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Generation and Approximation of Semigroups of Lipschitz Operators *

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Introduction

After a pioneering work by Kōmura [10] the generation theorem of quasi-contractive semigroups in Banach spaces has been studied intensively and applied to the well-posedness of Cauchy problems for porous medium equations, Hamilton-Jacobi equations and scalar first order equations. (Some of the main points of theory of quasi-contractive semigroups have been outlined in a review paper by Crandall [5].) However, Temple [18] showed that the theory of quasi-contractive semigroups could not be in general applied to solve genuinely nonlinear symmetric hyperbolic systems with initial data dense in the whole underlying Banach space based on the space of integrable functions. It is expected that solution operators of the Cauchy problem for first order systems of conservation laws are Lipschitz continuous with respect to L^1 norm. In this case, it is conjectured that such solution operators form a semigroup of Lipschitz operators. In fact, an attempt has recently been made by Bressan, Liu and Yang [1] to prove that a family of solution operators of the Cauchy problem for strictly hyperbolic systems of conservation laws is a semigroup of Lipschitz operators in the space of integrable functions if its domain is defined by the set of integrable functions whose total variation are sufficiently small.

Throughout this paper X denotes a real Banach space with norm $\|\cdot\|$ and D a closed subset of X. By a *semigroup of Lipschitz operators on* D we mean a one-parameter family $\{T(t); t \geq 0\}$ of Lipschitz operators from D into itself satisfying the following three conditions:

(S1) $T(0)x = x$, $T(t)T(s)x = T(t+s)x$ for $x \in D$ and $t, s \geq 0$.

(S2) For each $x \in D$, $T(\cdot)x : [0,\infty) \to X$ is continuous.

(S3) For $\tau > 0$ there exists $M_\tau \geq 1$ such that

$$\|T(t)x - T(t)y\| \leq M_\tau \|x - y\| \qquad \text{for } x, y \in D \text{ and } t \in [0,\tau].$$

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An operator A_0 in X defined by

$$\begin{cases} A_0x = \lim_{h\downarrow 0}(T(h)x - x)/h & \text{for } x \in D(A_0) \\ D(A_0) = \{x \in D; \lim_{h\downarrow 0}(T(h)x - x)/h \quad \text{exists in } X\} \end{cases}$$

is called the *infinitesimal generator* of $\{T(t); t \geq 0\}$.

We are interested in studying a basic property of semigroups of Lipschitz operators and a characterization of infinitesimal generators of such semigroups which are roughly stated as follows:

(i) A nonlinear analogue of Feller's theorem for semigroups of class (C_0) (Theorem 1.1): A semigroup of Lipschitz operators is a quasi-contractive semigroup with respect to a certain metric or metric-like functional.

(ii) A characterization of infinitesimal generators of semigroups of Lipschitz operators (Theorem 1.2): A continuous operator A from D into X is the infinitesimal generator of a semigroup of Lipschitz operators on D if and only if it satisfies the subtangential condition and a general type of dissipative condition that there is a metric-like functional with respect to which A is dissipative.

Our discussion is restricted to a special case in which infinitesimal generators are continuous. However this does not mean that the abstract theory obtained here cannot be applied to any partial differential equations. In fact, in Section 1 we show that the generation theorem of (C_0) semigroups of bounded linear operators can be derived from our theory, and we also give an application of our results to the Cauchy problem for quasi-linear wave equation with damping.

Section 2 contains an approximation of semigroups of Lipschitz operators; namely the problem of approximation of a semigroup of Lipschitz operators by a sequence of discrete parameter semigroups (Theorem 2.1). This was discussed by Trotter [19], Chernoff [3] and Kurtz [11] for semigroups of linear operators. In the case of nonlinear quasi-contractive semigroups, a number of results were obtained by Miyadera and Oharu [17], Brezis and Pazy [2], Kurtz [12], and Miyadera and Kobayashi [16]. Although our discussion is restricted to the special case as mentioned above, the results obtained here are not covered with the results for quasi-contractive semigroups and are applicable to the existence problem of the global solution of the quasi-linear wave equation of Kirchhoff type by using a finite difference scheme of Lax-Friedrichs type. As for the related topics, sufficient conditions for the convergence of Chernoff's formula were obtained by Marsden [14], and an approximation theorem of Lax type for semigroups of Lipschitz operators was obtained by Kobayashi *et al.* [7], which is an improvement of Chorin *et al.* [4,Theorem 5.1].

