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Global existence of smooth solutions for two dimensional Navier-Stokes equations with nondecaying initial velocity

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1 Introduction

We consider the nonstationary Navier-Stokes equations in the plane.

$$\text{(NS)} \qquad \left\{ \begin{array}{l} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad \text{in } (0,T) \times \mathbb{R}^2, \\ \operatorname{div} u = 0 \quad \text{in } (0,T) \times \mathbb{R}^2, \\ u|_{t=0} = u_0 \quad (\text{with } \operatorname{div} u_0 = 0) \quad \text{in } \mathbb{R}^2, \end{array} \right.$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and unknown scaler of its pressure, while $u_0 = u_0(x) = (u_0^1(x), u_0^2(x))$ is a given initial velocity vector field.

Our goal is to prove the unique existance of global-in-time smooth solution of (NS) when initial velocity u_0 belongs to merely bounded uniformly continious, i.e., $u_0 \in BUC = BUC(\mathbb{R}^2)$.

Theorem 1 *Assume that $u_0 \in BUC$ satisfies $\operatorname{div} u_0 = 0$ in $\mathbb{R}^2$ (in the sense of distribution). Then there exists $u \in C([0,\infty); BUC)$ such that $u(0) = u_0$*

and $(u(t), \nabla p(t))$ *is a unique classical solution of (NS) for* $t > 0$, *provided that* $\nabla p(t) = \sum_{i,j=1}^{2} \nabla R_i R_j u^i(t) u^j(t)$, *where* $R_j = (-\Delta)^{-1/2}\partial/\partial x_j$ *is the Riesz transform.*

We may note that we do not impose any smallness assumptions on u_0 in Theorem 1.

We consider that u_0 belongs to BUC. In this case the initial velocity does not decay at space infinity. For example u_0 can be taken such as a constant or a spatially periodic function.

There is a large literature on local solvability of Navier-Stokes equations even in a various domain of $\mathbb{R}^n$ $(n \geq 2)$. In particular Leray [Le] has already obtained the time global smooth solutions if $u_0 \in L^2(\mathbb{R}^2)$. The method of his proof is based on the energy estimate. The kinematic energy is defined by $||u||_{L^2}^2/2$. This method does not apply directly to our situation because the energy is infinite.

On the other hand the time local solution in our situation is constructed by Cannon-Knightly [CK] in 1970, Cannone [Ca] in 1995, and Giga-Inui-Matsui [GIM] in 1999. They show the time local solvability including higher dimensional problems.

The relation $\nabla p = \sum \nabla R_i R_j u^i u^j$ does not follow from the equations since u may not decay at space infinity. Recently work of Jun Kato [Ka] shows a sufficient condition on p to get this relation.

Remark 1 (Jun Kato) *Assume that the initial data* $u_0 \in BUC(\mathbb{R}^n)$ *satisfying* $\mathrm{div} u_0 = 0$. *Let* (u, p) *is classical solutions with* $u \in L^\infty((0,T) \times \mathbb{R}^n)$ *and* $p \in L^1_{loc}((0,T); BMO(\mathbb{R}^n))$, *where* BMO *denotes the space of bounded mean oscillation functions. Then* $(u(t), \nabla p(t))$ *is a unique for* $t > 0$, *and the relation*

$$\nabla p(t) = \sum_{i,j=1}^{n} \nabla R_i R_j u^i(t) u^j(t)$$

holds.

2 Sketch of the proof of Theorem1

Let us briefly explain main ideas of proving Theorem 1. We use the integral equation;

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)ds,$$

where the matrix operator $\mathbf{P} = (P_{ij})_{i,j=1,2}$, $P_{ij} = \delta_{ij} + R_iR_j$. We denote that δ_{ij} is Kronecker's delta, and $\mathbf{P}$ is formaly the orthogonal projector on divergence-free subspace. $e^{t\Delta}$ is the solution operator of heat equation. We call the solution of (INT) the *mild solution.*

In the literature [CK] and [GIM], the local solution $u \in C([0,T_0]; BUC)$, this T_0 is estimates by

$$T_0 \geq C/||u_0||_\infty^2,$$

where C is a numerical constant.

The main idea is to establish a priori bound for $||u(t)||_\infty$. Once we obtain it, the local solution can be extended globally.

Theorem 2 *Assume that* $u_0 \in BUC$, *and assume that* u *is the mild solution in* $[0,T]$. *Then there exists a positive constant* K *independent of* T *and* u, *such that*

$$\|u(t)\|_\infty \leq K\exp(Ke^{Kt}) \qquad \text{for} \quad t \in [0,T].$$

It is easy to see that Theorem 2 implies Theorem 1.

We give an outline of the proof of Theorem 2. It consists of 3 steps.

(i) Maximum principle for vorticity equation.

(ii) Estimate of bilinear terms.

(iii) Logarithmic type Gronwall inequality.

(Step i) We take rotation to (NS) to get

$$\omega_t - \Delta\omega + (u, \nabla)\omega = 0,$$

where $\omega(t) = \text{curl } u(t)$, which is a scalar function.
Since ω and u are bounded, we can apply the maximum principle for the vorticity equation. We obtain the following inequality:

$$||\omega(t)||_\infty \leq ||\omega_0||_\infty \qquad \text{for} \quad t \in [0, T],$$

where $\omega_0 = \text{curl } u_0$.

