \\
&\le C_2' \varphi_p(a) |t - s|^\beta + C_5 \varphi_p(a) \int_s^t (\sigma - s)^{\beta - 1} \left(\int \hat{p}(s, a; \sigma, b) db \right) d\sigma \\
&\le C_6 |t - s|^\beta \varphi_p(a).
\end{aligned}$$

This completes the proof. Q.E.D.

Since we know that our $L(\rho)$ lies in $\mathcal{K}^\beta$, we may resort to the general construction method for measure-valued processes with BRF $K = L(\rho)$ (cf. [7, 17, 18, 20]) to obtain

THEOREM 10. *Let $d \le 3$. There exists a unique $\mathcal{M}_p$- valued Markov process $X_t^{L(\rho)}$ (with BRF $L(\rho)$) whose Laplace transition functional is given by*

$$P_{s,\mu}^\rho \exp\langle X_t^{L(\rho)}, -\varphi \rangle = \exp\langle \mu, -v^{[\varphi]}(s, t, \cdot) \rangle \tag{36}$$

for an element φ of $C_{+,K}$, *where the function $v \equiv v^{[\varphi]}(\cdot, t, \cdot)$ is the unique solution of*

$$v(s, t, a) + \Pi_{s,a} \int_s^t v^2(r, t, \xi_r) L(\rho)(dr) = \Pi_{s,a} \varphi(\xi_t) \quad (for\ \ 0 \le s \le t, \ \ a \in \mathbf{R}^d). \tag{37}$$

Remark 1. It can be interpreted, in fact, as the particle view that a hidden L- diffusion particle at position $y = \xi_r \in \mathbf{R}^d$ at time r branches with rate $L(\xi, \rho)(dr)$.

In what follows we consider chiefly the one dimensional case. Let $d = 1$. For any $\mu \in \mathcal{M}_p$, the catalyst $\rho = X^\gamma$ possesses a jointly continuous density field $\tilde{\rho}_t(z)$ on $\mathbf{R}_+ \times \mathbf{R}$ such that

$$\rho_t(dz) = \tilde{\rho}_t(z)dz$$

holds for any $t \geq 0$ with $\mathbf{P}_\mu$- probability one. While, for any $\mu \in \mathcal{M}_p$, for $\mathbf{P}_\mu$- a.a. path realization $\rho(\omega)$, there exists a catalytic BRF $L(\rho) = L(\xi, \rho)$ and it follows that $L(\rho) \in \mathcal{K}^\beta$ for some $\beta > 0$. Moreover, when $d = 1$, $L(\rho)$ can be expressed precisely by using $\tilde{\rho}_t$, namely, a simple limiting computation leads to the representation

$$L(\rho)(dr) = \tilde{\rho}_r(\xi_r)dr, \quad \Pi_{s,a} - a.s. \quad a \in R. \tag{38}$$

Therefore, we can construct the corresponding catalytic SD $X^{\tilde{\rho}dr}$ which is a Markov process taking values in $\mathcal{M}_p$. Furthermore, its Laplace transition functional is given by

$$P^\rho_{s,\mu} \exp\langle X^{\tilde{\rho}dr}_t, -\varphi\rangle = \exp\langle \mu, -v^{[\varphi]}(s,t,\cdot)\rangle \tag{39}$$

for each $\mathbf{P}_\mu$- a.a. realization ρ, and for any $\varphi \in \mathcal{C}_{+,K}$. Here $P^\rho_{s,\mu}$ is the law of $X^{\tilde{\rho}dr}$, and this $X^{\tilde{\rho}dr}$ is called a one dimensional catalytic SD in the catalytic medium $\rho = X^\gamma$ distributed by $\mathbf{P}$. Here for any $\varphi \in \mathcal{C}^p_+$, $t > 0$, clearly the solution $v \equiv v^{[\varphi]}(\cdot, t, \cdot)$ of the log-Laplace equation

$$v(s,t,a) + \Pi_{s,a} \int_s^t v^2(r,t,\xi_r)\tilde{\rho}_r(\xi_r)dr = \Pi_{s,a}\varphi(\xi_t) \quad (\text{for } 0 \leq s \leq t, \; a \in \mathbf{R}) \tag{40}$$

solves uniquely the one dimensional nonlinear parabolic equation

$$\begin{cases} -\dfrac{\partial v}{\partial s} &= Lv - \tilde{\rho}_s v^2, \qquad (0 < s \leq t) \\ v|_{s=t} &= \varphi. \end{cases} \tag{41}$$

III.6 Moment Formulae

We have the following moment formulae for catalytic SD $X^{L(\rho)}$ with catalytic BRF $L(\rho)$.