1. Semigroups of Lipschitz Operators

We begin by stating a nonlinear analogue of Feller's theorem.

Theorem 1.1 ([8,Theorem 4.1]). *Let $\{T(t); t \geq 0\}$ be a one-parameter family of Lipschitz operators from D into itself satisfying two conditions* (S1) *and* (S2). *Then the following statements are mutually equivalent:*

(i) *$\{T(t); t \geq 0\}$ is a semigroup of Lipschitz operators on D.*

(ii) *There exist $M \geq 1$ and $\omega \in \mathbf{R}$ such that*

$$\|T(t)x - T(t)y\| \leq Me^{\omega t}\|x - y\| \quad \text{for } x, y \in D \text{ and } t \geq 0.$$

(iii) *There exist $\omega \in \mathbf{R}$ and a nonnegative and Lipschitz continuous functional V on $X \times X$, satisfying the property*

(V) *there exist $M \geq m > 0$ such that $m\|x - y\| \leq V(x, y) \leq M\|x - y\|$ for $x, y \in D$,*

such that

$$V(T(t)x, T(t)y) \leq e^{\omega t}V(x, y) \quad \text{for } x, y \in D \text{ and } t \geq 0. \tag{1.1}$$

Remark. In general, the functional V on $X \times X$ in Theorem 1.1 cannot be represented as $V(x, y) = N(x - y)$ for $(x, y) \in X \times X$, by using any norm $N(\cdot)$ equivalent to the original norm $\|\cdot\|$. Indeed, let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying the property that there exists $M > 1$ such that $1/M \leq f(r) \leq 1$ for $r \in \mathbf{R}$. The unique solution $u(\cdot; x) \in C^1([0, \infty); \mathbf{R})$ of the Cauchy problem

$$u'(t) = f(u(t)), \qquad u(0) = x \in \mathbf{R}$$

is given by $u(t; x) = g^{-1}(t + g(x))$ where $g(r) = \int_0^r \frac{d\sigma}{f(\sigma)}$ for $r \in \mathbf{R}$. A family $\{T(t); t \geq 0\}$ defined by $T(t)x = u(t; x)$ is a semigroup on $\mathbf{R}$, and (1.1) is satisfied with $\omega = 0$ and a functional V on $\mathbf{R} \times \mathbf{R}$ defined by $V(x, y) = |g(x) - g(y)|$ for $(x, y) \in \mathbf{R} \times \mathbf{R}$. The functional V also satisfies condition (V) of Theorem 1.1.

Now, let us consider the function $f(r) = (1/M + \sqrt{|r|}) \wedge 1$ for $r \in \mathbf{R}$. Then there exists no real number ω such that $|T(t)x - T(t)y| \leq e^{\omega t}|x - y|$ for $x, y \in \mathbf{R}$ and $t \geq 0$, because $\operatorname{sign}(x - y)(f(x) - f(y)) = \sqrt{x}$ for $0 = y \leq x \leq (1 - 1/M)^2$.

A characterization of the continuous infinitesimal generators of semigroups of Lipschitz operators is given by the following theorem which is a generalization of [15,Theorem 5] (see also [13]).

Theorem 1.2 ([8,Theorem 4.2]). *Let A be a continuous operator from D into X. Then A is the infinitesimal generator of a semigroup $\{T(t); t \geq 0\}$ of Lipschitz operators on D if and only if it satisfies the following two conditions.*

(A1) $\liminf_{h\downarrow 0} d(x + hAx, D)/h = 0$ *for all* $x \in D$.