It is well-known result that there is a regularizing effect, so that for all $t_0 \in (0, T)$, $\nabla u(t_0) \in BUC$. Thus we may assume that this t_0 is an initial time, so that $||\omega_0||_\infty$ is finite.

We note that in the case of the boundary exists the above estimate is not expected since the vorticity is created near the boundary. In higher dimensional cases without boundary it is not expected to have similar vorticity estimate because of vorticity stretching terms.

(Step ii) It is summarized in the following lemma:

Lemma 1 *There exists a numerical positive constant C such that*

$$||\nabla e^{t\Delta}(f \otimes f)||_\infty \leq C\{(1 + \frac{1}{\sqrt{t}} + \log R)||f||_\infty ||\text{curl} f||_\infty + \frac{1}{R}||f||_\infty^2\},$$

for all $t > 0, R > 1$, and $f \in C^1(\mathbb{R}^2)$; $\text{div} f = 0$.

We refer to the similar estimate holds in higher dimensional cases.

The proof of this lemma is not difficult, but not short. We estimate the Riesz transform by using duality, but we skip the detail.

(Step iii) This step is summarized by the logarithmic type Gronwall inequality:

Lemma 2 *Let nonnegative function $a(t, s)$ be continious in $\{(t, s); 0 \leq s < t \leq T\}$, and satisfies $a(t, \cdot) \in L^1(0, t)$ for $t \in (0, T]$, with some $T > 0$. Assume that there exists a positive constant ϵ_0 and a constant $A \in (0, 1)$ such that*

$$\sup_{0 \leq t \leq T} \int_{t-\epsilon_0}^{t} a(t, s) ds \leq 1 - A.$$

If positive constants $\alpha, \beta > 0$, *and nonnegative continious function* $f \in C([0,T])$ *satisfy that*

$$f(t) \le \alpha + \int_0^t a(t,s) f(s) ds + \beta \int_0^t \{1 + \log(1 + f(s))\} f(s) ds,$$

for $t \in [0,T]$. *Then the following inequality holds;*

$$f(t) \le -1 + \frac{1}{e}\left[(1 + \frac{\alpha}{A})e\right]^{\exp(\frac{\beta+\gamma}{A}t)},$$

for $t \in [0,T]$, *where the constant* γ *is defined by*

$$\gamma = \sup_{0 \le t \le T} \{ \sup_{0 \le s \le t - \epsilon_0} a(t,s) \}.$$

The Gronwall inequality with the logarithmic terms is shown by Wolibner [Wo] in 1933 and Brezis-Gallouet [BG] in 1980. But in our case there is a singular term $a(t,s)$, so ours is quite different from thiers.

Finally we estimate L^∞-norm of the mild solution explicitly.

$$||u(t)||_\infty \le ||e^{t\Delta}u_0||_\infty + \int_0^t ||\nabla e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)||_\infty ds.$$

We use the estimate $||e^{t\Delta}u_0||_\infty \le ||u_0||_\infty$, and by Lemma 1 to obtain

$$||u(t)||_\infty \le ||u_0||_\infty + \int_0^t C\{(1 + \frac{1}{\sqrt{t-s}} + \log R)||u(s)||_\infty ||\omega(s)||_\infty + \frac{1}{R}||u(s)||_\infty^2\} ds,$$

with some positive constant C. We now take $R = 1 + ||u(s)||_\infty$; this choice of R is similar to that of [BG]. By maximum principle for vorticity equation, we have

$$||u(t)||_\infty \le ||u_0||_\infty + C(1 + ||\omega_0||_\infty) \int_0^t \{1 + \frac{1}{\sqrt{t-s}} + \log(1 + ||u(s)||_\infty)\} ||u(s)||_\infty ds.$$

We apply the logarithmic type Gronwall inequality (Lemma 2) to obtain Theorem 2.

References

[BG] Brezis, H. and Gallouet, T. : *Nonlinear Schrödinger evolution equations,* J. Nonlinear Anal. **4**, (1980) 677-681.

[CK] Cannon, J. R. and Knightly, G. H. : *A note on the Cauchy problem for the Navier-Stokes equations,* SIAM J. Appl. Math. **18**, (1984) 641-644.

[Ca] Cannone, M. : *Paraproduits et Navier-Stokes Diderot Editeur,* Arts et Sciences Paris-New York-Amsterdam (1995).

[GIM] Giga, Y. , Inui, K. and Matsui, S. : *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data,* Quaderni di Matematica, **4**, (1999) 28-68.

[Ka] Kato, J. and : *On the uniqueness of the nondecaying solution of the Navier-Stokes equations,* (preprint)

[Le] Leray, J. : *Sur le mouvement d'un liquide visquex emplissant l'espace,* Acta. Math. **63**, (1934) 193-248.

[Wo] Wolibner, W. : *Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompreeible, pendant un temps infiniment long,* Math. Z. **37**, (1933) 698-726.