LEMMA 11. *For $0 \leq s \leq t$, $\mu \in \mathcal{M}_p$,* and $\varphi \in \mathcal{B}^p_+$, *we have the expectation formula*

$$P_{s,\mu}\langle X^{L(\rho)}_t, \varphi\rangle = \Pi_{s,\mu}\,\varphi(\xi_t) = \langle \mu, S_{t-s}\varphi\rangle = \langle S_{t-s}\mu, \varphi\rangle < +\infty \tag{42}$$

where $S = (S_t)_{t\geq 0}$ is the L- diffusion semigroup.

Proof. Take a sequence of approximating AFs $\{K_n\}$ from the Dynkin class $\mathcal{K}_0$ (cf. [20]) such that

$$K_n \nearrow L(\xi, \rho) \in \mathcal{K}.$$

From the known formulae (1.22) and (1.23) in [27] for the AF $K_n \in \mathcal{K}_0$, for $\varphi \in \mathcal{B}_+^p$, $m \in \mathcal{M}_F$ we can derive the relation

$$P_{s,m}^{K_n}\langle X_t^{K_n}, \varphi\rangle = \Pi_{s,m}\,\varphi(\xi_t) = \langle m, S_{t-s}\varphi\rangle \tag{43}$$

for $\mathcal{M}_F$- valued Markov processes X^{K_n}, $K_n \in \mathcal{K}_0$, by applying the usual technique [4]. It is possible to extend X^{K_n} to an $\mathcal{M}_p$- valued Markov process $X^{L(\rho)}$ by the domination properties

$$\begin{aligned}
\sup_{s\in I} S_{t-s}\varphi(x) &\le C_0\|\varphi\|\varphi_p(x) \quad (\text{for some } C_0 > 0) \\
\text{and} \quad 0 \le u_n(s,x) &= u_I^{[\varphi]}(K_n) \le C_1\varphi_p(x) \quad (\text{for some } C_1 > 0)
\end{aligned}$$

for $t, s \in I \subset \mathbf{R}_+$: given interval, $x \in \mathbf{R}^d$. Since $K_n \nearrow L(\rho)$, based on the monotone convergence

$$u_n \searrow v_I^{[\varphi]}(L(\rho)) = v(s,x)$$

(by continuity in $\mathcal{K}$ of the cumulant equation), we conclude the convergence of the corresponding moment formula from (43). Q.E.D.

Similarly we can easily show

LEMMA 12. *For $0 \le s \le t, u$, any $\mu \in \mathcal{M}_p$, and $\varphi, \psi \in \mathcal{B}_+^p$, we have the following covariance formula*

$$COV^{P_{s,\mu}}[\langle X_t^{L(\rho)}, \varphi\rangle, \langle X_u^{L(\rho)}, \psi\rangle] = 2\Pi_{s,\mu}\int_s^{t\wedge u} S_{t-r}\varphi(\xi_r)S_{u-r}\psi(\xi_r)L(\rho)(dr). \tag{44}$$

IV Longtime Asymptotic Behaviors

IV.1 Asymptotic Non-Degeneracy of Positive Solutions

In this section we shall introduce our main result of this paper, which is a limit theorem on lomg-time asymptotic non-degeneracy of the L^1- norm of positive solutions of nonlinear catalytic equation (41). Let B be a Borel subset of $\mathbf{R}$, and define ρ^B as the catalyst ρ starting with the restricted measure $\rho_0((\cdot) \cap B)$ for $\rho_0 \in \mathcal{M}_p$ given. Set

$$C_m := [m, m+1) \quad \text{for} \quad m \in Z,$$

and decompose ρ as

$$\rho = \sum_{m\in Z} \rho^{C_m}.$$

We may propose the following condition:

(A.2) $\mathbf{P}_\mu(\rho_t^{C_m}(B(0,|m|/2)) > 0 \quad \text{for some } t > 1) \approx \mathrm{O}(|m|^{-2}), \; m \neq 0 \; (m \nearrow \infty)$.

Now we are in a position to state the principal theorem in this paper.

THEOREM 13. *Assume* (A.1). *If SD X^γ has a finite time interference property with L- diffusion ξ, then for the positive solution $v^{[\varphi]}$, $\varphi \in C_{+,K}$ of* (41), *there can be found a positive constant $C(\varphi)$ depending on φ such that*

$$\lim_{t\to\infty} \|v(s,t,\cdot)\|_1 = C(\varphi) > 0. \tag{45}$$

Otherwise, we have

$$\lim_{t\to\infty} \|v(s,t,\cdot)\|_1 = 0.$$

Here $\|\cdot\|_1$ denotes the L^1-norm on $\mathbf{R}$.