(A2) *There exist* $\omega \in \mathbf{R}$ *and a nonnegative and Lipschitz continuous functional* V *on* $X\times X$ *satisfying property* (V) *of Theorem* 1.1 *such that*

$$D_+V(x,y)(Ax, Ay) \leq \omega V(x,y) \quad \textit{for } x, y \in D,$$

where D_+V *is a directional derivative defined by*

$$D_+V(x,y)(\xi,\eta) = \liminf_{h\downarrow 0}(V(x + h\xi, y + h\eta) - V(x,y))/h \quad \textit{for } (x,y), (\xi,\eta) \in X \times X.$$

In this case, for each $x \in D$ *the abstract Cauchy problem*

$$u'(t) = Au(t) \quad \textit{for } t \geq 0, \qquad \textit{and} \qquad u(0) = x$$

has a unique global solution $u \in C^1([0,\infty); X)$ *given by* $u(t) = T(t)x$ *for* $t \geq 0$.

We explain a way to find a functional V so that (A2) is satisfied, in an abstract fashion. A functional V must be chosen so that the solution operator $T(t)$ is quasi-contractive with respect to V, because the quasi-contractivity of $T(t)$ implies the dissipativity condition (A2). For this purpose, let $x, y \in D$, and assume that there exists a curve c lying in D such that $c(0) = x$ and $c(1) = y$ and that for each $\theta \in [0,1]$, the Cauchy problem

$$\left\{\begin{array}{ll} u'(t;\theta) = Au(t;\theta) & \text{for } t \geq 0, \\ u(0;\theta) = c(\theta) & \end{array}\right. \tag{1.2}$$

has a global "smooth" solution $u(t;\theta)$. Differentiating (1.2) in θ we have

$$\left\{\begin{array}{ll} \dot{u}'(t;\theta) = dA(u(t;\theta))\dot{u}(t;\theta) & \text{for } t \geq 0, \\ \dot{u}(0;\theta) = \dot{c}(\theta), & \end{array}\right.$$

where $dA(w)\xi = \lim_{h\downarrow 0}(A(w + h\xi) - A(w))/h$ and the limit is taken in some sense.

Moreover, we assume that for each $w \in D$, the operator $dA(w)$ generates a quasi-contractive (C_0) semigroup on the Banach space X equipped with norm $\|\cdot\|_w$ depending Lipschitz continuously on w in the sense of [6]; namely there exists a real Banach space E continuously embedded in X such that the set $\{Au; u \in D\}$ is bounded in E, and there exist $L > 0$ and $\beta \geq 0$ such that

$$\left\{\begin{array}{ll} \|u\|_w \leq (1 + L\|w - z\|_E)\|u\|_z & \text{for } u \in X, \text{ and } w, z \in D, \\ [u, dA(w)u]_w \leq \beta\|u\|_w & \text{for } u \in D(dA(w)) \text{ and } w \in D \end{array}\right.$$

where

$$[u,\xi]_w = \lim_{h\downarrow 0}(\|u\|_w - \|u-h\xi\|_w)/h \quad \text{for } u,\xi \in X \text{ and } w \in D.$$

By using the Lipschitz continuity of $\|\cdot\|_w$ with respect to w, we have

$$\begin{aligned}(d/dt)\|\dot{u}(t;\theta)\|_{u(t;\theta)} &\leq [\dot{u}(t;\theta),\dot{u}'(t;\theta)]_{u(t;\theta)} + L\|Au(t;\theta)\|_E\|\dot{u}(t;\theta)\|_{u(t;\theta)} \\ &\leq (\beta + L\|Au(t;\theta)\|_E)\|\dot{u}(t;\theta)\|_{u(t;\theta)}.\end{aligned}$$

