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A Study of the Relativistic Euler Equation

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1 Introduction

In this article we study the Cauchy problem to the one-dimensional relativistic Euler equation

$$\frac{\partial}{\partial t}\frac{\rho+Pu^2/c^4}{1-u^2/c^2}+\frac{\partial}{\partial x}\frac{(\rho+P/c^2)u}{1-u^2/c^2}=0,$$
$$\frac{\partial}{\partial t}\frac{(\rho+P/c^2)u}{1-u^2/c^2}+\frac{\partial}{\partial x}\frac{P+\rho u^2}{1-u^2/c^2}=0, \tag{1.1}$$

$$\rho|_{t=0}=\rho_0(x), \qquad u|_{t=0}=u_0(x). \tag{1.2}$$

Here c is a positive constant, the speed of light, and P is a given function of ρ. The equation (1.1) governs the one dimensional motion of a perfect gas in the Minkowski space-time. When $c\to\infty$, (1.1) tends to the usual Euler equation of gas dynamics

$$\rho_t+(\rho u)_x=0,$$
$$(\rho u)_t+(P+\rho u^2)_x=0. \tag{1.3}$$

Many mathematical investigations for this non-relativistic Euler equation were done. But the first mathematical investigation for the relativistic Euler equation (1.1) was done recently by Smoller and Temple [6]. They assume $P=\sigma^2\rho$, where σ is a positive constant $< c$. Under this assumption, they showed that if the initial data $\rho_0(x)$ and $u_0(x)$ satisfy

$$T.V.\log\rho_0<\infty, \qquad T.V.\log\frac{c+u_0}{c-u_0}<\infty,$$

then there exists a global weak solution to the Cauchy problem (1.1)(1.2). The result was obtained by Glimm's scheme and it is the relativistic version of Nishida's result [5] for the non-relativistic problem.

However we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

$$P = Kc^5 f(y), \qquad \rho = Kc^3 g(y)$$
$$f(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}}dq,$$
$$g(y) = 3\int_0^y q^2\sqrt{1+q^2}dq.$$

For this equation of state, we have $P \sim \frac{c^2}{3}\rho$ as $\rho \to \infty$ but $P \sim \frac{1}{5}K^{2/3}\rho^{5/3}$ as $\rho \to 0$. So we assume the following properties of the function $P(\rho)$:

(A):

$$P(\rho) > 0, \qquad 0 < dP/d\rho < c^2, \qquad 0 < d^2P/d\rho^2$$

for $\rho > 0$, and

$$P = A\rho^\gamma(1 + [\rho^{\gamma-1}/c^2]_1)$$

as $\rho \to 0$. Here A and γ are positive constants and

$$\gamma = 1 + \frac{2}{2N+1},$$

N being a positive integer, and $[X]_1$ denotes a convergent power series of the form $\sum_{k\geq 1} a_k X^k$.

The result which we want to generalize to the relativistic problem is those by G.-Q. Chen et al [2]. So we assume that the initial data $\rho_0(x)$, $u_0(x)$ satisfy

$$0 \leq \rho_0(x) \leq M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \leq M_0.$$

A weak solution of (1.1)(1.2) is defined as follows.

We write

$$\begin{aligned} E &= \frac{\rho + Pu^2/c^4}{1-u^2/c^2}, \\ F &= \frac{(\rho + P/c^2)u}{1-u^2/c^2}, \\ G &= \frac{P+\rho u^2}{1-u^2/c^2}, \\ U &= (E,F)^T, \qquad f(U) = (F,G)^T. \end{aligned}$$

Then (1.1) can be written as

$$U_t + f(U)_x = 0.$$

Let us denote by $U_0(x)$ the initial data. Then a weak solution $U(t,x)$ is a bounded measurable function which satisfies

$$\int\int (U\Phi_t + f(U)\Phi_x)dxdt + \int U_0(x)\Phi(0,x)dx = 0$$

for any test function $\Phi \in C_0^\infty([0,+\infty) \times R)$.

2 Riemann problems

The Riemann problem is the problem to the special initial data of the form

$$U_0(x) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases}$$

In order to slolve this we introduce the Riemann invariants

$$w = x + y, \qquad z = x - y$$

where

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho.$$

Then (1.1) is diagonarized as

$$w_t + \lambda_2 w_x = 0, \qquad z_t + \lambda_1 z_x = 0,$$

where

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}, \qquad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}.$$

the possible states $U = U_R$ connected to U_L on the right by rarefaction waves are

$$R_1: \qquad w = w_L, z > z_L$$

and

$$R_2: \qquad w > w_L, z = z_L.$$

The Rankine Hugoniot jump condition

$$\sigma[U] = [f(U)],$$

where $[U] = U_R - U_L, [f(U)] = f(U_R) - f(U_L)$, gives the shock curve

$$\frac{(u_R - u_L)^2}{(1 - u_R^2/c^2)(1 - u_L^2/c^2)} = \frac{(\rho_R - \rho_L)(P_R - P_L)}{(\rho_L + P_L/c^2)(\rho_R + P_R/c^2)}.$$

Along this curve we have shocks

$$S_1: \qquad \rho_L < \rho_R, u_R < u_L,$$

$$S_2: \qquad \rho_R < \rho_L, u_R < u_L.$$

The Riemann problem can be solved uniquely by using these rarefaction waves and shock waves and vaccuum state. The detailed discussion can be found in J. Chen [1].

If we look at a region of the form

$$\Sigma_B = \{(w,z) | -B \leq z \leq w \leq B\},$$

we have the following

Proposition 1 *If the initial data U_L, U_R belong to Σ_B for some large B, then the solution of the Riemann problem is confined to Σ_B.*

Moreover if we consider the image of Σ_B in the (E,F)-space, we have

Proposition 2 *The region Σ_B is convex in the (E,F)-plane.*

Proof. Let us consider the above hedge $F = F(E)$ which corresponds to $w = B, -B < z < B$. We have to show $d^2F/dE^2 < 0$. Along the hedge $w = B$, we have

$$u = c \tanh \frac{1}{c}(B - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho),$$

from which

$$\frac{du}{d\rho} = -(1 - u^2/c^2)\frac{\sqrt{P'}}{\rho + P/c^2}.$$

By a direct calculation we have

$$\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2} = \lambda_1.$$

Differentiating once more we have

$$\frac{d^2F}{dE^2} = -\frac{1 - u^2/c^2}{(1 - \sqrt{P'}u/c^2)^4}(\frac{P''}{2\sqrt{P'}} + (1 - \frac{P'}{c^2})\frac{\sqrt{P'}}{\rho + P/c^2}) < 0.$$

This was to be seen. QED.

From Proposition 2, we have

Proposition 3 *If $U(s), s \in [a,b]$, is confined to a region Σ_B, then the average*

$$\frac{1}{b-a}\int_a^b U(s)ds$$

belongs to Σ_B.

Let us look at the shock wave which connects the left state U_L to the right state U_R with the shock speed σ.

The right state U_R and σ are parametrized by $\rho = \rho_R$. Then we have the following fact, which will be used in Section 4.

Proposition 4 *Along*$S_1(\rho_L < \rho)$, *we have* $d\sigma/d\rho < 0$, *and along* $S_2(\rho < \rho_L)$ *we have* $d\sigma/d\rho > 0$.

Proof. Without loss of generality we can assume $u_L = 0$. Then $u = u_R$ is given by

$$u = -\sqrt{\frac{[\rho][P]}{(\rho_L + P/c^2)(\rho + P_L/c^2)}},$$

where $[\rho] = \rho - \rho_L, [P] = P - P_L$. We have

$$\sigma = \frac{[F]}{[E]} = \frac{(\rho + P/c^2)u}{\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2)}.$$

By a direct but tedious computations, we have

$$\begin{aligned}
\frac{d\sigma}{d\rho} &= \frac{(\rho + P/c^2)(\rho_L + P_L/c^2)[\rho]X}{2(\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2))^2 u(\rho_L + P/c^2)^2(\rho + P_L/c^2)^2},\\
X &= (\rho + P_L/c^2)(\rho + P/c^2)P'[\rho] + \\
&+ (\rho + P_L/c^2)(-(\rho + P_L/c^2) + [P]/c^2)[P] + \\
&- (\rho_L + P/c^2)[P]^2/c^2.
\end{aligned}$$

Since $P'' > 0$ we know $[P] \leq P'[\rho]$. Thus

$$\begin{aligned}
X &\geq (\rho + P_L/c^2)(\rho + P/c^2)[P] + \\
&+ (\rho + P_L/c^2)(-(\rho_L + P_L/c^2) + [P]/c^2)[P] + \\
&- (\rho_L + P/c^2)[P]^2/c^2 \\
&= [P]((\rho + P_L/c^2)([\rho] + [P]/c^2) + ([\rho] - [P]/c^2)[P]/c^2).
\end{aligned}$$

But

$$1 > \frac{[\rho] - [P]/c^2}{[\rho]} = 1 - P'(\rho_L + \theta(\rho - \rho_L))/c^2 > 0.$$

Using this, itis easy to see $X > 0$ both when $[\rho] > 0$ and when $[\rho] < 0$. Since $u < 0$, this completes the proof. QED.

3 Entropies

A pair of functions η and q is called an entropy- entropy flux if it satisfies the equation

$$D_U q = D_U \eta . D_U f. \tag{3.1}$$

Using the Riemann invariants, we can write (3.1) as

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

By eliminating q from the equation, we get the following second order equation:

$$\frac{\partial^2\eta}{\partial w\partial z}+Q(J\frac{\partial\eta}{\partial w}-\frac{1}{J}\frac{\partial\eta}{\partial z})=0, \tag{3.2}$$

where

$$\begin{aligned} Q &= \frac{1}{4\sqrt{P'}}(1-\frac{P'}{c^2}-\frac{\rho+P/c^2}{2P'}P''), \\ J &= \frac{1-\sqrt{P'}u/c^2}{1+\sqrt{P'}u/c^2}. \end{aligned}$$

Since this equation tends to the Euler-Poisson-Darboux equation

$$\frac{\partial^2\eta}{\partial w\partial z}+\frac{N}{w-z}(\frac{\partial\eta}{\partial w}-\frac{\partial\eta}{\partial z})=0 \tag{3.3}$$

as $c\to\infty$, we shall call (3.2) the relativistic Euler-Poisson-Darboux equation.

Among entropies of (3.3) when $c=\infty$ the kinetic energy

$$\eta=\frac{1}{2}\rho u^2+\frac{P}{\gamma-1} \tag{3.4}$$

plays an important role. Therefore we want to find an entropy of (3.2) which tends to (3.4) as $c\to\infty$. Let us look for an entropy-entropy flux of the form

$$\eta=H(\rho,u^2),\qquad q=Q(\rho,u^2)u.$$

Inserting this to the equation it is easy to find an entropy-entropy flux

$$\begin{aligned} \eta^* &= -\frac{\Psi(\rho)}{(1-u^2/c^2)^{1/2}}+c^2(\frac{\rho+Pu^2/c^4}{1-u^2/c^2}), && (3.5) \\ q^* &= (-\frac{\Psi(\rho)}{(1-u^2/c^2)^{1/2}}+c^2\frac{\rho+P/c^2}{1-u^2/c^2})u, && (3.6) \\ \Psi &= exp(\int_1^\rho\frac{d\rho}{\rho+P/c^2}+K_0), && (3.7) \end{aligned}$$

where K_0 is determined so that η^* tends to the kinetic energy (3.4) as $c=\infty$. We call the entropy η^* defined by (3.5) the relativistic standard entropy. The important fact is

Proposition 5 *The Hessian $D^2_U\eta^*$ is positive definite. For any fixed B there is a positive constant k such that*

$$(\xi|D^2_U\eta^*(U)\xi)\geq k|\xi|^2,$$

for any $U\in\Sigma_B$ and $\xi=(\xi_0,\xi_1)$ with $|\xi|^2=\xi_0^2+\xi_1^2$.

Proof. The proof is due to direct but tedious calculations. We note

$$\begin{aligned}
\frac{\partial \rho}{\partial E} &= \frac{1+u^2/c^2}{1-P'u^2/c^4}, \\
\frac{\partial u}{\partial E} &= -\frac{(1+P'/c^2)(1-u^2/c^2)u}{(\rho+P/c^2)(1-P'u^2/c^4)}, \\
\frac{\partial \rho}{\partial F} &= -\frac{2u/c^2}{1-P'u^2/c^4}, \\
\frac{\partial u}{\partial F} &= \frac{(1-u^2/c^2)(1+P'u^2/c^4)}{(\rho+P/c^2)(1-P'u^2/c4)}.
\end{aligned}$$

Using these, we have

$$\begin{aligned}
\frac{\partial \eta^*}{\partial E} &= -\frac{\Psi}{(\rho+P/c^2)(1-u^2/c^2)^{1/2}}+c^2, \\
\frac{\partial \eta^*}{\partial F} &= \frac{\Psi u/c^2}{(\rho+P/c^2)(1-u^2/c^2)^{1/2}}, \\
\frac{\partial^2 \eta^*}{\partial E^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(P'+2P'u^2/c^2+u^2), \\
\frac{\partial^2 \eta^*}{\partial E\partial F} &= \frac{-\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(2P'/c^2+1+P'u^2/c^4)u, \\
\frac{\partial^2 \eta^*}{\partial F^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(1+3P'u^2/c^4).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
(\xi|D_U^2\eta^*\xi) &= \eta^*_{EE}\xi_0^2+2\eta^*_{EF}\xi_0\xi_1+\eta^*_{FF}\xi_1^2 \\
&= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}Z, \\
Z &= (P'+2P'u^2/c^2+u^2)\xi_0^2-2(2P'/c^2+1+P'u^2/c^4)u\xi_0\xi_1+ \\
&+ \ (1+3P'u^2/c^4)\xi_1^2 \\
&\geq \frac{2P'(1-u^2/c^2)^2(1-P'u^2/c^4)}{A+C+\sqrt{(A-C)^2+4B^2}}(\xi_0^2+\xi_1^2), \\
A &= P'+2P'u^2/c^2+u^2, \\
B &= (2P'/c^2+1+P'u^2/c^4)u, \\
C &= 1+3P'u^2/c^4.
\end{aligned}$$

This completes the proof. QED.

4 Construction of approximate solutions

Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.

Suppose that the initial data $U_0(x)$ is confined to an invariant region Σ_B. Put $\Lambda_0 = \sup\{|\lambda_j(U)||j = 1, 2, U \in \Sigma_B\}$. Fixing $\Lambda_1 > \Lambda_0$, we take mesh lengths $\Delta x, \Delta t$ such that $\Delta x = \Lambda_1 \Delta t$. We denote $\Delta = \Delta x$.

