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Self-similar solutions to a parabolic system modelling chemotaxis

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We are concerned with the system of partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(\nabla u - u\nabla v) \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - \gamma v + u \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^N$ and $t > 0$, where $\tau > 0$ and $\gamma \geq 0$ are constants. This system is a mathematical model of chemotaxis (aggregation of organisms sensitive to the gradient of chemical substance) proposed by Keller and Segel [12] in 1970. The functions $u(x,t)$ and $v(x,t)$ in (1.1) denote the cell density of cellular slime molds and the concentration of the chemical substance at place x and time t, respectively. It is assumed that u and v are nonnegative.

We deal with a special class of solutions which are called self-similar solutions. If $\gamma = 0$, then the system (1.1) is invariant under the similarity transformation

$$u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t) \quad \text{and} \quad v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$$

for $\lambda > 0$, that is, if (u, v) is a solution of (1.1) globally in time, then so is (u_λ, v_λ). A solution (u, v) is said to be *self-similar*, when the solution is invariant under this transformation, that is,

$$u(x,t) = u_\lambda(x,t) \quad \text{and} \quad v(x,t) = v_\lambda(x,t) \quad \text{for all } \lambda > 0. \tag{1.2}$$

Letting $\lambda = 1/\sqrt{t}$ in (1.2), we see that (u, v) has the form

$$u(x,t) = \frac{1}{t}\phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad v(x,t) = \psi\left(\frac{x}{\sqrt{t}}\right) \tag{1.3}$$

for $x \in \mathbb{R}^N$ and $t > 0$. By a direct computation it is shown that (u, v) satisfies (1.1) if and only if (ϕ, ψ) satisfies

$$(1.4) \qquad \begin{cases} \nabla \cdot (\nabla\phi - \phi\nabla\psi) + \frac{1}{2}x \cdot \nabla\phi + \phi = 0 \\ \Delta\psi + \frac{\tau}{2}x \cdot \nabla\psi + \phi = 0 \end{cases}$$

for $x \in \mathbb{R}^N$. It follows that

$$\int_{\mathbb{R}^N} u(x,t)dx = t^{(N-2)/2} \int_{\mathbb{R}^N} \phi(y)dy$$

for $\phi \in L^1(\mathbb{R}^N)$. Therefore self-similar solution (u, v) preserves the mass $\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)}$ if and only if $N = 2$. Henceforth we study the case $N = 2$. We are concerned with the classical solutions $(\phi, \psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ of (1.4) satisfying

$$(1.5) \qquad \phi, \psi \geq 0 \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \phi(x), \psi(x) \to 0 \quad \text{as } |x| \to \infty.$$

Define the solution set $\mathcal{S}$ of (1.4) as

$$(1.6) \qquad \mathcal{S} = \{(\phi, \psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2) : (\phi, \psi) \text{ is a solution of (1.4) with (1.5)}\}.$$

The problem of existence of self-similar solutions has been studied extensively. The existence of radial solutions (ϕ, ψ) of (1.4) has been obtained by Mizutani and Nagai [15]. It has shown by Biler [1] that there is an upper bound on the mass of self-similar solutions. More precisely, the system (1.4) with $\tau = 1$ has no radial solutions (ϕ, ψ) satisfying $\|\phi\|_{L^1(\mathbb{R}^2)}/2\pi \geq 7.82\ldots$. Furthermore, for every $M \in (0, 8\pi)$, there exists a radial solution (ϕ, ψ) satisfying $\|\phi\|_{L^1(\mathbb{R}^2)} = M$. In this paper we investigate the structure of the solution set $\mathcal{S}$ defined by (1.6) more precisely.