Since the set $\{Au; u \in D\}$ is bounded in E, we have

$$\|\dot{u}(t;\theta)\|_{u(t;\theta)} \leq e^{\omega t}\|\dot{c}(\theta)\|_{c(\theta)} \quad \text{for } t \geq 0.$$

Therefore, the family $\{T(t); t \geq 0\}$ is quasi-contractive with respect to the non-negative functional V defined by

$$V(x,y) = \inf_{\text{path}}\left\{\int_0^1 \|\dot{c}(\theta)\|_{c(\theta)}; c(0)=x, c(1)=y\right\};$$

namely

$$V(T(t)x, T(t)y) \leq e^{\omega t}V(x,y) \qquad \text{for } x,y \in D \text{ and } t \geq 0.$$

We shall give two applications of Theorem 1.2.

Application 1. It will be shown that the generation theorem of contractive (C_0) semigroups can be derived from Martin's result [15,Theorem 5] which is a special case of Theorem 1.2.

If A_0 is a densely defined linear operator in X satisfying the Hille-Yosida condition

$$\|(I-\lambda A_0)^{-1}\| \leq 1 \quad \textit{for } \lambda > 0, \tag{1.3}$$

then the following assertions hold.

(i) *If D is defined as the closure of the set $D_0 = \{x \in D(A_0^2); \|A_0^2 x\| \leq r\}$ where $r > 0$, then there exists a Hölder continuous operator A from D into X such that $Ax = A_0 x$ for $x \in D_0$.*

(ii) *The limit $S(t)x = \lim_{\lambda\downarrow 0}(I-\lambda A_0)^{-[t/\lambda]}x$ exists in X, for every $x \in X$ and $t \geq 0$.*

(iii) *The family $\{S(t); t \geq 0\}$ is a contractive (C_0) semigroup on X whose infinitesimal generator is A_0.*

We show that Theorem 2.1 is applicable to prove assertion (ii). To this end, let $\lambda > 0$ and $x \in D(A_0^2)$. Since $(I-\lambda A_0)^{-1}z = z + \lambda(I-\lambda A_0)^{-1}A_0 z$ for $z \in D(A_0)$, we have $(I-\lambda A_0)^{-1}x = x + \lambda(A_0 x + \lambda(I-\lambda A_0)^{-1}A_0^2 x)$, and so

$$\lambda A_0 x = (I-\lambda A_0)^{-1}x - x - \lambda^2(I-\lambda A_0)^{-1}A_0^2 x.$$

We estimate this identity by (1.3). This yields $\lambda\|A_0x\| \le 2\|x\| + \lambda^2\|A_0^2x\|$, which implies $\|A_0x\|^2 - 8\|x\|\|A_0^2x\| \le 0$. We therefore obtain the generalized Landau inequality

$$\|A_0x\| \le 2\sqrt{2}\|A_0^2x\|^{1/2}\|x\|^{1/2} \quad \text{for } x \in D(A_0^2). \tag{1.4}$$

Now, to prove (i) let $r > 0$ and let D be the closure of D_0. Then we have by (1.4)

$$\|A_0x - A_0y\| \le 4\sqrt{r}\|x - y\|^{1/2} \quad \text{for } x, y \in D_0.$$

The operator A from D into X, constructed in the way that $Ax = \lim_{n\to\infty} A_0x_n$ if $x \in D$ and $x_n \in D_0$ satisfy $\lim_{n\to\infty} x_n = x$, is Hölder continuous. This means that assertion (i) is true. To prove (ii), we first show that

(1.5) $(I - \lambda A_0)^{-1}x \in D$ and $A(I - \lambda A_0)^{-1}x = A_0(I - \lambda A_0)^{-1}x$ for $x \in D$ and $\lambda > 0$,

(1.6) $\lim_{\lambda\downarrow 0} A_0(I - \lambda A_0)^{-1}x = Ax$ for $x \in D$.