Remark 2. The condition (A.2) is nothing but one of the sufficient conditions for the so-called finite time interference of density field to occur without any additional conditions.

Proof. We assume first that X^γ has the finite time interference property. Then note that there exists a random time $\tau(s,a,\tilde\omega)$ such that

$$\tilde\rho_r(\xi_r) = 0$$

holds for any r satisfying

$$r > \tau(s,a), \quad \Pi_{s,a} \times \mathbf{P}_\mu - a.s. \; \tilde\omega = (\omega,\rho)$$

from our major premise. We have the following Feynman-Kac equation

$$v(s,t,a) = \Pi_{s,a}\varphi(\xi_t)\exp\left\{-\int_s^{t\wedge\tau(s,a)} \tilde\rho_r(\xi_r)v(r,t,\xi_r)dr\right\}. \tag{46}$$

From (46) we can get upper estimates:

$$v(r,t,b) \le \Pi_{r,b}\varphi(\xi_t)$$

and hence

$$\|v(s,t,\cdot)\|_1 \le \|\varphi\|_1.$$

Consequently, there is a constant $C > 0$ such that

$$v(r,t,b) \le C(t-\tau(s,a))_+^{-1/2}\|\varphi\|_1 =: C\cdot\Phi(t;\tau,\varphi).$$

Therefore, a simple calculation leads to

$$v(s,t,a) \geq \Pi_{s,a}\varphi(\xi_t)\exp\left\{-C\int_s^{t\wedge\tau(s,a)} \tilde{\rho}_r(\xi_r)\Phi(t;\tau,\varphi)dr\right\}.$$

Integrating it with respect to the Lebesgue measure $\lambda(da)$ over $\mathbf{R}$, we obtain

$$\lim_{t\to\infty}\|v(s,t)\|_1 \geq \lim_{t\to\infty}\Pi_{s,\lambda}\varphi(\xi_t) = \|\varphi\|_1,$$

where we employed the Lebesgue type dominated convergence theorem and took advantage of the monotone convergence of the exponential term

$$\exp\{-C_0(t)\int_{[s,t\wedge\tau]}\tilde{\rho}_r(\xi_r)dr\}$$

with $C_0(t) = C\cdot\Phi(t)$ towards one as t approaches to infinity. Summing up, we conclude the assertion (45). Otherwise, the above-mentioned estimates together with the convergence theorem yield to the L^1-norm degeneracy of positive solution v as a longtime asymptotic behavior. This completes the proof. Q.E.D.

IV.2 Stochastic Convergence of Catalytic Superdiffusion

THEOREM 14. *Assume the same conditions as in Theorem 13. Let $d = 1$. For $\mathbf{P}_\lambda$-a.a. realization ρ of catalyst process, the catalytic SD $X_t^{L(\rho)}$ converges to the initial Lebesgue measure λ in $P_{s,\lambda}^\rho$- probability ($s \geq 0$) in the p-vague topology in $\mathcal{M}_p$ as t approaches to infinity.*

Proof. The convergence result (45) in Theorem 13 implies that

$$-\log P_{s,\lambda}^\rho \exp\langle X_t^{L(\rho)}, -\varphi\rangle \to \langle\lambda,\varphi\rangle$$

with $\langle\lambda,\varphi\rangle = \|\varphi\|_1$ if we take the Laplace functional characterization of $X^{L(\rho)}$ into consideration. Then the followings are true:

$$\begin{aligned}
\langle\lambda,\varphi_p\rangle P_{s,\lambda}^\rho\left(\langle\lambda,\varphi_p\rangle - \langle X_t^{L(\rho)},\varphi_p\rangle\right) &\to 0 \qquad (t\to\infty)\\
P_{s,\lambda}^\rho\left(\langle X_t^{L(\rho)},\varphi_p\rangle^2 - \langle X_t^{L(\rho)},\varphi_p\rangle\langle\lambda,\varphi_p\rangle\right) &\to 0 \qquad (t\to\infty).
\end{aligned}$$

Hence it follows immediately from the above convergence that

$$P_{s,\lambda}^\rho\left(|\langle X_t^{L(\rho)},\varphi_p\rangle - \langle\lambda,\varphi_p\rangle| > \varepsilon\right) \to 0 \quad \text{as} \quad t\to\infty.$$

This completes the proof. Q.E.D.

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