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A breif history of variational calculus in the first half of twentieth century

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1 Prehistory

The classical calculus of variations is originated in the work of Euler, Lagarange, Legendre, and developed by Jacobi and Wierstrass.

We may say that the calculus of variations has born in the year 1969 since the problem of determining Brachystochrone was generally publicized due to a rather bombbastic advertisement in Acta Eruditorum by Johann Bernouli (1667-1748). As is known the problem was solved by many persons; Newton, Leibnitz and Johann and Jacob Bernoulli.

Usually the birth year of variational calculus is considered as 1744 since Euler, Leonhard (1707-1783) published his@famous book *Methodus inveniendi lineas curvas maximi minive proprietate gaudentes* (A method of discovering curved lines that enjoy a maximum and minimum propertyC or the solution of the isoperimetric problem taken in its wide sense).

Naturally , the book contains the famous Euler equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0,$$

which is a necessary condition for $y(x)$ minimizing

$$J[y] = \int_{x_0}^{x_1} L(x, y, y')dx,$$

where $y(x_0) = y_0, y(x_1) = y_1,\ x_0 < x_1$.

It also contains a collection of 66 problems.

Here we mention the book, published in ninteen century, 1887-1896 *Lecons sur la Théorie générale des surfaces*, 1887-1896 , 4 volumes, Paris.

2 Contribution of Hilbert, Hadamard and Lévy

I. D. Hilbert (1862-1943)

At the international conference of Mathematicians in 1900, Hilbert (1862-1943) mentioned the Mathematical problems in which the variational calculus is the last one. His lecture during the period 1899-1901 at Gottingen was on variational calculus and we can see the influence on his students by thier papers related with variational calculus, for instance see Osgood(1901), Hedrick (1902). The dessertation of Gottingen people such as Bliss, Hahn, Noble were related on variational calculus.

Hilbert's paper on variational calculus *Zur Variationsrechnung* appeared in Math. Ann. Vol LXI, p351-370 in 1906.

II. J. Hadamard (1865-1963)

At the end of 19th century Hadamard first encountered the calculus of variations when working on Wave theory, Elasticity, and Geometrical Problems such as Geodesics.

He discussed the functional operation in 1903 in his paper *On the functional operations* Comptes@Rendus@136, 351-354.

In the preface of his book *Lecon's sur le calcul des variations,* Paris, published in 1910, we can see his concept as follows.

The calculus of variation is nothing else than the first chapter of the theory which is nowadays called the calculus of functionals, and whose development will undoubtedly be one of the first tasks of the future. It is this idea which inspired me above all, in the course of lectures I gave this topic at the Collége de France as well as in the preparation of this work.

Hadamard introduced the term "functional " to replace "functions of lines" , the eaarlier terminology of Volterra.

In the paper *On the functional operations,* Comptes@Rendus@136, 351-354, 1903, he showed that an arbitrary linear functional $U(f)$ on the space $\mathrm{C}[a,b]$ of continuous functions f on $[a,b]$ can be represented in the form

$$U[f] = \lim_{\lambda \to \infty} \int_a^b F(t,\lambda) f(t) dt,$$

where F is independent of f and defined by the functional U on the half strip $\{(t,\lambda) : a \leq t \leq b, \lambda > 0\}$ preceeded the well-known Riesz representation, obtained in 1909.

The representation of a linear functional $U(\omega)$ on the set of analytic functions $\omega(z)$ of a line integral was first obtained by Hadamard as follows;

$$U[\omega] = \frac{1}{2\pi i} \int_C \omega(\zeta)\varphi(\zeta) d\zeta,$$

using the indicator of a functional, which is the function

$$\varphi(\zeta) = U\left[\frac{1}{\zeta - z}\right].$$

This can be seen in his book (1910), however the outline is given in the 1903 paper.

(It is generally accepted that Italian Mathematician Fantappie has done in 1920, by using another indicator.)

In this paper he took a closed surface S, and the two interior points A and B, then

$$\delta g_A^B = \frac{1}{4\pi}\int\int_S \lambda \frac{dg_A^M}{dn}\frac{dg_B^M}{dn} dS_M.$$

λ is normal distance.