To demonstrate (1.5), let $\lambda > 0$ and $x \in D$. Then there exists a sequence $\{x_n\}$ in D_0 such that $\lim_{n\to\infty} x_n = x$. By the definition of D_0 we have $(I - \lambda A_0)^{-1}x_n \in D(A_0^2)$ and

$$\|A_0^2(I - \lambda A_0)^{-1}x_n\| = \|(I - \lambda A_0)^{-1}A_0^2x_n\| \le \|A_0^2x_n\| \le r$$

for $n \ge 1$, which implies that $(I - \lambda A_0)^{-1}x_n \in D_0$ for $n \ge 1$. Since $\lim_{n\to\infty}(I - \lambda A_0)^{-1}x_n = (I - \lambda A_0)^{-1}x$, we have $(I - \lambda A_0)^{-1}x \in D$ and $A(I - \lambda A_0)^{-1}x = \lim_{n\to\infty} A_0(I - \lambda A_0)^{-1}x_n = A_0(I - \lambda A_0)^{-1}x$. Assertion (1.6) is derived from the continuity of A from D into X, (1.5) and $\lim_{\lambda\downarrow 0}(I - \lambda A_0)^{-1}x = x$ for $x \in D$.

The subtangential condition (A1) follows from (1.5) and (1.6), because $d(x + hAx, D)/h \le \|x + hAx - (I - hA_0)^{-1}x\|/h = \|A_0(I - hA_0)^{-1}x - Ax\| \to 0$ as $h \downarrow 0$. Since

$$\begin{aligned}
&(\|x + hAx - (y + hAy)\| - \|x - y\|)/h \\
&\quad \le (\|(I - hA_0)^{-1}x - (I - hA_0)^{-1}y\| - \|x - y\|)/h \\
&\qquad + \|x + hAx - (I - hA_0)^{-1}x\| + \|y + hAy - (I - hA_0)^{-1}y\|,
\end{aligned}$$

it follows from (1.3) that (A2) is satisfied with $V(x, y) = \|x - y\|$ and $\omega = 0$. Therefore, we deduce from Theorem 1.2 that A is the infinitesimal generator of a contractive semigroup $\{T(t); t \ge 0\}$ on D.

Now, we turn to the proof of assertion (ii). Let $\lambda > 0$ and $x \in D$. For simplicity, set $x_i^\lambda = (I - \lambda A_0)^{-i}x$ for $i = 1, 2, \ldots$. Then we have by (1.5)

$$(x_i^\lambda - x_{i-1}^\lambda)/\lambda = Ax_i^\lambda$$

for $i = 1, 2, \ldots$. Since $(d/dt)T(t)x = AT(t)x$ for $t \ge 0$, we have

$$(T(i\lambda)x - T((i-1)\lambda)x)/\lambda = AT(i\lambda)x + \varepsilon_i^\lambda,$$

where

$$\varepsilon_i^\lambda = \frac{1}{\lambda}\int_{(i-1)\lambda}^{i\lambda} (AT(r)x - AT(i\lambda)x)\,dr$$

for $i = 1, 2, \ldots$. By the dissipativity of A we have

$$(\|x_i^\lambda - T(i\lambda)x\| - \|x_i^\lambda - T(i\lambda)x - \lambda(Ax_i^\lambda - AT(i\lambda)x)\|)/\lambda \le 0,$$

which implies that $\|x_i^\lambda - T(i\lambda)x\| \le \lambda \sum_{k=1}^{i} \|\varepsilon_k^\lambda\|$ for $i = 0, 1, 2, \ldots$. Since $AT(\cdot)x : [0,\infty) \to X$ is continuous, it is concluded that $\lim_{\lambda\downarrow 0}(I - \lambda A_0)^{-[t/\lambda]}x = T(t)x$ uniformly on every compact subinterval of $[0,\infty)$. Assertion (ii) is proved by a density argument, since $D(A_0^2)$ is dense in X.