First we show the system (1.4) is reduced to a single ordinary differential equation. Put

$$(1.7) \qquad \phi(x) = \sigma e^{-|x|^2/4} e^{\psi(x)},$$

where σ is a positive constant. Then ϕ satisfies the first equation of (1.4), and so if we find a solutions ψ of

$$(1.8) \qquad \Delta\psi + \frac{\tau}{2}x \cdot \nabla\psi + \sigma e^{-|x|^2/4} e^{\psi} = 0 \qquad \text{in } \mathbb{R}^2,$$

we can obtain the solution (ϕ, ψ) of (1.4). In [15, 16] they have shown the existence of radial solutions ψ of (1.8) satisfying

$$(1.9) \qquad \psi(x) \to 0 \quad \text{as } |x| \to \infty$$

by investigating the corresponding ordinary differential equation. Conversely, we have following:

Theorem 1. *Assume that* (ϕ, ψ) *is a nonnegative solution of* (1.4) *satisfying* ϕ, $\psi \in L^\infty(\mathbb{R}^2)$. *Then* ϕ *and* ψ *are positive and satisfy* (1.7), *where* $\sigma > 0$ *is a constant. Assume furthermore that* (1.9) *holds. Then* ϕ *and* ψ *are radially symmetric about the origin, and satisfy* $\phi_r < 0$ *and* $\psi_r < 0$ *for* $r > 0$, *and*

$$\phi(x) = O(e^{-|x|^2/4}) \quad \textit{and} \quad \psi(x) = O(e^{-\min\{\tau,1\}|x|^2/4}) \quad \textit{as } |x| \to \infty.$$

The proof of Theorem 1 consists of two steps. First we reduce the system (1.4) to the equation (1.8) by employing the Liouville type result essensially due to Meyers and Serrin [14]. Then we show the radial symmetry of solutions by the method of moving planes. This device was first developed by Serrin [23] in PDE theory, and later extended and generalized by Gidas, Ni, and Nirenberg [6, 7]. With a change of variables we are still able to obtain a symmetry result for the equation (1.8) as in [18].

Next we investigate the structure of the solution set $\mathcal{S}$ defined by (1.6). From Theorem 1, the set $\mathcal{S}$ contains nonnegative solutions (ϕ, ψ) satisfying $\phi \in L^\infty(\mathbb{R}^2)$ and (1.9). For $(\phi, \psi) \in \mathcal{S}$, ϕ and ψ are radially symmetric about the origin, and satisfy $\phi, \psi \in L^1(\mathbb{R}^2)$.

Theorem 2. *The solution set* $\mathcal{S}$ *is written by one parameter families* $(\phi(s), \psi(s))$ *on* $s \in \mathbb{R}$, *that is,* $\mathcal{S} = \{(\phi(s), \psi(s)) : s \in \mathbb{R}\}$. *The solutions* $(\phi(s), \psi(s))$ *and* $\lambda(s) = \|\phi(s)\|_{L^1(\mathbb{R}^2)}$ *satisfy the following properties:*

(i) $s \mapsto (\phi(s), \psi(s)) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ *and* $s \mapsto \lambda(s) \in \mathbb{R}$ *are continuous*;

(ii) $(\phi(s), \psi(s)) \to (0,0)$ *in* $C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ *and* $\lambda(s) \to 0$ *as* $s \to -\infty$;

(iii) $\|\psi(s)\|_{L^\infty(\mathbb{R}^2)} \to \infty$ *as* $s \to \infty$, *and*

$$\lambda(s) \to 8m\pi \quad \textit{and} \quad \phi(s)dx \to 8m\pi\delta_0(dx) \quad \textit{as } s \to \infty \textit{ in the sense of measure}$$

for some integer m *satisfying* $1 \leq m \leq \max\{1, [\pi^2\tau^2/6]\}$, *where* $[a]$ *is the greatest integer not exceeding* a *and* $\delta_0(dx)$ *denotes Dirac's delta function with the support in origin. Moreover,*

$$\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(s)} dy \to \infty \quad \textit{as } s \to \infty;$$

(iv) *Let* $\lambda^* = \sup_{s \in \mathbb{R}} \lambda(s)$. *Then* $8m\pi \leq \lambda^* \leq \max\{4\pi^3/3, 4\pi^3\tau^2/3\}$;

(v) *If* $0 < \tau \leq 1/2$ *then* $0 < \lambda(s) < 8\pi$ *for* $s \in \mathbb{R}$.