Let us construct the approxomate solution $U^\Delta(t, x)$. First we put

$$U_0^\Delta(x) = U_0(x)\chi_{[-1/\Delta, 1/\Delta]}.$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta x}\int_{2j\Delta x}^{(2j+2)\Delta x} U_0^\Delta(x)dx$$

for $2j\Delta x < x \le (2j+2)\Delta x$. Solving the Riemann problem on each interval $[2(j-1)\Delta, 2(j+1)\Delta]$, we define $U^\Delta(t, x)$ for $0 \le t < \Delta t$. Since the Courant-Friedrichs-Levi condition is satisfied, the wave from the center $2j\Delta$ does not intersect. If $U^\Delta(t, x)$ for $0 \le t < n\Delta t$ has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t - 0, x)dx$$

for $2j\Delta < x \le (2j+2)\Delta$. Solving the Riemann problem, we define $U^\Delta(t, x)$ for $n\Delta t \le t < (n+1)\Delta t$.

By Proposition 1 and 3, it is inductively guaranteed that U^Δ remains in Σ_B, say,

Proposition 6 *The approximate solution $U^\Delta(t, x)$ satisfies $U^\Delta(t, x) \in \Sigma_B$, therefore,*

$$0 \le \rho^\Delta(t, x) \le M, \qquad |\frac{c}{2}\log\frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)}| \le M.$$

Moreover we shall prove

Proposition 7 *For any test function Φ it holds that*

$$\int\int(\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0, x)U_0^\Delta(x)dx = O(\Delta^{1/2}).$$

In order to prove Proposition 7, we prepare

Proposition 8 *For any shock wave from U_L to U_R with the shock speed σ and for any convex entropy η, we have*

$$\sigma[\eta] - [q] \ge 0,$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$.

Proof. The right state of shocks can be parametrized by $\rho = \rho_R$. Putting

$$Q(\rho) = \sigma[\eta] - [q],$$

we shall see $dQ/d\rho \geq 0$ along $S_1 : [\rho] > 0$ and $dQ/d\rho \leq 0$ along $S_2 : [\rho] < 0$. Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

$$\begin{aligned} \frac{dQ}{d\rho} &= \frac{d\sigma}{d\rho}([\eta] - D_U\eta(U).[U]) \\ &= -\frac{d\sigma}{d\rho}\int_0^1 \theta(U - U_L|D_U^2\eta(U_L + \theta(U - U_L).(U - U_L))d\theta. \end{aligned}$$

We supposed $D_U^2\eta \geq 0$. By Proposition 4, we know $d\sigma/d\rho < 0$ on S_1 and $d\sigma/d\rho > 0$ on S_2. QED.

Proof of Proposition 7.

We fix T to consider U^Δ on $0 \leq t \leq T$. First we shall show

$$\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, (2j+1)\Delta)|^2 dx \leq C. \tag{4.1}$$

Let us consider the standard entropy η^*. Then we have

$$\begin{aligned} 0 &= \int \eta^*(U(T,x))dx - \int \eta^*(U(0,x))dx + L + \Sigma, \\ L &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} (\eta^*(U(n\Delta t - 0, x)) - \eta^*(U(n\Delta t + 0, (2j+1)\Delta)))dx, \\ \Sigma &= \int_0^T \sum_{shocks} (\sigma[\eta^*] - [q^*])dt. \end{aligned}$$

We write $U_0 = U(n\Delta t + 0, (2j+1)\Delta), U_1 = U(n\Delta t - 0, x)$. Since

$$U_0 = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,$$

we see

$$\begin{aligned} L &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}\int_0^1 (1-\theta)(U_1 - U_0|D_U^2\eta^*(U_0 + \theta(U_1 - U_0)).(U_1 - U_0))d\theta dx \\ &\geq 0. \end{aligned}$$

On the other hand we have $\Sigma \geq 0$ from Proposition 8. Thus $L \leq C, \Sigma \leq C$. But from Proposition 5, we have $D_U^2\eta^* \geq k$. Therefore

$$C \geq L \geq \frac{k}{2}\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} |U_1 - U_0|^2 dx.$$

Thus we get (4.1).

Now let us consider a test function Φ. Put

$$J = \int\int (\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0,x) U_0^\Delta dx.$$

Since U^Δ is a weak solution on each time strip $n\Delta t < t < (n+1)\Delta t$, we have

$$\begin{aligned} J &= \sum_n \int \Phi(n\Delta t, x)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx \\ &= J_1 + J_2, \\ J_1 &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, j\Delta)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx, \\ J_2 &= \sum_{j.n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t,x) - \Phi(n\Delta t, j\Delta))(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx. \end{aligned}$$

Since

$$U(n\Delta t + 0, x) = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t - 0, x)dx$$

for $2j\Delta < x < (2j+2)\Delta$, we see $J_1 = 0$. It follows from (4.1) that

$$\begin{aligned} |J_2| &\leq C\Delta^{1/2}||\Phi||_{C^1}(\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, x)|^2 dx)^{1/2} \\ &\leq C'\Delta^{1/2}. \end{aligned}$$

Here we have used $T/\Delta t = O(1/\Delta)$. QED.

Summing up, we have the following theorem.

Theorem 1 *The approximate solution $U^\Delta(t,x)$ satisfies*

$$0 \leq \rho^\Delta(t,x) \leq M, \qquad |\frac{c}{2}\log\frac{c + u^\Delta(t,x)}{c - u^\Delta(t,x)}| \leq M$$

and

$$\int\int (\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0,x) U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function Φ.

We expect that U^Δ tends to a weak solution everywhere. For the non-relativistic gas dynamics, this was done by DiPerna [3] and G.Q.Chen et al [2]. In their proof the Darboux formula

$$\eta = \int_z^w ((w-s)(s-z))^N \phi(s)ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3) , ϕ being arbitrary, plays an important role. Section 6 will be devoted to find such an integral formula for the relativistic Euler-Poisson-Darboux equation (3.2).

5 Remark

We note that

$$\begin{aligned}
\lambda_2 - \lambda_1 &= \frac{\sqrt{P'}(1-u^2/c^2)}{1-u^2P'/c^4} > 0, \\
\frac{\partial \lambda_1}{\partial z} &= \frac{1-u^2/c^2}{2(1-\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) > 0, \\
\frac{\partial \lambda_2}{\partial w} &= \frac{1-u^2/c^2}{2(1+\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) > 0
\end{aligned}$$

for $\rho > 0$ and $|u| < c$.

This says that the system is strictly hyperbolic and genuinely nonlinear on $\rho > 0$. Therefore the Glimm's theory can be applied if

$$||U_0(x) - U^*||_{L^\infty} + T.V.U_0$$

is sufficiently small, where U^* is a constant state such that $\rho^* > 0, |u^*| < c$. But the vaccum may not be covered by this application of the general theorem.

6 Generalized Darboux formula

In this section we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. Let us introduce the variables

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho.$$

Then the relativistic Euler-Poisson-Darboux equation is

$$(EPD) \qquad \eta_{xx} - \eta_{yy} + A(x,y)\eta_y + B(x,y)\eta_x = 0,$$

where

$$\begin{aligned}
A(x,y) &= \frac{1}{\sqrt{P'}}(1-\frac{P'}{c^2}-\frac{\rho+P/c^2}{2P'}P'')\frac{1+P'u^2/c^4}{1-P'u^2/c^4}, \\
B(x,y) &= -\frac{2u/c^2}{1-P'u^2/c^4}(1-\frac{P'}{c^2}-\frac{\rho+P/c^2}{2P'}P'').
\end{aligned}$$

The coefficients A and B are of the form

$$\begin{aligned}
A &= \frac{2N}{y} + a, \qquad a = \frac{y}{c^2}(a_0 + [x^2/c^2, y^2/c^2]_1), \\
B &= -\frac{4N}{N+1}\frac{x}{c^2}(1 + [x^2/c^2, y^2/c^2]_1),
\end{aligned}$$

where $[X,Y]_1$ denotes a convergent power series $\sum_{j+k\geq 1} c_{jk}X^jY^k$. In order to remove the singularity in A, we use the trick of Weinstein [7]. We introduce the sequence of variables $\eta_j, j = 0, 1, , ..., N$ by

$$\frac{\partial \eta_j}{\partial y} = y\eta_{j+1},$$

or

$$\eta_j(x,y) = I\eta_{j+1}(x,y) = \int_0^y Y\eta_{j+1}(x,Y)dY,$$

where $\eta_0 = \eta$. The sequence of formal integro-differential operators L_j is defined by

$$\begin{aligned} L_jV &= V_{xx} - V_{yy} + (\frac{2(N-j)}{y} + a)V_y + BV_x + \\ &+ j\tilde{a}V + \sum_{k=1}^{j} F_{jk}I^kV_x + \sum_{k=1}^{j} H_{jk}I^kV, \end{aligned}$$

where

$$\tilde{a} = \frac{\partial a}{\partial y} + \frac{a}{y} = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0.$$

The coefficients F_{jk} and H_{jk} are determined inductively by

$$F_{j+1,k} = \begin{cases} F_{j1} + \frac{1}{y}\frac{\partial B}{\partial y} & \text{if } k = 1 \\ F_{jk} + \frac{1}{y}\frac{\partial}{\partial y}F_{j,k-1} & \text{if } k \geq 2 \end{cases}$$

$$H_{j+1,k} = \begin{cases} H_{j1} + j\frac{1}{y}\frac{\partial \tilde{a}}{\partial y} & \text{if } k = 1 \\ H_{jk} + \frac{1}{y}\frac{\partial}{\partial y}H_{j.k-1} & \text{if } k \geq 2 \end{cases}$$

It is easy to see that F_{jk} are of the form $\frac{x}{c^2}[x^2/c^2, y^2/c^2]_0$ and H_{jk} are of the form $\frac{1}{c^2}[x^2/c^2, y^2/c^2]_0$. By the definition we have formally

$$\frac{1}{y}\frac{\partial}{\partial y}(L_j\eta_j) = L_{j+1}\eta_{j+1}.$$

Now we consider the equation $L_NV = 0$ for $V = \eta_N$ with the initial conditions

$$V = 0, \qquad V_y = 2^{N+1}N!\phi(x), \qquad \text{at } y = 0.$$

The problem is

(Q)
$$\begin{aligned} V_{yy} - V_{xx} &= aV_y + BV_x + N\tilde{a}V + \\ &+ \sum_{k=1}^{N} F_kI^kV_x + \sum_{k=1}^{N} H_kI^kV, \end{aligned}$$

$$V = 0, \qquad V_y = 2^{N+1}N!\phi(x) \qquad \text{at } y = 0.$$

Proposition 9 *If $\phi \in C^1(R)$, then the problem (Q) admits a unique solution V in $C^2(R \times [0,\infty))$.*

Proof. Let us denote by $H(x,y,V)$ the right hand side of the equation $L_N = 0$. Then (Q) is transformed to the integral equation

$$V(x,y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2}\int_0^y \int_{x-y+Y}^{x+y-Y} H(X,Y,V)dXdY.$$

We can solve this integral equation by the iteration

$$\begin{aligned} V_0(x,y) &= 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi, \\ V^{n+1}(x,y) &= 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2}\int_0^y \int_{x-y+Y}^{x+y-Y} H(X,Y,V^n)dXdY. \end{aligned}$$

Fixing L arbitrarily, we consider $|x| \leq L$. Then it is easy to get the estimates

$$|V^{n+1}(x,y) - V^n(x,y)| \leq \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$

Therefore V^n tends to a limit V uniformly on $|x| \leq L, 0 \leq y \leq L$. The limit is the unique solution of (Q). QED.

Now we put

$$\eta_N = V, \qquad \eta_{N-k} = I\eta_{N-k+1}.$$

Since η_{N-k} and its derivatives of order ≤ 2 all vanish on $y = 0$ for $k \geq 1$, we see $\eta = \eta_0$ gives a solution of the relativistic Euler-Poisson-Darboux equation (EPD).

Next we give an integral formula for the solution V of (Q).

Proposition 10 *There is a C^{N+2}-function $G(x,y,\xi)$ of $|x| < \infty, y \geq 0, x-y \leq \xi \leq x+y$ such that the solution V of (Q) satisfies*

$$V(x,y) = \int_{x-y}^{x+y} G(x,y,\xi)\phi(\xi)d\xi. \qquad (6.1)$$

Moreover

$$\begin{aligned} G &= 2^N N! + O(y/c^2), \\ \partial_x^{p_1}\partial_\xi^{p_2}\partial_y^{p_3} G &= O(1/c^2) \qquad \textit{for } 1 \leq p_1 + p_2 + p_3 \leq N+2 \end{aligned}$$