III. Paul Lévy (1886-1971)

We can see the influence of Hadamard on Lévy in his desertation (1911), where the generalization of Hadamard equation and integrability was discussed.

In his paper "Sur les équations aux dériv'ees fonctionelles et leur application a —'a la phisique, mathematique", Rendicont del Circolo Matemaatics di Palermo Vol. 33, p281-312, 1912, he discussed the integrability of Hadamard equation, equillibrium problem of elastic plate and Dirichlet problem. In the same journal he discussed Green function in the same volume and general variational equation and analogy of Cauchy problem in volume 37.

Before them three short papers on variational calculus appeared in Comptes Redus.

Later, topics related to variational calculus for Green's function and Neumann's function appeared in Acta math. 42, 1919 (65 pages). However, he did not go into details on functionals of curve or surface.

We note that in Part I and Part II of monograph, published in 1951, he devoted many pages to the variation of such functionals. There was a long pause on this subject until 1971, just before he passed away he mentioned the Hadamard equation in his paper "*Fonctions de lignes et équations aux dérivées fonctionelles*".

In "**Cour de Mechanic**" we can find a section dealing with a flexible system where curves are deforming. There he discussed the solutions to the Euler equation.

A curve C is deformed to a curve $C+\delta C$; that may be represented by a system $\{\delta n(s)\}$ of functions defined on C, where $\delta n(s)$ stands for the normal distance from C to $C+\delta C$. Note that the choice of functions $\{\delta n(s)\}$ depends on C and $C+\delta C$. For a visualized expression of deformation, we can directly see a geometric change from $C \to C + \delta C$.

Example 1. Let L be the length of curve C.

$$L[C+\delta C] - L(C) = -\kappa \int_C \delta n ds + o(\delta C)$$

Example 2. The variation of the integral over a curve is as follows.@

$$\begin{aligned} \mathbf{I} &= \int_C uds \\ \delta I &= \int_C (\delta uds + u\delta ds) \\ &= \int_C (\frac{du}{dn} - \kappa u)\delta uds \end{aligned}$$

In his paper "*On the variation of the distribution of electricity over a conductor, the surface of which is deformed*" Bull. Soc. Math. 1918 France 46, Dirichlet extension problem was discussed.

Let g_B^A be Green's function and **f** be a (harmonic) field between charged surfaces S and S' such that $f = 0$ at ∞. Let A and B be the points between the two surfaces S and S', P be a point on the surface S and M be a boundary point of S.

$$f(A) = \frac{1}{4\pi} \int \frac{\partial g_M^A}{\partial n} f(M) ds$$

By deforming S and S', the variation of Green's function is obtained as

$$\delta g_B^A = -\frac{1}{4\pi} \int_{S \cup S'} \frac{\partial g_M^A}{\partial n} \frac{\partial g_B^M}{\partial n} \delta n ds.$$

In addition, the variations of the total electricity on S and S' are also discussed.

3 Current topic on Variational calculus

We are interested in variation of random fields $\mathbf{X}(\mathbf{C})$. For the random field Lévy's infinitesimal equation can be generalized as

$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C)$$

where $C' < C$ means that C' is inside of C, the domain (C') enclosed by a contour, is a subset of (C), and where Φ is, as before, a nonrandom function and the system

$$Y = \{Y(s), s \in C; C \in \mathbf{C}\}$$

is the innovation.

Here C= $\{C\}$ has to be taken as a class

$$\begin{aligned} &\mathbf{C} = \{C; C \in C^2, \text{diffeomorphic to } S^1, (C) \text{ is convex}\}, \\ &(C): \text{ being the domain enclosed by } C. \end{aligned}$$

The classical variation theory can be applied by using the $S-$transform in white noise theory.

Before we discuss the variation of Gaussian random fields depending on a contour, it is essential to consider a non-random function $G(C)$ of C in $\mathbf{C}$.

I. Non-random function

First consider a non-random function $G(C)$ defined on $\mathbf{C}$, where $G(C)$ is in R^1 and $\mathbf{C}$ is defined in the previous section. Take $C + \delta C \in \mathbf{C}$ which is a slight deformation of C. We write δC as only a symbolic expression of a contour sitting outside of C determined by

$$\delta C = \{\delta n(s); s \in C\} \tag{3.1}$$

in which s is the arc length which represents the parameter of C, $\delta n(s)$ denotes the normal vector to C to the outward direction at the point s and $|\delta n(s)|$ denotes the distance from s to $C + \delta C$.