As a consequence of Theorem 2 we obtain the following:

Corollary 1. *There exist an integer m and a constant λ^* satisfying*

$$1 \le m \le \max\{1, [\pi^2\tau^2/6]\} \quad \textit{and} \quad 8m\pi \le \lambda^* \le \max\{4\pi^3/3, 4\pi^3\tau^2/3\},$$

respectively, such that

(i) *for every $\lambda \in (0, \lambda^*)$, there exists a solution $(\phi, \psi) \in \mathcal{S}$ satisfying $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$;*

(ii) *for $\lambda > \lambda^*$, there exists no solution $(\phi, \psi) \in \mathcal{S}$ satisfying $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$;*

(iii) *there exists a sequence $(\phi_k, \psi_k) \subset \mathcal{S}$ satisfying $\phi_k dx \to 8m\pi\delta_0(dx)$ as $k \to \infty$ in the sense of measure.*

Moreover, if $0 < \tau \le 1/2$ then $m = 1$ and $\lambda^* = 8\pi$.

The proof of Theorem 2 is based on the ODE arguments. Furthermore, we employ the blow-up analysis by Brezis-Merle [2] and Li-Shafrir [13] to investigate the asymptotic behavior of $(\phi, \psi) \in \mathcal{S}$ as $\|\psi\|_{L^\infty(\mathbb{R}^2)} \to \infty$. To show the upper bounds of the mass of ϕ we use the techniques due to Bilar [1]. Let $\lambda = \|\phi\|_{L^1(\mathbb{R}^2)}$. From (1.7) it follows that

$$\lambda = \sigma \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy.$$

Then (1.8) is rewritten as the elliptic equation with a non-local term

$$\Delta\psi + \frac{\tau}{2} x \cdot \nabla\psi + \lambda e^{-|x|^2/4} e^{\psi} \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy = 0 \tag{1.10}$$

for $x \in \mathbb{R}^2$. This equation plays an important role to investigate the blow-up properties of $(\phi, \psi) \in \mathcal{S}$.

Finally, we obtain the result concerning the existence of solutions to (1.8) with (1.9). This refines the previous results [15, Theorem 1], [16, Theorems 1 and 2], and [17, Theorem 1.1].

Theorem 3. *For any $\tau > 0$ there exists $\sigma^* > 0$ such that*

(i) *if $\sigma > \sigma^*$, then (1.8) with (1.9) has no solution;*

(ii) *if $\sigma = \sigma^*$, (1.8) with (1.9) has at least one solution;*

(iii) *if $0 < \sigma < \sigma^*$, then (1.8) with (1.9) has at least two distinct solutions $\underline{\psi}_\sigma$, $\overline{\psi}_\sigma$ satisfying $\lim_{\sigma\to 0} \underline{\psi}_\sigma(0) = 0$ and $\lim_{\sigma \to 0} \overline{\psi}_\sigma(0) = \infty$.*

Recently, attentions have been paid to blowup problems for the system (1.1) for $(x,t) \in \Omega \times (0, T)$ subject to the boundary and initial condition

$$\begin{cases} \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \quad \text{and} \quad v(x,0) = v_0 & \text{on } \Omega, \end{cases} \tag{1.11}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and ν is the outer normal unit vector. Childress and Percus [4] and Childress [3] have studied the stationary problem and have conjectured that there exists a threshold $||u_0||_{L^1(\Omega)} = 8\pi$ of blowup, that is, if $||u_0||_{L^1(\Omega)} < 8\pi$ then the solution (u, v) exists globally in time, and if $||u_0||_{L^1(\Omega)} > 8\pi$ then $u(x,t)$ can form a delta function singularity in finite time. Their arguments were heuristic, while recent studies are supporting their validity rigorously, see, [21]. We also refer to [9], [19], and [22].

On the other hand, it is asserted that self-similar solutions take an important role for the Cauchy problem for the semilinear parabolic equations on the whole space, see, e.g., [5], [10], and [11]. By the definition, self-similar solutions are global in time, and they are expected to describe large time behavior of general solutions generically. From Corollary 1, we are led to the following conjectures for the Cauchy problem (1.1) with $\gamma = 0$.

For $0 < \tau \leq 1/2$, if $||u_0||_{L^1(\mathbb{R}^2)} < 8\pi$ then the solution of the Cauchy problem exists globally in time, and if $||u_0||_{L^1(\Omega)} > 8\pi$ then the solution can blowup in a finite time.

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