Proof. We consider the approximate solution $V^n(x,y)$ which appeared in the iteration of the proof of Proposition 9. By writing H as

$$H = (aV)_y + (BV)_x + bV + \sum (F_k I^k V)_x + \sum \tilde{H}_k I^k V,$$

where

$$\begin{aligned} b &= N\tilde{a} - a_y - B_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \\ \tilde{H}_k &= H_k - (F_k)_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \end{aligned}$$

it is easy to see inductively that there is a kernel $G^n(x,y,\xi)$ such that

$$V^n(x,y) = \int_{x-y}^{x+y} G^n(x,y,\xi)\phi(\xi)d\xi.$$

In fact $G^0 = 2$ and G^n are determined inductively by the formula

$$\begin{aligned} G^{n+1} &= 2 + \frac{1}{2}(G_I^n + G_{II}^n + G_{III}^n + \sum G_{IVk}^n + \sum G_{Vk}^n), \\ G_I &= \int_{(-x+y+\xi)/2}^{y} a(x-y+Y,Y)G(x-y+Y,Y,\xi)dY + \\ &+ \int_{(x+y-\xi)/2}^{y} a(x+y-Y,Y)G(x+y-Y,Y,\xi)dY, \\ G_{II} &= \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)G(x+y-Y,Y,\xi)dY + \\ &- \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)G(x-y+Y,Y,\xi)dY, \\ G_{III} &= \int\int_{D(x,y,\xi)} b(X,Y)G(X,Y,\xi)dXdY, \end{aligned}$$

where

$$\begin{aligned} D(x,y,\xi) &= \{(X,Y) | X - Y \le \xi \le X + Y, x - y + Y \le X \le x + y - Y, 0 \le Y \le y\}, \\ G_{IVk} &= \int_{(x-y+\xi)/2}^{y} F_k(x+y-Y,Y)J^k G(x+y-Y,Y,\xi)dY + \\ &- \int_{(-x+y+\xi)/2}^{y} F_k(x-y+Y,Y)J^k G(x-y+Y,Y,\xi)dY, \end{aligned}$$

where

$$JG(x,y,\xi) = \int_{|x-\xi|}^{y} YG(x,Y,\xi)dY,$$

and

$$G_{Vk} = \int\int_{D(x,y,\xi)} \tilde{H}_k(X,Y)J^k G(X,Y,\xi)dXdY.$$

It is easy to see inductively that

$$|G^{n+1}(x,y,\xi) - G^n(x,y,\xi)| \le \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$

therefore G^n converges to a limit G uniformly and (6.1) holds. Moreover we can differentiate G^{n+1} supposing that G^n is differentiable. In fact we have

$$
\begin{aligned}
G_{I,x} &= \frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) \\
&- \frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+ \\
&+ \int_{(-x+y+\xi)/2}^{y}(aG)_x(x-y+Y,Y,\xi)dY \\
&+ \int_{(x+y-\xi)/2}^{y}(aG)_x(x-Y+Y,Y,\xi)dY, \\
G_{I,\xi} &= -\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+ \\
&+ \frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+ \\
&+ \int_{(-x+y+\xi)/2}^{y}aG_\xi(x-y+Y,Y,\xi)dY+ \\
&+ \int_{(-x+y+\xi)/2}^{y}aG_\xi(x+y-Y,Y,\xi)dY, \\
G_{I,y} &= -\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+ \\
&- \frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+ \\
&+ 2aG(x,y,\xi)+ \\
&- \int_{(-x+y+\xi)/2}^{y}(aG)_x(x-y+Y,Y,\xi)dY+ \\
&+ \int_{(-x+y+\xi)/2}^{y}(aG)_x(x+y-Y,Y,\xi)dY; \\
G_{II,x} &= -\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+ \\
&- \frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+ \\
&+ \int_{(x+y-\xi)/2}^{y}(BG)_x(x+y-Y,Y,\xi)dY+ \\
&- \int_{(-x+y+\xi)/2}^{y}(BG)_x(x-y+Y,Y,\xi)dY, \\
G_{II,\xi} &= \frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+ \\
&+ \frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+ \\
&+ \int_{(x+y-\xi)/2}^{y}BG_\xi(x+y-Y,Y,\xi)dY+ \\
&- \int_{(-x+y+\xi)/2}^{y}BG_\xi(x-y+Y,Y,\xi)dY,
\end{aligned}
$$

$$
\begin{aligned}
G_{II,y} &= -\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ \frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&+ \int_{(x+y-\xi)/2}^{y}(BG)_x(x+y-Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}(BG)_x(x-y+Y,Y,\xi)dY;\\
G_{III,x} &= \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY-\int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY,\\
G_{III,\xi} &= \int_{0}^{(x+y-\xi)/2} bG(\xi+Y,Y,\xi)dY+\int_{0}^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY+\\
&+ \int\int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY,\\
G_{III,y} &= \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY+\int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY;
\end{aligned}
$$

and the derivatives of G_{IVk} are similar to G_{II} and the derivatives of G_{IVk} are similar to G_{III}. Then it is easy to see inductively that

$$|G_x^{n+1}-G_x^n|+|G_\xi^{n+1}-G_\xi^n|+|G_y^{n+1}-G_y^n|\le\frac{M^ny^n}{n!}.$$

Thus the limit G is differentiable. In a smilar manner we see

$$
\begin{aligned}
|G_{xx}^{n+1}-G_{xx}^n| &+ |G_{x\xi}^{n+1}-G_{x\xi}^n|+|G_{xy}^{n+1}-G_{xy}^n|+\\
&+ |G_{\xi\xi}^{n+1}-G_{\xi\xi}^n|+|G_{\xi y}^{n+1}-G_{\xi y}^n|+|G_{yy}^{n+1}-G_{yy}^n|\le\\
&\le \frac{M^{n-1}y^{n-1}}{(n-1)!}.
\end{aligned}
$$

Thus G is twice continuously differentiable. In a similar manner we see that G is $N+2$-times continuously differentiable. The rough estimates stated in the propositions is obvious since the coefficients are all of $O(1/c^2)$. QED.

The solution η_{N-k} enjoyes an integral representation

$$\eta_{N-k}=\int_{x-y}^{x+y}K_{N-k}(x,y,\xi)\phi(\xi)d\xi,$$

where

$$K_{N-k}(x,y,\xi)=JK_{N-k+1}(x,y,\xi)=J^kG(x,y,\xi).$$

So the solution η of the relativistic Euler-Poisson-Darboux equation is given by

$$\eta(x,y)=\int_{x-y}^{x+y}K(x,y,\xi)\phi(\xi)d\xi,$$

where

$$K(x,y,\xi) = J^N G(x,y,\xi).$$

By induction we see

$$J^k G(x,y,\xi) = \frac{2^N N!}{2^k k!}(y^2 - (x-\xi)^2)^k (1 + O(y/c^2)).$$

Thus we have

Proposition 11 *There is a kernal $K(x,y,\xi)$ which is of C^{N+2}-class in $|x| < \infty, 0 \leq y, x-y \leq \xi \leq x+y$ such that*

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi$$

gives a solution of the relativistic Euler- Poisson-Darboux equation for any smooth ϕ. Moreover

$$K(x,y,\xi) = (y^2 - (x-\xi)^2)^N (1 + O(y/c^2)).$$

But in order to apply this integration formula, the generalized Darboux formula, to the study of the relativistic Euler equation, more detailed estimates of the remainder are necessary.

Proposition 12 *We have*

$$G_y = O(y/c^2).$$

Proof. Since $a = O(y/c^2)$, it is clear that $G_{I,y} = O(y/c^2)$. Next we see

$$G_{II,y} = -B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2)) + O(y/c^2).$$

On the other hand we can write

$$B = \frac{1}{c^2}B_0(x) + O(y^2/c^2)$$

and

$$\frac{x+y+\xi}{2} = x + \frac{y+Z}{2}, \qquad \frac{x-y+\xi}{2} = x + \frac{-y+Z}{2}, \qquad Z = \xi - x.$$

Therefore we see $G_{II,y} = O(y/c^2)$. It is clear that $G_{III,y} = O(y/c^2)$ and $G_{IVk,y}, G_{Vk,y} = O(y^2/c^2)$. QED.

Proposition 13 *We have*

$$G = 2^N N! + \frac{1}{c^2}C_0(x,c)(\xi - x) + O(y^2/c^2),$$

where $C_0(x,c)$ is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.$$

Proof. It is clear that $G_I = O(y^2/c^2)$ since $a = O(y/c^2)$. Next we see

$$G_{II} = 2^N N! \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - 2^N N1 \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY + O(y^2/c^2),$$

since $G = 2^N N! + O(y/c^2)$. If we write

$$B = \frac{1}{c^2}B_0(x) + O(y^2/c^2), \qquad Z = \xi - x$$

, then we see

$$\begin{aligned}\int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY \quad &- \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY = \\ &= \frac{1}{c^2}\Big(\int_{x}^{x+\frac{y+Z}{2}} B_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x} B_0(s)ds\Big) + O(y^2/c^2) \\ &= \frac{1}{c^2}B_0(x)Z + O(y^2/c^2).\end{aligned}$$

Note $|Z| \leq y$. It is clear that $G_{III}, G_{IVk}, G_{Vk} = O(y^2/c^2)$. QED.

Proposition 14 *We have*

$$G_x + G_\xi = O(y/c^2).$$

Proof. First we see

$$\begin{aligned} G_{I,x} + G_{I,\xi} &= \int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_\xi)(x-y+Y,Y,\xi)dY + \\ &+ \int_{(x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x+y-Y,Y,\xi)dY \\ &= O(y^2/c^2), \end{aligned}$$

since $a, a_x = O(y/c^2)$. Next we see

$$\begin{aligned} G_{II,x} + G_{II,\xi} &= \int_{(x+y-\xi)/2}^{y} ((BG)_x) + BG_\xi)(x+y-Y,Y,\xi)dY + \\ &- \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY \\ &= O(y/c^2). \end{aligned}$$

It is clear that $G_{III,x}, G_{III,\xi}, G_{Vk,x}, G_{Vk,\xi} = O(y/c^2)$. $G_{IVk,x} + G_{IVk,\xi}$ is estimated in a similar manner as $G_{II,x} + G_{II,\xi}$. QED.

Proposition 15 *We have*

$$(G_x + G_\xi)_y = O(y/c^2).$$

Proof. First we see

$$\begin{aligned}(G_{I,x}+G_{I,\xi})_y &= 2((aG)_x+aG_\xi)(x,y,\xi)+\\ &- \frac{1}{2}((aG)_x+aG_\xi)((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\ &- \frac{1}{2}((aG)_x+aG_\xi)((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\ &- \int_{(-x+y+\xi)/2}^{y}((aG)_x+aG_\xi)_x(x-y+Y,Y,\xi)dY+\\ &+ \int_{((x+y-\xi)/2}^{y}((aG)_x+aG_\xi)_x(x+y-Y,Y,\xi)dY\\ &= O(y/c^2),\end{aligned}$$

since $a, a_x = O(y/c^2)$. Next we see

$$\begin{aligned}(G_{II,x}+G_{II,\xi})_y &= -\frac{1}{2}((BG)_x+BG_\xi)((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\ &+ \frac{1}{2}((BG)_x+BG_\xi)((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\ &+ \int_{(x+y-\xi)/2}^{y}((BG)_x+BG_\xi)_x(x+y-Y,Y,\xi)dY+\\ &+ \int_{(-x+y+\xi)/2}^{y}((BG)_x+BG_\xi)_x(x-y+Y,Y,\xi)dY\\ &= 2^{N-1}N!B_x((x-y+\xi)/2,(-x+y+\xi)/2)\\ &- 2^{N-1}N!B_x((x+y+\xi)/2,(x+y-\xi)/2)+\\ &+ O(y/c^2),\end{aligned}$$

since $G = 2^N N! + O(y/c^2)$ and $G_x + G_\xi = O(y/c^2)$. But

$$B_x = \frac{1}{c^2}B_0'(x) + O(y^2/c^2)$$

and

$$\begin{aligned}B_x((x-y+\xi)/2,(-x+y+\xi)/2) &- B_x((x+y+\xi)/2,(x+y-\xi)/2) =\\ &= \frac{1}{c^2}B_0''(x)(-y)+O(y^2/c^2)\\ &= O(y/c^2).\end{aligned}$$

It is clear that

$$\begin{aligned}(G_{III,x}+G_{III,\xi})_y &= \int_{(x+y-\xi)/2}^{y}((bG)_x+bG_\xi)(x+y-Y,Y,\xi)dY+\\ &+ \int_{(-x+y+\xi)/2}^{y}((bG)_x+bG_\xi)(x-y+Y,Y,\xi)dY\\ &= O(y/c^2).\end{aligned}$$

Similarly we can estimate $(G_{IVk,x}+G_{IVk,\xi})_y, (G_{Vk,x}+G_{Vk,\xi})_y$ bearing in mind that $(JG)_x+(JG)_\xi = J(G_x+G_\xi)$. QED.

Proposition 16 *We have*

$$G_x+G_\xi = \frac{1}{c^2}C_1(x,c)(\xi-x)+O(y^2/c^2),$$

where $C_1(x,c)$ *is a function of the form*

$$[x^2/c^2]_0+\frac{x}{c^2}[x^2/c^2]_0.$$

Proof. We already observed that $G_{Ix}+G_{I\xi}=O(y^2/c^2)$. Next we look at

$$\begin{aligned}
G_{II,x} \quad + \quad G_{II,\xi} &= \int_{(x+y-\xi)/2}^{y}((BG)_x+BG_\xi)(x+y-Y,Y,\xi)dY + \\
&- \int_{(-x+y+\xi)/2}^{y}((BG)_x+BG_\xi)(x-y+Y,Y,\xi)dY \\
&= 2^N N!\int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - 2^N N!\int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY + \\
&+ O(y^2/c^2),
\end{aligned}$$

since $G=2+O(y/c^2)$ and $G_x+G_\xi = O(y/c^2)$. Bearing in mind that $B_y=O(y/c^2)$, we see

$$\begin{aligned}
\int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY \quad &- \quad \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY = \\
=-\int_{(x+y-\xi)/2}^{y}(-B_x+B_y)(x+y-Y,Y)dY \quad &- \quad \int_{(-x+y+\xi)/2}^{y}(B_x+B_y)(x-y+Y,Y)dY \\
&+ \quad O(y^2/c^2) \\
=-2B(x,y) \quad &+ \quad B((x+y+\xi)/2,(x+y-\xi)/2)+ \\
+B((x-y+\xi)/2,(-x+y+\xi)/2) \quad &+ \quad O(y^2/c^2) \\
=\frac{1}{c^2}(-2B_0(x)+B_0(x+\frac{y+Z}{2}) \quad &+ \quad B_0(x+\frac{-y+Z}{2}))+O(y^2/c^2) \\
&= \quad \frac{1}{c^2}B_0'(x)Z+O(y^2/c^2).
\end{aligned}$$

Next we look at

$$\begin{aligned}
G_{III,x}+G_{III,\xi} \quad &= \quad \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY + \\
&+ \quad \int_0^{(x+y-\xi)/2} bG(\xi+Y,Y,\xi)dY - \int_0^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY + \\
&+ \quad \int\int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY.
\end{aligned}$$

Putting

$$b(x,y) = \frac{1}{c^2}b_0(x) + O(y^2/c^2),$$

we see

$$\begin{aligned} G_{III,x} + G_{III,\xi} &= 2^N N!\Big(\int_x^{x+\frac{y+Z}{2}} b_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x} b_0(s)ds + \\ &+ \int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s)ds\Big) + O(y^2/c^2) \\ &= \frac{2^N N!}{c^2} b_0(x)\Big(\frac{y+Z}{2} - \frac{y-Z}{2} + \frac{y-Z}{2} - \frac{y+Z}{2}\Big) + O(y^2/c^2) \\ &= O(y^2/c^2). \end{aligned}$$

$G_{IVk,x} + G_{IVk,\xi}$ can be estimated in a similer manner as $G_{II,x} + G_{II,\xi}$. Finally $G_{Vk,x}, G_{Vk,\xi} = O(y^3/c^2)$ since $J^k G = O(y^2/c^2)$ for $k \geq 1$. QED.