Application 2. We next give an application of Theorem 1.2 to the Cauchy problem for quasi-linear wave equation with damping

$$\begin{cases} u_t = v_x \\ v_t = \varphi'(u)_x - \nu v, \end{cases} \tag{1.7}$$

where $\nu > 0$ and $\varphi \in C^4(\mathbf{R})$ satisfies $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(r) \ge c_0 > 0$ for $r \in \mathbf{R}$.

Remark. The problem of existence and uniqueness of global solutions of (1.7) has been studied in different ways, by several authors (for example, see [20]). Our approach is operator-theoretic and based on Theorem 1.2.

Theorem 1.3 ([8,Theorem 5.1]). *There is an $r_0 > 0$ such that for each $(u_0, v_0) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ with $\|(u_0, v_0)\|_{H^2\times H^2} \le r_0$, the problem (1.7) has a unique solution (u, v) in the class*

$$C^1([0,\infty); L^2(\mathbf{R}) \times L^2(\mathbf{R})) \cap L^\infty(0,\infty; H^2(\mathbf{R}) \times H^2(\mathbf{R}))$$

satisfying the initial condition $(u,v)|_{t=0} = (u_0, v_0)$. Moreover, there exist constants $M \ge 1$ and $\omega \ge 0$ such that if (u,v) and $(\hat{u}, \hat{v})$ are solutions with initial data (u_0, v_0) and $(\hat{u}_0, \hat{v}_0) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ satisfying $\|(u_0, v_0)\|_{H^2\times H^2} \le r_0$ and $\|(\hat{u}_0, \hat{v}_0)\|_{H^2\times H^2} \le r_0$ respectively, then we have

$$\|(u(t,\cdot), v(t,\cdot)) - (\hat{u}(t,\cdot), \hat{v}(t,\cdot))\|_{L^2\times L^2} \le Me^{\omega t}\,\|(u_0, v_0) - (\hat{u}_0, \hat{v}_0)\|_{L^2\times L^2} \qquad \textit{for } t \ge 0.$$

Now, let X be a real Hilbert space $L^2(\mathbf{R}) \times L^2(\mathbf{R})$ with norm $\|(u,v)\| = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$. It is shown that a nonnegative functional V on $X \times X$ defined by

$$V((u,v),(\hat{u},\hat{v})) = \left(\int_{-\infty}^{\infty} \left(\int_u^{\hat{u}} \sqrt{\varphi''(r)}\,dr\right)^2 + (\hat{v} - v)^2\,dx\right)^{1/2}$$

is Lipschitz continuous and satisfies property (V). We use a functional H defined by

$$H(u,v) = \tfrac{1}{2}\int_{-\infty}^{\infty} v^2 + |\nu u + \partial_x v|^2 + |\nu \partial_x u + \partial_x^2 v|^2 \, dx \\ + \tfrac{1}{2}\int_{-\infty}^{\infty} \varphi''(u)(|\partial_x u|^2 + |\partial_x^2 u|^2)\, dx + \int_{-\infty}^{\infty} \varphi(u)\, dx$$

for $(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$.

Theorem 1.3 is deduced from the following theorem by virtue of Theorem 1.2.

Theorem 1.4 ([8,Theorem 5.2]). *There is an $r_0 > 0$ such that a nonlinear operator A in X defined by*

$$\begin{cases} A(u,v) = (\partial_x v, \partial_x \varphi'(u) - \nu v) & \text{for } (u,v) \in D \\ D = \{(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R}); H(u,v) \le r_0\} \end{cases}$$

satisfies the following four properties:

(a) *The set D is closed in X.*

(b) *The operator $A : D \to X$ is continuous.*

(c) *There exists $\omega \in \mathbf{R}$ such that A is ω-dissipative with respect to V; namely*

$$D_+ V\left((u,v),(\hat{u},\hat{v})\right)\left(A(u,v), A(\hat{u},\hat{v})\right) \le \omega V\left((u,v),(\hat{u},\hat{v})\right)$$

for $(u,v), (\hat{u},\hat{v}) \in D$.