Definition

If $\|\delta n\| = \sup_s |\delta n(s)| \to 0$ then we say that $C + \delta C$ tends to C .

We can now assume that $\delta n(s)$ is continuous.

Let us assume that G satisfies the following.

$$G(C + \delta C) - G(C) = \delta G(C) + g(C, \delta C) \tag{3.2}$$

such that

1. $\delta G(C)$ is continuous and linear in $\delta n(s)$ and
2. $g(C, \delta C)$ is $o(\|\delta n\|)$;

According to the fact (1), there is φ such that $\delta G(C)$ can be expressed as

$$\delta G(C) = \int_C \varphi(s)\delta n(s)ds. \tag{3.3}$$

Denote $\varphi(s)$ by $\frac{\partial G(C)}{\partial n}(s)$. Thus we have

$$\delta G(C) = \int_C \frac{\partial G(C)}{\partial n}(s)\delta n(s)ds. \tag{3.4}$$

Note. It is to note that the normal vector $\delta n(s)$ is taken to the outward direction from C, since the interior of C is tacitly understood to be the past in a sense so that δC is taken towards the future.

II. Random fields

Like as in the case of the non random function $G(C)$, the variation of $Y(C)$ is given by the following proposition.

Proposition 5.1 *The variation of $Y(C)$, expressed in the form (4.8) is*

$$\delta Y(C) = \int_C g(s)x(s)\delta n(s)ds, \tag{3.5}$$

where $g(s)$ is the restriction of g on C.

Let us define the functional of manifold $\Phi(C)$ as a linear function of R^d parameter white noise $x(u)$ as follows :

$$\Phi(C) = \int_{(C)} F(C,u)x(u)du,$$

where F is in $L^2(R^d)$ kernel.
Then, by using the S-transform, its variation is obtained as

$$\begin{aligned}\delta\Phi(C) &= \int_C F(C,s)x(s)\delta n(s)ds \\ &\quad + \int_{(C)}\int_C F'_n(C,u)(s)x(u)\delta_n(s)duds.\end{aligned}$$

4 Literatures on variational calculus

In 1900 Kneser published the book *Lehrbuch der Variationsrechnung,* (Braunschweig) which is the only modern text book at that time.

The other interesting literatures are

1. Bolza, *Lectures on the calculus of variations*, 1904, (Chicago, 1904 reprinted by Dover Publ.)

2. Hancock, *Lectures on the calculus of variations,* 1904, Cincinnati.

The mathematicians and their interesting literatures , contributed on variational calculus, are listed in the following.

L. Tonelli (1885-1946)

1923-1924 Tonelli *Fondamenti di Calcolo delle Variazioni,* 2 vols Bologna

R. Goursat (1858-1936)

1927 Goursat *Integral equations Calculus of Variations,* Cours de analyse vol 3.

R. Courant (1888-1972)

1931 Courant-Hilbert *Methods of Mathematical Physics Vol I.*

1934 Morse *The Calculus of Variations in the Large,* New York

Carathéodory (1873-1950)

1904 Dessertation, Gottingen

1935 *Calculus of variations and partial differential equations of the first order,* (in German) English translation Vol I, II (1965-67)

V. Volterra (1860-1940)

Volterra *Collected papers Vol 5.*

Fomin-Volterra *Calculus of variations, Princeton-Hall,* (English translation)(1961)

Main terms of Lagarange function is similar to the potential equation of electromagnetic field.

References

[1] J. Hadamard, On the functional operations , Comptes@Rendus@136, 351-354

[2] P. Lévy, On the variation of the distribution of electricity over a conductor, the surface of which is deformed Bull. Soc. Math. France 46 , 1918.

[3] P. Lévy, Cours de mécanique, 1928.

[4] P. Lévy, Problémes concrets d'analyse fonctonnelle, 1951.

[5] T. Hida, Lecture notes, 2000.