Proposition 17 *We have*

$$(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2).$$

Proof. First we see

$$\begin{aligned} (G_{I,x} + G_{I,\xi})_x &+ (G_{I,x} + G_{I,\xi})_\xi = \\ &= \int_{(-x+y+\xi)/2}^{y} ((aG)_{xx} + 2(aG_\xi)_x + aG_{\xi\xi})(x-y+Y,Y,\xi)dY + \\ &+ \int_{(x+y-\xi)/2}^{y} ((aG)_{xx} + 2(aG_\xi)_x + aG_{\xi\xi})(x+y-Y,Y,\xi)dY \\ &= O(y^2/c^2), \end{aligned}$$

since $a, a_x, a_{xx} = O(y/c^2)$. Next

$$\begin{aligned} (G_{II,x} &+ G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi = \\ &= \int_{(x+y-\xi)/2}^{y} (((BG)_x + BG_\xi)_x + ((BG)_x + BG_\xi)_\xi)(x+y-Y,Y,\xi)dY + \\ &+ \int_{(-x+y+\xi)/2}^{y} (((BG)_x + BG_\xi)_x + ((BG)_x + BG_\xi)_\xi)(x+y-Y,Y,\xi)dY \\ &= O(y/c^2). \end{aligned}$$

It is easy to see

$$(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi = O(y/c^2).$$

The estimates of G_{IVk} and G_{Vk} can be seen similarly. QED.

Proposition 18 *We have*

$$(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = \frac{1}{c^2} C_2(x,c)(\xi - x) + O(y^2/c^2),$$

where $C_2(x,c)$ is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.$$

Proof. We already observed that

$$(G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi = O(y^2/c^2).$$

Next, bearing in mind that $G_x + G_\xi = O(y/c^2)$ and $(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2)$, we see

$$\begin{aligned}
(G_{II,x} &+ G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi = \\
&= \int_{(x+y-\xi)/2}^{y} (B_{xx}G + 2B_x(G_x + G_\xi) + \\
&+ B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x + y - Y, Y, \xi)dY + \\
&- \int_{(-x+y+\xi)/2}^{y} (B_{xx}G + 2B_x(G_x + G_\xi) + \\
&+ B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi)(x - y + Y, Y, \xi)dY \\
&= 2^N N! \int_{(x+y-\xi)/2}^{y} B_{xx}(x + y - Y, Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B_{xx}(x - y + Y, Y)dY + \\
&+ O(y^2/c^2).
\end{aligned}$$

The same discussion to that of the proof of Proposition 16 can be applied by replacing B by B_x. Let us look at $(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi$. Note that

$$\begin{aligned}
(bG)_x + bG_\xi &= b_x G + b(G_x + G_\xi) \\
&= 2^N N! b_x + O(y/c^2), \\
bG &= 2^N N! b + O(y/c^2).
\end{aligned}$$

Applying the discussion of the proof of Proposition 16 by replacing b by b_x, we see

$$\begin{aligned}
(G_{III,x} &+ G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi = \\
&= 2^N N! (\int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s)ds) + \\
&+ O(y^2/c^2) \\
&= -2^N N! b_0(x) Z + O(y^2/c^2).
\end{aligned}$$

The estimates of G_{IVk}, G_{Vk} are paralell. QED.

Proposition 19 *We have*

$$G_\xi = \frac{1}{c^2} C_3(x,c) + O(y/c^2).$$

Proof. It is sufficient to note that

$$\begin{aligned} G_{II,\xi} &= 2^{N-1}N!(B((x+y+\xi)/2,(x+y-\xi)/2) + B((x-y+\xi)/2,(-x+y+\xi)/2)) + \\ &+ O(y/c^2) \\ &= \frac{2^{N-1}N!}{c^2}(B_0(x+\frac{y+Z}{2}) + B_0(x+\frac{-y+Z}{2})) + O(y/c^2) \\ &= \frac{2^N N!}{c^2}B_0(x) + O(y/c^2). \end{aligned}$$

QED.

Proposition 20 *We have*

$$(G_x + G_\xi)_\xi = \frac{1}{c^2} C_4(x,c) + O(y/c^2).$$

Proof. We see

$$(G_{I,x} + G_{I,\xi})_\xi = O(y/c^2)$$

by $a, a_x = O(y/c^2)$. Next we see

$$\begin{aligned} (G_{II,x} &+ G_{II,\xi})_\xi = \\ &= 2^{N-1}N!(B_x((x+y+\xi)/2,(x+y-\xi)/2) + \\ &+ B_x((x-y+\xi)/2,(-x+y+\xi)/2)) + O(y/c^2) \\ &= \frac{2^N N!}{c^2}B_0'(x) + O(y/c^2). \end{aligned}$$

And we see

$$\begin{aligned} (G_{III,x} &+ G_{III,\xi})_\xi = \\ &= 2^N N! b((x-y+\xi)/2,(-x+y+\xi)/2) + O(y/c^2) \\ &= \frac{2^N N!}{c^2} b_0(x) + O(y/c^2). \end{aligned}$$

Other terms can be estimated similarly. QED.

7 Estimates of the derivatives of entropies

Let us consider the entropy η generated by ϕ of C^3-class, that is,

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi.$$

In this section we will find estimates of the derivatives of η with respect to E, F. As auxiliary variables we introduce

$$R = y^{2N+1}, \qquad M = xy^{2N+1}. \tag{7.1}$$

We are going to prove the following

Proposition 21 *We have*

$$\begin{aligned}
\frac{\partial \eta}{\partial M} &= 2^{2N+1}\int_0^1 (s-s^2)^N D\phi(x+(2s-1)y)ds + O(y^2/c^2), & (7.2)\\
\frac{\partial \eta}{\partial R} &= 2^{2N+1}\int_0^1 (s-s^2)^N \phi ds + \\
& \quad 2^{2N+1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds + \\
& \quad O(y^2/c^2), & (7.3)\\
\frac{\partial^2 \eta}{\partial M^2} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi ds + O(y^{-2N+1}/c^2), & (7.4)\\
\frac{\partial^2 \eta}{\partial R \partial M} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D^2\phi ds + \\
& \quad O(y^{-2N+1}/c^2), & (7.5)\\
\frac{\partial^2 \eta}{\partial R^2} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N((-x+\frac{y}{2N+1}(2s-1))^2 + \\
& \quad \frac{4}{(2N+1)^2}s(1-s)y^2)D^2\phi(x+(2s-1)y)ds + O(y^{-1}/c^2) & (7.6)
\end{aligned}$$

Proof. We write

$$\eta = 2R^{\frac{1}{2N+1}}\int_0^1 K(\frac{M}{R}, R^{\frac{1}{2N+1}}, \frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}})\phi(\frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}})ds.$$

Differentiating η with respect to M, we have

$$\begin{aligned}
\frac{\partial \eta}{\partial M} &= (1)+(2),\\
(1) &= 2R^{\frac{-2N}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,x+(2s-1)y)\phi(x+(2s-1)y)ds,\\
(2) &= 2R^{\frac{-2N}{2N+1}}\int_0^1 K(x,y,x+(2s-1)y)D\phi(x+(2s-1)y)ds.
\end{aligned}$$

Since $K(x,y,\xi) = J^N G(x,y,\xi)$,i.e.

$$K(x,y,\xi) = \int_{|x-\xi|}^{y} Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1}\cdots\int_{|x-\xi|}^{Y_2} Y_1 G(x,Y_1,\xi)dY_1\cdots Y_N,$$

by Proposition 16 we see

$$\begin{aligned}(K_x &+ K_\xi)(x,y,x+(2s-1)y) \\ &= \int_{|2s-1|y}^{y} Y_N \int_{|2s-1|y}^{Y_N} Y_{N-1}\cdots\int_{|2s-1|y}^{Y_2} Y_1(G_x+G_\xi)(x,Y_1,x+(2s-1)y)dY_1\cdots Y_N \\ &= \frac{C_1(x,c)}{2^N N!c^2}y^{2N+1}(2s-1)(1-(2s-1)^2)^N + O(y^{2N+2}/c^2) \\ &= -\frac{2^N C_1(x,c)}{(N+1)!c^2}y^{2N+1}\frac{d}{ds}(s-s^2)^{N+1} + O(y^{2N+2}/c^2).\end{aligned}$$

Therefore by integration by part we get

$$\begin{aligned}(1) &= R^{\frac{-2N}{2N+1}}y^{2N+2}\frac{2^{N+1}C_1(x,c)}{(N+1)!c^2}\int_0^1 (s-s^2)^{N+1}D\phi ds + O(y^2/c^2) \\ &= O(y^2/c^2).\end{aligned}$$

By Proposition 13 we see

$$\begin{aligned}K(x,y,\xi) &= \int_{|x-\xi|}^{y} Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1}\cdots\int_{|x-\xi|}^{Y_2} Y_1 G(x,Y_1,\xi)dY_1\cdots Y_N, \\ &= 2^{2N}(s-s^2)^N y^{2N} + \frac{2^N C_0(x,c)}{N!c^2}(2s-1)(s-s^2)^N y^{2N+1} + O(y^{2N+2}/c^2).\end{aligned}$$

Therefore by integration by parts we get

$$\begin{aligned}(2) &= 2^{2N+1}R^{\frac{-2N}{2N+1}}y^{2N}\int_0^1 (s(1-s))^N D\phi(x+(2s-1)y)ds \\ &+ R^{\frac{-2N}{2N+1}}O(y^{2N+2}/c^2).\end{aligned}$$

Thus we have (7.2). Next we show (7.3). We have

$$\begin{aligned}\frac{\partial\eta}{\partial R} &= (3)+(4)+(5), \\ (3) &= \frac{2}{2N+1}R^{\frac{-2N}{2N+1}}\int_0^1 K(x,y,x+(2s-1)y)\phi(x+(2s-1)y)ds, \\ (4) &= 2R^{\frac{-2N}{2N+1}}\int_0^1 (-x(K_x+K_\xi)+\frac{1}{2N+1}y(K_y+(2s-1)K_\xi))\times \\ &\quad \phi(x+(2s-1)y)ds, \\ (5) &= 2R^{\frac{-2N}{2N+1}}\int_0^1 K(x,y,x+(2s-1)y)(-x+\frac{y}{2N+1}(2s-1))D\phi(...)ds.\end{aligned}$$

By Proposition 13 we get

$$(3) = \frac{2^{2N+1}}{2N+1}\int_0^1 (s-s^2)^N\phi(...)ds + O(y^2/c^2).$$

As for (4) we use Proposition 16 and

$$K_y + (2s-1)K_\xi = yJ^{N-1}G - (2s-1)(\xi-x)G(x,|\xi-x|,\xi)J^{N-1}1 + (2s-1)J^N G_\xi$$

$$= 2^{2N+1}N(s-s^2)^N y^{2N-1} + \frac{2^{N-1}C_0(x,c)}{(N-1)!c^2}(2s-1)(s-s^2)^N y^{2N} +$$

$$+ \frac{2^N C_3(x,c)}{N!c^2}(2s-1)(s-s^2)^N y^{2N} + O(y^{2N+1}/c^2)$$

(See Proposition 19). Then by integration by parts we have

$$(4) = \frac{2^{2N+2}N}{2N+1}\int_0^1 (s-s^2)^N \phi(...)ds + O(y^2/c^2).$$

As (2) we get

$$(5) = 2^{2N+1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi(...)ds + O(y^2/c^2).$$

Thus we get (7.3).

Next we show (7.4). We have

$$\begin{aligned}
\frac{\partial^2\eta}{\partial M^2} &= (6)+(7)+(8),\\
(6) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 ((K_x+K_\xi)_x + (K_x+K_\xi)_\xi)(x,y,...)\times\phi(...)ds,\\
(7) &= 4R^{\frac{-4N-1}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,...)D\phi(...)ds,\\
(8) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K(x,y,...)D^2\phi(...)ds.
\end{aligned}$$

By Proposition 18 we have

$$\begin{aligned}
((K_x+K_\xi)_x &+ (K_x+K_\xi)_\xi)(x,y,x+(2s-1)y) =\\
&= \frac{2^N C_2(x,c)}{N!c^2}(s-s^2)^N(2s-1)y^{2N+1} + O(y^{2N+2}/c^2).
\end{aligned}$$

Thus by integration by parts we get

$$(6) = O(y^{-2N+1}/c^2).$$

By the same discussion as (1) we see $(7) = O(y^{-2N+1}/c^2)$. By the same discussion as (2) we see

$$(8) = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi(...)ds + O(y^{-2N+1}/c^2).$$

Thus we get (7.4).

Next we show (7.5). We see

$$\frac{\partial^2\eta}{\partial M\partial R} = (9)+(10)+(11)+(12)+(13)+(14),$$

$$
\begin{aligned}
(9) &= -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,...)\phi(...)ds,\\
(10) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1(-x(K_x+K_\xi)_x+(K_x+K_\xi)_\xi)+\\
&+ \frac{y}{2N+1}((K_x+K_\xi)_y+(2s-1)(K_x+K_\xi)_\xi))\phi(...)ds,\\
(11) &= 2R^{\frac{4N}{2N+1}}\int_0^1(K_x+K_\xi)(-x+\frac{y}{2N+1}(2s-1))D\phi ds,\\
(12) &= -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 KD\phi ds,\\
(13) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1(-x(K_x+K_\xi)+\frac{y}{2N+1}(K_y+(2s-1)K_\xi))D\phi ds\\
(14) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K(-x+\frac{y}{2N+1}(2s-1))D^2\phi ds.
\end{aligned}
$$

We already know that $(9)=O(y^{-2N+1}/c^2)$. (Recall (1).) Next we look at (10). The first term is $O(y^{-2N+1}/c^2)$. (Recall (6)). By Proposition 16 and 20 we see

$$
\begin{aligned}
(K_x+K_\xi)_y &+ (2s-1)(K_x+K_\xi)_\xi =\\
&= \frac{2^{N-1}C_1(x,c)}{(N-1)!c^2}y^{2N}(2s-1)(s-s^2)^N+\\
&+ \frac{2^N C_4(x,c)}{N!c^2}y^{2N}(2s-1)(s-s^2)^N+\\
&- \frac{2^{N-1}C_1(x,c)}{(N-1)!c^2}y^{2N}(2s-1)^3(s-s^2)^{N-1}+O(y^{2N+1}/c^2).
\end{aligned}
$$

By integration by parts we see $(10)=O(y^{-2N+1}/c^2)$. We already know $(11)=O(y^{-2N+1}/c^2)$. Clearly

$$(12) = -\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N D\phi ds + O(y^{-2N+1}/c^2).$$

We see

$$(13) = O(y^{-2N+1}/c^2)+\frac{2}{2N+1}R^{\frac{-4N}{2N+1}}\int_0^1(K_y+(2s-1)K_\xi)D\phi ds.$$

As (4) we have

$$(13) = \frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N D\phi ds + O(y^{-2N+1}/c^2).$$

Finally we see

$$(14) = 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+(2s-1)\frac{y}{2N+1})D^2\phi ds+O(y^{-2N+1}/c^2).$$

(Recall (5)). Summing up we get (7.5).

Next we show (7.6).

$$
\begin{aligned}
\frac{\partial^2 \eta}{\partial R^2} &= \frac{\partial}{\partial R}(3) + \frac{\partial}{\partial R}(4) + \frac{\partial}{\partial R}(5), \\
\frac{\partial}{\partial R}(3) &= (15) + (16) + (17), \\
(15) &= -\frac{4N}{(2N+1)^2} R^{\frac{-4N-1}{2N+1}} \int_0^1 K\phi ds, \\
(16) &= \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\phi ds, \\
(17) &= \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))D\phi ds, \\
\frac{\partial}{\partial R}(4) &= (18) + (19) + (20), \\
(18) &= -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\phi ds, \\
(19) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K''\phi ds,
\end{aligned}
$$

where

$$
\begin{aligned}
K'' &= x(K_x + K_\xi) + x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) + \\
&+ \frac{y}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) + \\
&- \frac{xy}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) + \\
&+ \frac{y^2}{2N+1}((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi) + \\
&+ \frac{y}{(2N+1)^2}(K_y + (2s-1)K_\xi), \\
(20) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \\
&\times (-x + \frac{y}{2N+1}(2s-1))D\phi ds, \\
\frac{\partial}{\partial R}(5) &= (21) + (22) + (23) + (24), \\
(21) &= -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))D\phi ds, \\
(22) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \\
&\times (-x + \frac{y}{2N+1}(2s-1))D\phi ds, \\
(23) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(x + \frac{y}{(2N+1)^2}(2s-1))D\phi ds,
\end{aligned}
$$

$$(24) \;=\; 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))^2 D^2\phi ds.$$

First we see

$$(15) \;=\; -\frac{-2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$
$$(16) \;=\; \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$
$$(17) \;=\; \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N(-x + \frac{y}{2N+1}(2s-1))D\phi ds + O(y^{-2N+1}/c^2).$$

Thus we have

$$\frac{\partial}{\partial R}(3) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds + O(y^{-2N+1}/c^2).$$

Since (18) is similar to (16), we have

$$(18) = -\frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).$$

Next let us look at (19). We already know

$$2R^{\frac{-4N-1}{2N+1}} \int_0^1 x(K_x + K_\xi)\phi ds \;=\; O(y^{-2N+1}/c^2),$$
$$2R^{\frac{-4N-1}{2N+1}} \int_0^1 x^2((K_x+K_\xi)_x + (K_x+K_\xi)_\xi)\phi ds \;=\; O(y^{-2N+1}/c^2).$$

Recalling (10), we see

$$\frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} y \int_0^1 ((K_x+K_\xi)_y + (2s-1)(K_x+K_\xi)_\xi)\phi ds \;=\; O(y^{-2N+1}/c^2),$$
$$\frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} xy \int_0^1 ((K_x+K_\xi)_y + (2s-1)(K_x+K_\xi)_\xi)\phi ds \;=\; O(y^{-2N+1}/c^2).$$

When $N = 1$, we have

$$\begin{aligned}(K_y \;&+\; (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi \\ &= \; 8(s-s^2) + \frac{C_3}{c^2}(2s-1)y - \frac{C_0}{c^2}(2s-1)^3 y - \\ &- \; \frac{2C_3}{c^2}(2s-1)^3 y + O(y^2/c^2).\end{aligned}$$

When $N \geq 2$, there are bounded functions $F_j(x,c)$ such that

$$\begin{aligned}(K_y \;&+\; (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi = \\ &= \; 2^{2N+1}N(2N-1)(s-s^2)^N y^{2N-2} + \frac{F_1(x,c)}{c^2}(2s-1)(s-s^2)^{N-1}y^{2N-1} + \\ &+ \; \frac{F_2(x,c)}{c^2}(2s-1)(s-s^2)^{N-2}y^{2N-1} + \frac{F_3(x,c)}{c^2}(2s-1)^3(s-s^2)^{N-2}y^{2N-1} + \\ &+ \; \frac{F_4(x,c)}{c^2}(2s-1)^3(s-s^2)^{N-1}y^{2N-1} + \frac{F_5(x,c)}{c^2}(2s-1)^5(s-s^2)^{N-2}y^{2N-1} + \\ &+ \; O(y^{2N}/c^2).\end{aligned}$$

Thus we see

$$
\begin{aligned}
&2R^{\frac{-4N-1}{2N+1}}\frac{y^2}{(2N+1)^2}\int_0^1((K_y+(2s-1)K_\xi)_y+(2s-1)(K_y+(2s-1)K_\xi)_y)\phi ds\\
=\;&\frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds-\frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds\\
&+O(y^{-2N+1}/c^2).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{2}{(2N+1)^2}R^{\frac{-4N-1}{2N+1}}y\int_0^1(K_y+(2s-1)K_\xi)\phi ds\\
&=\frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+O(y^{-2N+1}/c^2).
\end{aligned}
$$

Therefore

$$
(19)=\frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+O(y^{-2N+1}/c^2).
$$

We see

$$
(20)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2).
$$

Therefore

$$
\frac{\partial}{\partial R}(4)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2).
$$

Next we see

$$
\begin{aligned}
(21)\;&=\;-\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2),\\
(22)\;&=\;\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2),\\
(23)\;&=\;2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds+O(y^{-2N+1}/c^2),\\
(24)\;&=\;2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+}(2s-1))^2D^2\phi ds+O(y^{-2N+1}/c^2).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\frac{\partial}{\partial R}(5)\;&=\;2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds+\\
&+\;2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds+O(y^{-2N+1}/c^2).
\end{aligned}
$$

Summing up, we have

$$
\frac{\partial^2\eta}{\partial R^2}\;=\;2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds
$$

$$\begin{aligned}
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds\\
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds\\
&= \quad \frac{2^{2N+2}(N+1)}{(2N+1)^2}y^{-2N-1}\int_0^1 (s-s^2)^N(2s-1)yD\phi ds +\\
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds\\
&= \quad \frac{2^{2N+3}}{(2N+1)^2}y^{-2N-1}\int_0^1 (s-s^2)^{N+1}y^2D^2\phi ds +\\
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds.
\end{aligned}$$

Thus we get (7.6). QED.

Let us recall the standard entropy η^*. This is generated by

$$\phi^*(x) = A'c^2(\frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}}),$$

where

$$A' = (2N+1)^{-2N}((2N+1)/(2N+3)A)^{\frac{2N+1}{2}}(2N-1)!!/2^{N+1}N!.$$

We note that

$$D^2\phi^*(x) = A'(1+\frac{u^2/c^2}{1-u^2/c^2})(2-\sqrt{1-u^2/c^2}) \geq A'.$$

We are going to show that the Hessian $D^2_U\eta^*$ dominates any $D^2_U\eta$.

Proposition 22 *For each ϕ fixed in C^3 we have on each compact subset of $\{\rho \geq 0\}$*

$$|(\xi|D^2_U\eta.\xi)| \leq C(\xi|D^2_U\eta^*.\xi),$$

provided that c is sufficiently large.

By the assumption we have

$$\begin{aligned}
R &= y^{2N+1} = K\rho(1+[\rho^{\frac{2}{2N+1}}/c^2]_1),\\
\frac{dR}{d\rho} &= K + [\rho^{\frac{2}{2N+1}}/c^2]_1,\\
\frac{d^2R}{d\rho^2} &= \frac{\rho^{\frac{1-2N}{2N+1}}}{c^2}[\rho^{\frac{2}{2N+1}}/c^2]_0,
\end{aligned}$$

where $K = ((2N+3)(2N+1)A)^{\frac{2N+1}{2}}$. Using these, we have

$$\frac{\partial R}{\partial E} = \frac{dR}{d\rho}\frac{1+u^2/c^2}{1-P'u^2/c^4}$$

$$
\begin{aligned}
&= K(1+u^2/c^2)+O(y^2/c^2),\\
\frac{\partial R}{\partial F} &= -\frac{dR}{d\rho}\frac{2u/c^2}{1-P'u^2/c^4}\\
&= -K\frac{2u}{c^2}+O(y^2/c^2),\\
\frac{\partial M}{\partial E} &= -\frac{R}{\rho+P/c^2}\frac{1+P'/c^2}{1-P'u^2/c^4}u + x\frac{dR}{d\rho}\frac{1+u^2/c^2}{1-P'u^2/c^4}\\
&= K(-u+x(1+u^2/c^2))+O(y^2/c^2),\\
\frac{\partial M}{\partial F} &= \frac{R}{\rho+P/c^2}\frac{1+P'u^2/c^4}{1-P'u^2/c^4}-\frac{dR}{d\rho}2xu/c^2\frac{1}{1-P'u^2/c^4}\\
&= K(1-2xu/c^2)+O(y^2/c^2). \qquad (7.7)
\end{aligned}
$$

Differentiating once more, we see

$$
\begin{aligned}
\frac{\partial^2 R}{\partial E^2} &= -\frac{K^2}{y^{2N+1}}2u^2(1-u^2/c^2)/c^2+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 M}{\partial E^2} &= \frac{K^2}{y^{2N+1}}u(-2u^2/c^2-2ux(1-u^2/c^2)/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 R}{\partial E\partial F} &= \frac{K^2}{y^{2N+1}}\frac{2u}{c^2}(1-u^2/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 M}{\partial E\partial F} &= \frac{K^2}{y^{2N+1}}(2u^2/c^2+2xu(1-u^2/c^2)/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 R}{\partial F^2} &= -\frac{2}{c^2}\frac{K^2}{y^{2N+1}}(1-u^2/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 M}{\partial F^2} &= -\frac{K^2}{y^{2N+1}}2(u+x(1-u^2/c^2))/c^2+O(y^{-2N+1}/c^2). \qquad (7.8)
\end{aligned}
$$

The chain rule gives

$$
\begin{aligned}
\frac{\partial^2\eta}{\partial E^2} &= (\frac{\partial R}{\partial E})^2\frac{\partial^2\eta}{\partial R^2}+2\frac{\partial R}{\partial E}\frac{\partial M}{\partial E}\frac{\partial^2\eta}{\partial R\partial M}\\
&+ (\frac{\partial M}{\partial E})^2\frac{\partial^2\eta}{\partial M^2}+\frac{\partial^2 R}{\partial E^2}\frac{\partial\eta}{\partial R}+\frac{\partial^2 M}{\partial E^2}\frac{\partial\eta}{\partial M}, \qquad (7.9)
\end{aligned}
$$

and so on. Inserting (7.7)nd (7.8) into (7.9), and using Proposition 21, we have

$$
\begin{aligned}
(\xi|D_U^2\eta.\xi) &= \frac{2^{2N+1}K^2}{y^{2N+1}}\int_0^1(s-s^2)^N Z[\xi]D^2\phi ds+\\
&- \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial R}+\\
&- \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial M}+\\
&+ O(y^{-2N+1}/c^2),
\end{aligned}
$$

where

$$
Z[\xi] = Z_{00}\xi_0^2+2Z_{01}\xi_0\xi_1+Z_{11}\xi_1^2,
$$

$$
\begin{aligned}
Z_{00} &= (1+u^2/c^2)^2((-x+\frac{y}{2N+1}(2s-1))^2+\frac{4}{(2N+1)^2}s(1-s)y^2)+\\
&+ 2(1+u^2/c^2)(-u+x(1+u^2/c^2))(-x+\frac{y}{2N+1}(2s-1))\\
&+ (-u+x(1+u^2/c^2))^2,\\
Z_{01} &= -2(1+u^2/c^2)u/c^2((-x+\frac{y}{2N+1}(2s-1))^2+\frac{4}{(2N+1)^2}s(1-s)y^2)+\\
&+ (1+3u^2/c^2-4x(1+u^2/c^2)u/c^2)(-x+\frac{y}{2N+1}(2s-1))+\\
&+ (-u+x(1+u^2/c^2))(1-2xu/c^2),\\
Z_{11} &= \frac{4u^2}{c^4}((-x+\frac{y}{2N+1}(2s-1))^2+\frac{4}{(2N+1)^2}s(1-s)y^2)+\\
&- \frac{4u}{c^2}(1-2xu/c^2)(-x+\frac{y}{2N+1}(2s-1))+\\
&+ (1-2xu/c^2)^2.
\end{aligned}
$$

It can be shown that

$$Z[\xi]\geq \kappa s(1-s)y^2,$$

where κ is a positive constant depending on the compact subset of $\{\rho\geq 0\}$. In fact we see

$$Z_{00}Z_{11}-Z_{01}^2=(1-u^2/c^2)\frac{4}{(2N+1)^2}s(1-s)y^2.$$

On the other hand, we can estimate

$$
\begin{aligned}
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)\frac{\partial\eta}{\partial R}| &\leq \frac{\epsilon}{y^{2N+1}},\\
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))\frac{\partial\eta}{\partial M}| &\leq \frac{\epsilon}{y^{2N+1}},
\end{aligned}
$$

where $\epsilon=K'/c^2$. Let us introduce the parameters

$$\zeta_0=\xi_0, \qquad \zeta_1=\xi_1-u\xi_0.$$

Then we have

$$Z[\xi]=Q_{00}\zeta_0^2+2Q_{01}\zeta_0\zeta_1+Q_{11}\zeta_1^2,$$

and

$$
\begin{aligned}
Q_{00} &= Q_{00}^{(1)}(x)(2s-1)y+Q_{00}^{(2)}(x,s)y^2,\\
Q_{01} &= Q_{01}^{(1)}(x)(2s-1)y+Q_{01}^{(2)}(x,s)y^2,\\
Q_{11} &= Z_{11}=1+O(1/c^2)>0.
\end{aligned}
$$

Therefore if $|D^2\phi|\leq C$, we see

$$|(\xi|D_U^2\eta\xi)| \leq \frac{2^{2N+1}K^2C}{y^{2N+1}}\int_0^1(s-s^2)^N Z[\xi]ds$$

$$
\begin{aligned}
&+ \frac{12\epsilon}{y^{2N+1}}\int_0^1 (s-s^2)^N\zeta^2 ds + O(y^{-2N+1}/c^2)\\
&\leq \frac{2^{2N+1}K^2C}{y^{2N+1}}\int_0^1 (s-s^2)^N(Q_{11}(1+\epsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00}\zeta_0^2)ds\\
&+ O(y^{-2N+1}/c^2).
\end{aligned}
$$

But since $Q_{00}^{(0)} = Q_{01}^{(0)} = 0$, $\int_0^1 (s-s^2)^N(2s-1)ds = 0$, we see

$$
\int_0^1 (s-s^2)^N(-2\epsilon' Q_{01}\zeta_0\zeta_1 - \epsilon' Q_{00}\zeta_0^2)ds = O(y^{-2N+1}/c^2).
$$

Therefore we get

$$
|(\xi|D_U^2\eta\xi)| \leq \frac{2^{2N+1}K^2C(1+\epsilon')}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + O(y^{-2N+1}/c^2).
$$

Similarly, if $D^2\phi^* \geq \mu$, we have

$$
(\xi|D_U^2\eta^*\xi) \geq \frac{2^{2N+1}K^2\mu(1-\epsilon'')}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + O(y^{-2N+1}/c^2).
$$

Thus we get

$$
|(\xi|D_U^2\eta\xi)| \leq \frac{C(1+\epsilon')}{\mu(1-\epsilon'')}(\xi|D_U^2\eta^*\xi) + O(y^{-2N+1}/c^2).
$$

But we know

$$
(\xi|D_U^2\eta^*\xi) \geq \kappa|\xi|^2 y^{-2N+1}.
$$

Hence if c is sufficiently large we get the required estimate. QED.

As for the first derivatives, the following conclusion is now clear.

Proposition 23 *On each compact subset of* $\{\rho \geq 0\}$, *we have*

$$
|\frac{\partial\eta}{\partial E}| + |\frac{\partial\eta}{\partial F}| \leq C.
$$

8 Usefull entropies

Let us consider an entropy η generated by ϕ, that is,

$$
\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi. \tag{8.1}
$$

The corresponding entropy flux q is given by integrating the differential equations

$$
\frac{\partial q}{\partial w} = \lambda_2\frac{\partial\eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1\frac{\partial\eta}{\partial z}.
$$

We can solve these equations as

$$\begin{aligned} q &= \lambda_2\eta - \int_z^w \frac{\partial\lambda_2}{\partial w}\eta dw \\ &= \lambda_1\eta + \int_z^w \frac{\partial\lambda_1}{\partial z}\eta dz. \end{aligned}$$

Thus we get the formula

$$q(x,y) = \int_{x-y}^{x+y} L(x,y,\xi)\phi(\xi)d\xi, \tag{8.2}$$

where

$$\begin{aligned} L(x,y,\xi) &= \lambda_1 K(x,y,\xi) + L_1(x,y,\xi) \\ &= \lambda_2 K(x,y,\xi) + L_2(x,y,\xi), \\ L_1(x,y,\xi) &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y,Y,)K(x+y-Y,Y,\xi)dY, \\ L_2(x,y,\xi) &= -2\int_{(-x+y+\xi)/2}^{y} \mu_2(x-y+Y,Y)K(x-y+Y,Y,\xi)dY, \\ \mu_1(x,y) &= \frac{\partial\lambda_1}{\partial z} \\ &= \frac{1-u^2/c^2}{2(1-\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) \\ &= \frac{N}{2N+1} + O(1/c^2), \\ \mu_2(x,y) &= \frac{\partial\lambda_2}{\partial w} \\ &= \frac{1-u^2/c^2}{2(1+\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) \\ &= \frac{N}{2N+1} + O(1/c^2). \end{aligned}$$

In this section we will construct various kinds of usefull entropies.

1) Let us put

$$\begin{aligned} \eta_k^1(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)k^{N+1}e^{k\xi}d\xi, \\ \eta_k^2(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)k^{N+1}e^{-k\xi}d\xi. \end{aligned}$$

Proposition 24 *If $1/c^2$ is sufficiently small, we have*

$$\eta_k^1 > 0, \qquad \eta_k^2 > 0 \qquad \textit{for } y > 0, \tag{8.3}$$

$$\begin{aligned} \eta_k^1 &= 2^N N! y^N(1+O(y/c^2))e^{k(x+y)}(1+O(1/k)), \\ \eta_k^2 &= 2^N N! y^N(1+O(y/c^2))e^{-k(x-y)}(1+O(1/k)) \end{aligned} \tag{8.4}$$

uniformly on each compact subset of $\{y > 0\}$. *Moreover*

$$\begin{aligned} q_k^1 &= \eta_k^1(\lambda_2 + O(1/k)), \\ q_k^2 &= \eta_k^2(\lambda_1 + O(1/k)) \qquad (8.5) \end{aligned}$$

uniformly on each compact subset of $\{y \geq 0\}$ *and*

$$\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} (\frac{1}{2N+1} + O(1/c^2)) e^{2ky} (y + O(1/k))^3. \quad (8.6)$$

Proof. Since $K = (1 + O(y/c^2))(y^2 - (x-\xi)^2)^N$, we see

$$\begin{aligned} \eta_k^1 &= (1 + O(y/c^2)) \int_{x-y}^{x+y} (y^2 - (x-\xi)^2)^N k^{N+1} e^{k\xi} d\xi \\ &= (1 + O(y/c^2)) 2^{2N+1} y^N e^{kx} f(ky) \end{aligned}$$

where

$$\begin{aligned} f(r) &= r^{N+1} e^{-r} \int_0^1 (s(1-s))^N e^{2rs} ds \\ &= e^r \int_0^r (\sigma(1 - \frac{\sigma}{r}))^N e^{-2\sigma} d\sigma. \end{aligned}$$

It is easy to see

$$e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r)$$

This implies (8.4). We note

$$\begin{aligned} \eta^1 &= (1 + O(1/c^2)) 2^N N! y^{N-1} e^{k(x+y)} (y + O(1/k)) \\ \eta^2 &= (1 + O(1/c^2)) 2^N N! y^{N-1} e^{-k(x-y)} (y + O(1/k)) \end{aligned}$$

uniformly on $\{y \geq 0\}$. Let us consider the flux. We have

$$\begin{aligned} L_2(x, y, \xi) &= -2 \int_{(-x+y+\xi)/2}^{y} \mu_2(x - y + Y, Y) K(x - y + Y, Y, \xi) dY \\ &= -2(\frac{N}{2N+1} + O(1/c^2)) \int_{(-x+y+\xi)/2}^{y} (Y^2 - (x - y + Y - \xi)^2)^N dY \\ &= -(\frac{N}{(2N+1)(N+1)} + O(1/c^2))(y - x + \xi)^N (y + x - \xi)^{N+1}, \\ q^1 - \lambda_2 \eta^1 &= -(\frac{N}{(2N+1)(N+1)} + O(1/c^2)) \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi. \end{aligned}$$

But

$$\begin{aligned} 0 &\leq \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi \\ &= (N+1) k^N \int_{x-y}^{x+y} (y^2 - (x-\xi)^2)^N e^{k\xi} d\xi \end{aligned}$$

$$- \quad Nk^N \int_{x-y}^{x+y} (y-x+\xi)^{N-1}(y+x-\xi)^{N+1}e^{k\xi}d\xi$$
$$\leq \quad (N+1)\frac{1}{k}\int_{x-y}^{x+y} (y^2-(x-\xi)^2)^N k^{N+1}e^{k\xi}d\xi.$$

Thus

$$q^1 - \lambda_2\eta^1 = O(1/k)\eta^1.$$

Since

$$\lambda_2 - \lambda_1 = \frac{\sqrt{P'}(1-u^2/c^2)}{1-P'u^2/c^4} = (\frac{1}{2N+1} + O(1/c^2))y,$$

we have

$$\eta^2 q^1 - \eta^1 q^2 = \eta^1\eta^2((\frac{1}{2N+1} + O(1/c^2))y + O(1/k)).$$

This implies (8.6). QED.

2) Let ψ be a function in $C_0^\infty(-1,1)$ such that $\psi \geq 0, \int \psi = 1$. We put

$$\begin{aligned}
\phi_n^3(x) &= \psi_n(x) = n\psi(n(x-a)),\\
\phi_n^4(x) &= -D\psi_n(x),\\
\eta_n^3(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)\phi_n^3(\xi)d\xi,\\
\eta_n^4(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)\phi_n^4(\xi)d\xi.\\
\eta^3(x,y) &= K(x,y,a)X,\\
\eta^4(x,y) &= K_\xi(x,y,a)X,\\
q^3(x,y) &= L(x,y,a)X,\\
q^4(x,y) &= L_\xi(x,y,a)X,\\
X &= 1 \qquad (x-y<a<x+y)\\
&= \frac{1}{2} \qquad (|x-a|=y)\\
&= 0 \qquad (|x-a|>y).
\end{aligned}$$

Proposition 25 *As $n \to \infty$, we have*

$$\eta_n^3 \to \eta^3, \qquad q_n^3 \to q^3, \qquad \eta_n^4 \to \eta^4, \qquad q_n^4 \to q^4.$$

Moreover

$$\begin{aligned}
|\eta_n^3| &\leq My^{2N}, \qquad |q_n^3| \leq My^{2N}(|x|+y), && (8.7)\\
|\eta_n^4| &\leq My^{2N-1}, \qquad |q_n^4| \leq My^{2N-1}(|x|+y), && (8.8)\\
\eta^3 q^4 - \eta^4 q^3 &= \frac{N}{(2N+1)(N+1)}(1+O(1/c^2))(y^2-(x-a)^2)^{2N} && (8.9)
\end{aligned}$$

Proof. We note

$$
\begin{aligned}
K_\xi &= -(\xi - x)G(x,|\xi-x|,\xi)\frac{1}{2^{N-1}(N-1)!}(y^2-(x-\xi)^2)^{N-1} + J^N G_\xi \\
&= (2N(x-\xi) + O(1/c^2)(\xi-x)^2)(y^2-(x-\xi)^2)^{N-1} + O(1/c^2)(y^2-(x-\xi)^2)^N, \\
L_{1,\xi} &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y,Y)K_\xi(x+y-Y,Y,\xi)dY.
\end{aligned}
$$

The estimates (8.7), (8.8) can be seen easily. Let us consider

$$\eta^3 q^4 - \eta^4 q^3 = (KL_\xi - LK_\xi)(x,y,a).$$

Suppose $x - a \geq 0$. Then

$$
\begin{aligned}
\frac{1}{2}(KL_\xi - LK_\xi) &= K\int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y,Y,a)dY - \\
&- K_\xi \int_{(x+y-a)/2}^{y} \mu_1 K(x+y-Y,Y,a)dY.
\end{aligned}
$$

We note

$$0 \leq \frac{x+y-a}{2} \leq x-y+Y-a \leq x-a \leq y.$$

Hence we have

$$
\begin{aligned}
&\int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y,Y,a)dY \\
= &\left(\frac{N}{2N+1} + O(1/c^2)\right)2N\int_{(x+y-a)/2}^{y}(x+y-Y-a)(Y^2-(x+y-Y-a)^2)^{N-1}dY + \\
+ &O(1/c^2)\int_{(x+y-a)/2}^{y}(Y^2-(x+y-Y-a)^2)^N dY \\
= &\left(\frac{N^2}{2(2N+1)} + O(1/c^2)\right)(x+y-a)^{N-1}(-x+y+a)^N\frac{1}{N(N+1)}(y+(2N+1)(x-a)) \\
+ &O(1/c^2)(y^2-(x-a)^2)^N.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&K\int_{(x+y-a)/2}^{y} \mu_1 K_\xi dY \\
= &\left(\frac{N}{2(2N+1)(N+1)} + O(1/c^2)\right)(y^2-(x-a)^2)^{2N-1}(-x+y+a)(y+(2N+1)(x-a)) \\
+ &O(1/c^2)(y^2-(x-a)^2)^{2N}.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
&K_\xi\int_{(x+y-a)/2}^{y} \mu_1 K dY \\
= &\left(\frac{N^2}{(2N+1)(N+1)} + O(1/c^2)\right)(x-a)(-x+y+a)(y^2-(x-a)^2)^{2N-1} \\
+ &O(1/c^2)(-x+y+a)(y^2-(x-a)^2)^{2N}.
\end{aligned}
$$

Hence

$$\frac{1}{2}(KL_\xi - LK_\xi) = (\frac{N}{2(2N+1)(N+1)} + O(1/c^2))(y^2 - (x-a)^2)^{2N}.$$

Here we have used

$$\begin{aligned} 0 &\le (x-a)(y-(x-a)) \le y^2 - (x-a)^2, \\ 0 &\le (y-x+a)(y+(2N+1)(x-a)) \\ &\le (2N+1)(y^2-(x-a)^2) \end{aligned}$$

provided that $0 \le x - a \le y$. When $x - a \le 0$, we can discuss in a similar manner by using L_2. QED.

3) Let Φ be a function in $C_0^\infty(-1,1)$ such that $\int \Phi = 0$ and the support $supp\Phi$ is $[-1+\alpha, 1+\alpha]$, where α is a small positive number. We put

$$\begin{aligned} \psi_n(x) &= n\Phi(n(x-a)), \\ \eta_n^5(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi, \\ q_n^5(x,y) &= \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi; \\ \hat{\Phi}(x) &= \frac{d}{dx}(x\int_{-1}^{x}\Phi), \\ \hat{\psi}_n(x) &= n\hat{\Phi}(n(x-a)), \\ \eta_n^6(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi d\xi, \\ q_n^6(x,y) &= \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi) d\xi; \\ B_n^3 &= \eta^3 q_n^5 - \eta_n^5 q^3, \\ B_n^4 &= \eta^4 q_n^5 - \eta_n^5 q^4, \\ B_n &= \eta_n^5 q_n^6 - \eta_n^6 q_n^5. \end{aligned}$$

Let us divide the domain $\Sigma = \{-B \le x - y \le x + y \le B\}$ into the following 5 parts.

$$\begin{aligned} S_0 &= \{-\frac{1}{n} < x+y-a \le \frac{1}{n}, -\frac{1}{n} \le x-y-a < \frac{1}{n}\} \cap \Sigma, \\ S_1 &= \{\frac{1}{n} < x+y-a, x-y-a < -\frac{1}{n}\} \cap \Sigma, \\ S_L &= \{-\frac{1}{n} < x+y-a \le \frac{1}{n}, x-y-a < -\frac{1}{n}\} \cap \Sigma, \\ S_R &= \{\frac{1}{n} < x+y-a, -\frac{1}{n} \le x-y-a < \frac{1}{n}\} \cap \Sigma, \\ S &= \Sigma - (S_0 \cup S_1 \cup S_1 \cup S_L \cup S_R). \end{aligned}$$

Proposition 26 *We have*

$$|B_n^3| \leq M/n, \qquad |B_n^4| \leq M \tag{8.10}$$

on Σ*, and*

$$|B_n| \leq M/n \tag{8.11}$$

on $S_0 \cup S_1 \cup S$*. Moreover, on* S_L*, we have*

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3, \tag{8.12}$$

where

$$\begin{aligned} A_1 &= (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2, \\ |A_2| &\leq M(|\int_{-1}^{n(x+y-a)} \Phi| + |\Phi(n(x+y-a))|), \\ |A_3| &\leq \frac{M}{n}. \end{aligned}$$

On S_R*, we have*

$$\begin{aligned} B_n &= ny^{2N}C_1 + y^N C_2 + C_3, \\ C_1 &= (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x-y-a)} \Phi)^2, \\ |C_2| &\leq M(|\int_{-1}^{n(x-y-a)} \Phi| + |\Phi(n(x-y-a))|), \\ |C_3| &\leq \frac{M}{n}. \end{aligned}$$

Proof. For the simplicity, we write $\eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6$. It is easy to see inductively that, for $G_j = J^j G = K_{N-j}$, we have

$$\partial_\xi^p G_j = J \partial_\xi^p G_{j-1}$$

for $j \geq p+1$ and

$$\partial_\xi^p G_p = (-1)^p (\xi - x)^p G(x, |\xi - x|, \xi) + J\partial_\xi^p G_{p-1}.$$

Therefore

$$\partial_\xi^p K = \partial_\xi^p G_N(x, y, \xi) = 0$$

for $p \leq N-1$ and $y = |x - \xi|$. Thus by integration by parts we have

$$\begin{aligned} \eta_n &= (-1)^N \partial_\xi^N K(x, y, x+y)\psi_n(x+y) + \\ &- (-1)^N \partial_\xi^N K(x, y, x-y)\psi_n(x-y) + \\ &+ F_n^1(x, y), \\ F_n^1(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi)\psi_n(\xi) d\xi. \end{aligned}$$

We see

$$\partial_\xi^p L_2(x,y,\xi) = -2\int_{(-x+y+\xi)/2}^{y} \mu_2 \partial_\xi^p K(x-y+Y,Y,\xi)dY$$

for $p \leq N-1$. Therefore

$$\partial_\xi^p L_2(x,y,x+y) = \partial_\xi^p L_2(x,y,x-y) = 0$$

for $p \leq N-1$. Moreover we see

$$\partial_\xi^N L_2(x,y,x+y) = 0.$$

Therefore by integration by parts we have

$$\begin{aligned}
\sigma_n(x,y) &= q_n(x,y) - \lambda_2\eta_n(x,y) \\
&= -(-1)^N \partial_\xi^N L_2(x,y,x-y)\psi_n(x-y) + \\
&+ F_n^2(x,y), \\
F_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x,y,\xi)\psi_n(\xi)d\xi.
\end{aligned}$$

Similarly

$$\begin{aligned}
\bar{\sigma}_n(x,y) &= q_n(x,y) - \lambda_1\eta_n(x,y) = \\
&= (-1)^N \partial_\xi^N L_1(x,y,x+y)\psi_n(x+y) + \\
&+ \bar{F}_n^2(x,y), \\
\bar{F}_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} L_1(x,y,\xi)\psi_n(\xi)d\xi.
\end{aligned}$$

We note

$$\partial_\xi^N K(x,y,\xi) = (-1)^N(\xi-x)^N G(x,|x-\xi|,\xi) + J\partial_\xi^N G_{n-1}.$$

It is easy to see inductively that

$$\begin{aligned}
\partial_\xi^{p+1} G_p(x,y,\xi) &= (-1)^p \frac{p(p+1)}{2}(\xi-x)^{p-1}G(x,|x-\xi|,\xi) + \\
&+ (\xi-x)^p H_p(x,\xi) + J\partial_\xi^p G_{p-1},
\end{aligned}$$

where $H_p = O(1/c^2)$. Therefore

$$\begin{aligned}
\partial_\xi^{N+1} K(x,y,\xi) &= (-1)^N \frac{N(N+1)}{2}(\xi-x)^{N-1}G(x,|\xi-x|,\xi) + \\
&+ (\xi-x)^N H_N(x,\xi) + J\partial_\xi^N G_{N-1}.
\end{aligned}$$

1) Suppose $(x,y) \in S$. Then it is clear that $\eta^3, \eta^4, q^3, q^4, \eta_n, q_n, \hat{\eta}_n, \hat{q}_n, B_n^3, B_n^4, B_n$ all vanish.

2) Suppose $(x, y) \in S_0$. Then we see

$$\begin{aligned}
\eta^3 &= K(x,y,a) \\
&= O((y^2-(x-a)^2)^N) \\
&= O(n^{-2N}), \\
\eta^4 &= K_\xi(x,y,a) \\
&= O(|x-a|(y^2-(x-a)^2)^{N-1}) + O((y^2-(x-a)^2)^N) \\
&= O(n^{-2N+1}), \\
\sigma^3 &= L_2(x,y,a) \\
&= -2\int_{(-x+y+a)/2}^{y} \mu_2 K(x-y+Y,Y,a)dY \\
&= O(n^{-2N-1}), \\
\sigma^4 &= L_{2,\xi}(x,y,a) \\
&= -2\int_{(-x+y+a)/2}^{y} \mu_2 K_\xi(x-y+Y,Y,a)dY \\
&= O(n^{-2N}).
\end{aligned}$$

Since $y = O(1/n)$ and $\psi_n = O(n)$, we see

$$\begin{aligned}
& (-1)^N\partial_\xi^N K(x,y,x+y)\psi_n(x+y) + \\
- \; & (-1)^N\partial_\xi^N K(x,y,x-y)\psi_n(x-y) = \\
= \; & O(n^{-N+1}).
\end{aligned}$$

Since $F_n^1 = O(1)$, we have $\eta_n = O(1)$. We see

$$\partial_\xi^N L_2(x,y,x-y) = -2\int_0^y \mu_2\partial_\xi^N K(x-y+Y,Y,x-y)dY = O(n^{-N-1}).$$

Therefore

$$-(-1)^N\partial_\xi^N L_2(x,y,x-y)\psi_n(x-y) = O(n^{-N}).$$

Since

$$\begin{aligned}
\partial_\xi^{N+1}L_2(x,y,\xi) &= \mu_2\partial_\xi^N K((x-y+\xi)/2,(-x+y+\xi)/2,\xi) + \\
&- 2\int_{(-x+y+\xi)/2}^{y} \partial_\xi^{N+1}K(x-y+Y,Y,\xi)dY \\
&= O((-x+y+\xi)^N) + O(x+y-\xi),
\end{aligned}$$

we see

$$\begin{aligned}
F_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1}L_2(x,y,\xi)\psi_n(\xi)d\xi \\
&= O(n^{-1}).
\end{aligned}$$

Hence $\sigma_n = O(n^{-1})$. Therefore

$$\begin{aligned} B_n^3 &= \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-2N-1}), \\ B_n^4 &= \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{-2N}), \\ B_n &= \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_n = O(n^{-1}). \end{aligned}$$

3) Suppose $(x, y) \in S_1$, where $x + y > a + \frac{1}{n}$ and $x - y < a - \frac{1}{n}$. Then $\psi_n(x+y) = \psi_n(x-y) = \hat{\psi}_n(x+y) = \hat{\psi}_n(x-y) = 0$. So, $\eta_n = F_n^1, \sigma_n = F_n^2$, and so on. But

$$\begin{aligned} F_n^1(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1}K(x,y,\xi)\psi_n(\xi)d\xi \\ &= (-1)^{N+1}\int_{-1}^{1}(\partial_\xi^{N+1}K(x,y,a+\frac{s}{n}) - \partial_\xi^{N+1}K(x,y,a))\Phi(s)ds \\ &= O(1/n) \end{aligned}$$

since $\int\Phi = 0$ and $\partial_\xi^{N+1}K$ is Lipschitz continuous. Same estimates hold for $F_n^2, \hat{F}_n^1, \hat{F}_n^2$. Thus

$$\begin{aligned} B_n^3 &= \eta^3 F_n^2 - F_n^1\sigma^3 = O(1/n), \\ B_n^4 &= \eta^4 F_n^2 - F_n^1\sigma^4 = O(1/n), \\ B_n &= F_n^1\hat{F}_n^2 - \hat{F}_n^1 F_n^2 = O(1/n^2). \end{aligned}$$

4) Suppose $(x, y) \in S_L$, where $|x + y - a| \leq 1/n$. It is easy to see $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$. Since $n(x-y-a) < -1$, we have $\psi_n(x-y) = 0$. Thus $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$. Therefore

$$\begin{aligned} B_n^3 &= \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-N}), \\ B_n^4 &= \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{1-N}). \end{aligned}$$

Let us estimate $B_n = \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_N$. Since

$$\begin{aligned} \partial_\xi^{N+1}K &= (-1)^N\frac{N(N+1)}{2}(\xi - x)^{N-1}G(x, |x-\xi|, \xi) + \\ &+ (\xi - x)^N H_N(x,\xi) + J\partial_\xi^N G_{N-1}, \end{aligned}$$

we have

$$\begin{aligned} F_n^1 &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}K(x,y,\xi)\psi_n(\xi)d\xi = \\ &= (-1)^{N+1}((-1)^N\frac{N(N+1)}{2}2^N N!(a-x)^{N-1} + F'(x,a))\int_{-1}^{n(x+y-a)}\Phi + \\ &+ O(1/n) = \\ &= -\frac{N(N+1)}{2}2^N N! y^{N-1}(1 + F''(x,y,a))\int_{-1}^{n(x+y-a)}\Phi + \\ &+ O(1/n), \end{aligned}$$

where $F' = O(1/c^2)|x-a|^N, F'' = O(1/c^2)$. On the other hand

$$\partial_\xi^N K(x,y,x+y) = (-1)^N y^N G(x,y,x+y).$$

Hence

$$\begin{aligned}\eta_n &= ny^N G(x,y,x+y)\Phi(n(x+y-a)) + \\ &- \frac{N(N+1)}{2} 2^N N! y^{N-1}(1+F''(x,y,a)) \int_{-1}^{n(x+y-a)} \Phi + \\ &+ O(1/n).\end{aligned}$$

Since

$$\begin{aligned}\partial_\xi^{N+1} L_2(x,y,\xi) &= \mu_2 \partial_\xi^N K((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\ &- 2\int_{(-x+y+\xi)/2}^{y} \mu_2 \partial_\xi^{N+1} K(x-y+Y, Y, \xi) dY = \\ &= (\frac{N}{2N+1} + O(1/c^2))(-1)^N (\frac{-x+y+\xi}{2})^N \times \\ &\times G((x+y+\xi)/2, (-x+y+\xi)/2, \xi) + \\ &+ O(x+y-\xi),\end{aligned}$$

we see

$$\begin{aligned}\sigma_n &= F_n^2 = \\ &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x,y,\xi)\psi_n(\xi) d\xi = \\ &= -\frac{N}{2N+1} 2^N N! y^N (1+L'(x,y,a)) \int_{-1}^{n(x+y-a)} \Phi \\ &+ O(1/n),\end{aligned}$$

where $L' = O(1/c^2)$. Here we have used

$$(\frac{-x+y+a}{2})^N = (y - \frac{x+y-a}{2})^N = y^N + O(1/n).$$

Similar estimates hold for $\hat{\eta}_n, \hat{\sigma}_n$. Thus

$$B_n = ny^{2N} A_1 + y^N A_2 + A_3,$$

where

$$\begin{aligned}A_1 &= -G\frac{N}{2N+1} 2^N N! (1+L')\Phi(\beta) \int_{-1}^{\beta} \hat{\Phi} + \\ &+ G\frac{N}{2N+1} 2^N N! (1+L')\hat{\Phi}(\beta) \int_{-1}^{\beta} \Phi = \\ &= \frac{N}{2N+1} 2^N N! G(1+L')(\int_{-1}^{\beta} \Phi)^2, \\ \beta &= n(x+y-a).\end{aligned}$$

The estimates on S_R can be obtained in a similar manner considering $\bar\sigma^3, \bar\sigma^4, \bar\sigma_n$. QED.

If we put

$$\begin{aligned}\hat B^3_n &= \eta^3\eta^6_n - \eta^6_n q^3,\\ \hat B^4_n &= \eta^4 q^6_n - \eta^6_n q^4,\end{aligned}$$

then the same estimates hold.

9 Compactness of $\eta_t + q_x$

Let us consider an entropy η generated by ϕ through the generalized Darboux formula and its flux q. In this section we will prove

Lemma 1 *Let U^Δ be the approximate solutions constructed in Section 4. Then $\eta(U^\Delta)_t + q(U^\Delta)_x$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$, Ω being a bounded open subset of $\{t \geq 0\}$.*

Proof. Let Φ be a test function and we consider

$$\begin{aligned}J &= \int\int (\eta(U^\Delta)\Phi_t + q(U^\Delta)\Phi_x)dxdt\\ &= N + L + \Sigma,\\ N &= -\int \eta(U^\Delta(+0,x)\Phi(0,x)dx,\\ L &= \sum_n \int [\eta(U^\Delta(t,x)]^{t=n\Delta t-0}_{t=n\Delta t+0}\Phi(n\Delta t, x)dx,\\ \Sigma &= \int \sum_{shock}(\sigma[\eta]-[q])\Phi dt.\end{aligned}$$

Since U^Δ is bounded, we see

$$|N| \leq M||\Phi||_C.$$

Let us look at L. We see

$$\begin{aligned}L &= L_1 + L_2,\\ L_1 &= \sum_{j,n}\Phi(n\Delta t, (2j+1)\Delta x)\int_{2j\Delta x}^{(2j+2)\Delta x}[\eta(U^\Delta)]^{t=n\Delta t-0}_{t=n\Delta t+0}dx,\\ L_2 &= \sum_{j,n}\int_{2j\Delta x}^{(2j+2)\Delta x}(\Phi(n\Delta t,x) - \Phi(n\Delta t,(2j+1)\Delta x)\times\\ &\times\ [\eta(U^\Delta)]^{t=n\Delta t-0}_{t=n\Delta t+0}dx.\end{aligned}$$

We note

$$\begin{aligned}[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} &= D_U\eta(U^\Delta(n\Delta t+0,x))[U^\Delta] \\ &+ \int_0^1 (1-\theta)([U^\Delta]|D_U^2(U^\Delta(n\Delta t+0)+\theta[U^\Delta]).[U^\Delta])d\theta.\end{aligned}$$

and

$$\int_{2j\Delta x}^{(2j+2)\Delta x}[U^\Delta]dx = 0$$

by the scheme. Therefore

$$|L_1| \le M||\Phi||_C \sum_{j,n}\int\int_0^1 (1-\theta)|F(\theta,\eta)|d\theta dx,$$

where

$$F(\theta,\eta) = ([U^\Delta]|D_U^2\eta(U^\Delta(n\Delta t+0)+\theta[U^\Delta]).[U^\Delta]).$$

By Proposition 22 we know $|F(\theta,\eta)| \le MF(\theta,\eta^*)$. But in the proof of Proposition 7 we know

$$\sum_{j,n}\int\int_0^1 (1-\theta)F(\theta,\eta^*)d\theta dx \le C.$$

Thus we know

$$|L_1| \le M||\Phi||_C.$$

In the proof of Proposition 7 we know

$$\sum_{j,n}\int_{2j\Delta x}^{(2j+2)\Delta x}|[U^\Delta]|^2dx \le C.$$

Therefore

$$\begin{aligned}|L_2| &\le 2^\alpha||\Phi||_{C^\alpha}\sum_n\int(\Delta x)^\alpha|[\eta(U^\Delta)]|dx \\ &\le 2^{\alpha-1}||\Phi||_{C^\alpha}\sum_n\int((\Delta x)^{\alpha+\frac{1}{2}}+(\Delta x)^{\alpha-\frac{1}{2}}|[\eta(U^\Delta)]|^2)dx \\ &\le M||\Phi||_{C^\alpha}((\Delta x)^{\alpha-\frac{1}{2}}+(\Delta x)^{\alpha-\frac{1}{2}}\sum\int|[U^\Delta]|^2dx \\ &\le M'(\Delta x)^{\alpha-\frac{1}{2}}||\Phi||_{C^\alpha},\end{aligned}$$

where we use the boundedness of $D_U\eta$ and $n = O(1/(\Delta x))$. Next we look at Σ. Along the shock we have

$$\begin{aligned}\sigma[\eta(U)] &- [q(U)] \\ &= \int_{\rho_L}^{\rho_R}(-\frac{d\sigma}{d\rho}\int_0^1\theta(U-U_L|D_U^2\eta(U_L+\theta(U-U_L))(U-U_L))d\theta)d\rho.\end{aligned}$$

This implies

$$|\sigma[\eta] - [q]| \le M(\sigma[\eta^*] - [q^*]).$$

But we know

$$\int \sum_{shock} (\sigma[\eta^*] - [q^*])dt \le C$$

in the proof of Proposition 7. Therefore

$$|\Sigma| \le M||\Phi||_C.$$

Summing up , we know the compactness. QED.

10 Convergence of approximate solutions

We consider the approximate solutions U^Δ constructed in Section 4. Since U^Δ is bounded, there is a sequence U^{Δ_n} and a family of Young measures $\nu_{t,x}$ such that $supp\nu_{t,x} \subset \Sigma = \Sigma_B$ and for any continuous function f

$$f(U^{\Delta_n}(t,x)) \to \bar{f} = <\nu_{t,x}, f>$$

in L^∞ weak star topology. By Lemma 1, we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta' q)(U^{\Delta_n}) \to <\nu, q><\nu, q'> - <\nu, \eta'><\nu, q>$$

in L^∞ weak star. Here $\eta, q; \eta', q'$ are arbitrary Darboux entropy pairs. Thus we have

Lemma 2 *For any pairs $(\eta, q), (\eta', q')$ of Darboux entropies-entropy flux, the identity*

$$<\nu, \eta q' - \eta' q> = <\nu, \eta><\nu, q'> - <\nu, \eta'><\nu, q>$$

holds a.e.-(t,x), where $\nu = \nu_{t,x}$.

Since entropies we will use are countably many, we can assune that the above identity holds outside a null set which is common to all η. We fix (t,x) at which the identity holds, and we write $\nu = \nu_{t,x}$. Of course $supp.\nu \subset \Sigma$. Suppose that $supp.\nu \cap \{\rho > 0\} \neq \phi$. Let Σ_0 be the smallest triangle $\{z_0 \le z \le w \le w_0\}$ such that $supp.\nu \cap \{\rho > 0\} \subset \Sigma_0$. Let us denote by P_0 the state (w_0, z_0). It will be verified that $\nu = \delta_{P_0}$. (the Dirac measure). First we show

Proposition 27

$$P_0 \in supp.\nu.$$

Proof. Suppose $P_0 \notin supp.\nu$. Since Σ_0 is the smallest triangle containing $supp.\nu \cap \{\rho > 0\}$, $w = w_0$ and $z = z_0$ intersect with $supp.\nu \cap \{\rho > 0\}$. On neighborhoods of these intersection points we have

$$\begin{aligned} \eta^1 &\geq \frac{1}{M} e^{k(w_0-\epsilon)}, \\ \eta^2 &\geq \frac{1}{M} e^{-k(z_0+\epsilon)}. \end{aligned}$$

(See Proposition 24). Since ν, η^1, η^2 are nonnegative, we see

$$\begin{aligned} < \nu, \eta^1 > &\geq \frac{1}{M} e^{k(w_0-\epsilon)}, \\ < \nu, \eta^2 > &\geq \frac{1}{M} e^{-k(z_0+\epsilon)}. \end{aligned}$$

Since$P_0 \notin supp.\nu$, we have

$$< \nu, \eta^2 q^1 - \eta^1 q^2 > \leq M e^{k(w_0 - z_0 - \delta)}.$$

Taking $2\epsilon < \delta$, we have

$$\begin{aligned} \left| \frac{< \nu, q^1 >}{< \nu, \eta^1 >} - \frac{< \nu, q^2 >}{< \nu, \eta^2 >} \right| &= \left| \frac{< \nu, \eta^2 q^1 - \eta^1 q^2 >}{< \nu, \eta^1 >< \nu, \eta^2 >} \right| \\ &\leq M e^{-k(\delta - 2\epsilon)} \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Let β be a sufficiently small positive number, and we put

$$\begin{aligned} \Sigma_2 &= \{z_0 \leq z \leq w < w_0 - \beta\} \\ \Sigma_3 &= \{z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w\}. \end{aligned}$$

Then

$$\eta^1 e^{-kw} = (1 + O(1/c^2)) 2^N N! y^{N-1} (y + O(1/k))$$

is bounded on Σ_0 and we have

$$< \nu|_{\Sigma_2}, \eta^1 > \leq M e^{k(w_0 - \beta)}.$$

Taking $\epsilon = \beta/2$, we know

$$\frac{< \nu|_{\Sigma_2}, \eta^1 >}{< \nu, \eta^1 >} \leq M e^{-\beta k/2} \to 0.$$

Since $\partial \lambda_2 / \partial w > 0$, we know

$$\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0)$$

on Σ_3. Therefore we have

$$\begin{aligned}\frac{<\nu,q^1>}{<\nu,\eta^1>} &= \frac{<\nu|_{\Sigma_2},\eta^1\lambda_2>}{<\nu,\eta^1>}+\frac{<\nu|_{\Sigma_3},\eta^1\lambda_2>}{<\nu,\eta^1>}+\\ &+ O(1/k)\\ &\geq o(1)+\lambda_2(w_0-\beta,z_0)\end{aligned}$$

Similarly we see

$$\frac{<\nu,q^2>}{<\nu,\eta^2>}\leq o(1)+\lambda_1(w_0,z_0+\beta).$$

Therefore we have

$$\lambda_2(w_0-\beta,z_0)-\lambda_1(w_0,z_0+\beta)\leq 0+o(1).$$

Passing to the limit, we know

$$\lambda_2(w_0,z_0)\leq\lambda_1(w_0,z_0).$$

But this means $P_0\in\{\rho=0\}$, a contradiction. QED.

Let us fix a such that $z_0<a<w_0$. We have

$$\begin{aligned}<\nu,B_n^3> &= <\nu,\eta^3><\nu,q_n^5>-<\nu,\eta_n^5><\nu,q^3>,\\ <\nu,B_n^4> &= <\nu,\eta^4><\nu,q_n^5>-<\nu,\eta_n^5><\nu,q^4>,\\ <\nu,\eta^3q^4-\eta^4q^3> &= <\nu,\eta^3><\nu,q^4>-<\nu,\eta^4><\nu,q^3>,\\ <\nu,B_n> &= <\nu,\eta_n^5><\nu,q_n^6>-<\nu,\eta_n^6><\nu,q_n^5>.\end{aligned}$$

From (8.8) we know

$$<\nu,\eta^3q^4-\eta^4q^3>>0$$

and from(8.10) we know

$$<\nu,B_n^3>\to 0$$

Using these we can prove the following propositions. Proofs can be found in Chen et al [2].

Proposition 28 *As $n\to\infty$, $<\nu,\eta_n^5>,<\nu,q_n^5>,<\nu,q_n^6>,<\nu,q_n^6>$ are bounded.*

Proposition 29 *As $n\to\infty$, we have $<\nu,B_n>\to 0$.*

Now, taking

$$\Phi_0(x)=\begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x|<1\\ 0 & \text{if } |x|\geq 1\end{cases}$$

we put

$$\Phi(x) = \frac{1}{\beta}(\Phi_0(\frac{x+\beta}{\beta}) - \Phi_0(\frac{x-\beta}{\beta}))$$

for the generating function of η_n^5. Here $\beta = (1-\alpha)/2$. We put

$$\begin{aligned} S_+ &= \{z \leq w, |w-a| \leq \frac{1-3\alpha}{n}\}, \\ S_- &= \{z \leq w, |z-a| \leq \frac{1-3\alpha}{n}\}. \end{aligned}$$

Proposition 30 *As $n \to \infty$, we have*

$$< \nu|_{S_+}, ny^{2N} > + < \nu|_{S_-}, ny^{2N} > \to 0.$$

Proof. Put $S'_L = S_+ \cap S_L, S'_R = S_- \cap S_R$. It is sufficient to prove that

$$< \nu|_{S'_L}, ny^{2N} > + < \nu|_{S'_R}, ny^{2N} > \to 0.$$

From (8.11) we have

$$< \nu|_{S_L}, ny^{2N}A_1 + y^N A_2 > + < \nu|_{S_R}, ny^{2N}C_1 + y^N C_2 > \to 0.$$

Note

$$A_1 = (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2 \geq \frac{1}{M_0} > 0$$

on S'_L. Put

$$E_n = \{0 \leq y \leq (\frac{1}{n})^{\mu}\},$$

where μ is a positive parameter. Then $|y^N A_2| \leq M(1/n)^{\mu N} = o(1)$ on $S_L \cap E_n$ and $|y^N A_2| \leq Mny^{2N}(1/n)^{1-\mu N}$ on $S_L - E_n$. Choose $d_n \searrow 0$ such that

$$\int_{-1+\alpha}^{1-\alpha-d_n} \Phi = -\int_{1-\alpha-d_n}^{1-\alpha} \Phi \geq (1/n)^{\mu_0}.$$

Then

$$(\int_{-1}^{H} \Phi)^2 \geq (1/n)^{2\mu_0}$$

for $|H| \leq 1-\alpha-d_n$, and

$$|\Phi(H)| + |\int_{-1}^{H} \Phi| = o(1)$$

for $1-\alpha-d_n \leq |H| \leq 1$. Put

$$S_+^n = S_L \cap \{|w-a| \leq \frac{1-\alpha-d_n}{n}\}.$$

Then $S'_L \subset S_+^n \subset S_L$ and

$$|y^N A_2| = o(1)$$

on $S_L - S_+^n$ and

$$\begin{aligned} ny^{2N}A_1 + y^N A_2 &\geq ny^{2N}(\frac{1}{M}(1/n)^{2\mu_0} - M(1/n)^{1-\mu N}) \\ &\geq 0 \end{aligned}$$

on $S_+^n - E_n$. Here we take $0 < 2\mu_0 < 1 - \mu N$. Then

$$\begin{aligned} < \nu|_{S_L}, ny^{2N}A_1 + y^N A_2 > &= < \nu|_{S_L \cap E_n}, ny^{2N}A_1 > + \\ &+ < \nu|_{S_L - E_n}, ny^{2N}A_1 + y^N A_2 > + \\ &+ o(1) \\ &\geq \frac{1}{M_0} < \nu|_{S'_L \cap E_n}, ny^{2N} > + \\ &+ < \nu|_{S_L - S_+^n \cap E_n}, ny^{2N}A_1 > + \\ &+ < \nu|_{S'_L - E_n}, ny^{2N}A_1 + y^N A_2 > + \\ &+ < \nu|_{S_+^n - S'_L - E_n}, ny^{2N}A_1 + y^N A_2 > + \\ &+ o(1) \\ &\geq \frac{1}{M_0} < \nu|_{S'_L \cap E_n}, ny^{2N} > + \\ &+ < \nu|_{S'_L - E_n}, ny^{2N}(\frac{1}{M_0} - M(1/n)^{1-\mu N} > + \\ &+ o(1) \\ &\geq \frac{1}{2M_0} < \nu|_{S'_L}, ny^{2N} > + \\ &+ o(1). \end{aligned}$$

Similarly we know

$$< \nu|_{S_R}, ny^{2N}C_1 + y^N C_2 >\geq \frac{1}{2M_0} < \nu|_{S'_R}, ny^{2N} > +o(1)$$

. Thus we see

$$< \nu|_{S'_L}, ny^{2N} > + < \nu|_{S'_R}, ny^{2N} >\to 0.$$

QED.

Proposition 31 *We have*

$$\nu|_{\{\rho>0\}} = \delta_{P_0}.$$

Proof. Proposition 30 says that the projections $P_w\tilde{\nu}, P_z\tilde{\nu}$ of the measure $\tilde{\nu} = y^{2N}\nu$ admits the Lebesgue lower derivatives which vanish at any a. Therefore we can claime that

$$supp.\nu \cap \{\rho > 0\} = \{P_0\}.$$

Since ν is a probability measure, we have

$$\nu|_{\{\rho>0\}} = C\delta_{P_0}.$$

But

$$C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3)$$

at P_0. Hence $C = 1$. QED.

Summing up we get the final

Theorem 2 *For any M_0 there is a positive number ϵ_0 such that if the initial data satisfy*

$$0 \leq \rho_0(x) \leq M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \leq M_0.$$

and if $1/c^2 \leq \epsilon_0$, then a subsequence of the approximate solutions U^Δ converges a.e. to a limit U which is a weak solution of the relativistic Euler equation.

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