(d) $\liminf_{\lambda \downarrow 0} d\left((u,v) + \lambda A(u,v), D\right)/\lambda = 0$ *for $(u,v) \in D$.*

We explain only the way to choose a number r_0 and to find a functional V so that four properties (a) through (d) of Theorem 1.4.

Suppose that the problem (1.7) has a global smooth solution $(u(t,\cdot), v(t,\cdot))$. A number $r_0 > 0$ must be chosen such that $(u(t,\cdot), v(t,\cdot)) \in D$ for $t \ge 0$, since the subtangential condition (A1) is deduced from this property. For this purpose, it is first proved that there exist $c > 0$ and a continuous function ρ satisfying $\rho(0) = 0$ such that

$$(d/dt)H(u(t,\cdot), v(t,\cdot)) + \{c - \rho(H(u(t,\cdot), v(t,\cdot)))\}H(u(t,\cdot), v(t,\cdot)) \le 0$$

for $t \ge 0$. The property that

$$(u(t,\cdot), v(t,\cdot)) \in D \qquad \text{for } t \ge 0$$

is satisfied, if $r_0 > 0$ is chosen such that $H(u,v) \le r_0$ implies $\rho(H(u,v)) < c$. Indeed, if $(u_0, v_0) \in D$ then

$$(d/dt)H(u(t,\cdot), v(t,\cdot)) \le 0$$

for $t \ge 0$; hence $H(u(t,\cdot), v(t,\cdot)) \le H(u_0, v_0) \le r_0$, namely $(u(t,\cdot), v(t,\cdot)) \in D$ for $t \ge 0$.

According to the abstract idea mentioned before, let us find a functional V associated with the differential equation (1.7). For this purpose, let us consider the equation

$$\begin{cases} \dot{u}_t = \dot{v}_x \\ \dot{v}_t = (\varphi''(u)\dot{u})_x - \nu\dot{v}, \end{cases}$$

where $\dot{u}$ denotes the derivative of u with respect to θ. Note that the set D is bounded in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$, and so the set $\{A(u,v); (u,v) \in D\}$ is bounded in $E := H^1(\mathbf{R}) \times H^1(\mathbf{R})$. For $(w,z) \in D$, let us define a linear operator $dA(w,z)$ in $L^2(\mathbf{R}) \times L^2(\mathbf{R})$ by

$$dA(w,z)(u,v) = (v_x, (\varphi''(w)u)_x - \nu v) \quad \text{for } (u,v) \in H^1(\mathbf{R}) \times H^1(\mathbf{R}).$$

Then, the operator $dA(w,z)$ generates a contractive (C_0) semigroup on X equipped with the norm defined by

$$\|(u,v)\|_{(w,z)} = \left(\int_{-\infty}^{\infty} \varphi''(w)u^2 + v^2\,dx\right)^{1/2}.$$

Since

$$\|(u,v)\|_{(w,z)} - \|(u,v)\|_{(\hat{w},\hat{z})} \le \left(\int_{-\infty}^{\infty} \left(\sqrt{\varphi''(w)} - \sqrt{\varphi''(\hat{w})}\right)^2 u^2\,dx\right)^{1/2}$$

by Minkowski's inequality, we have by the boundedness of D in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$

$$\|(u,v)\|_{(w,z)} - \|(u,v)\|_{(\hat{w},\hat{z})} \le L\|(u,v)\|_{(\hat{w},\hat{z})}\|w - \hat{w}\|_{H^1},$$

which implies that the norm $\|(u,v)\|_{(w,z)}$ depends Lipschitz continuously on (w,z) with $E = H^1(\mathbf{R}) \times H^1(\mathbf{R})$. Therefore, the functional $V((u,v),(\hat{u},\hat{v}))$ given by

$$\inf_{\text{path}} \left\{\int_0^1 \left(\int_{-\infty}^{\infty} \varphi''(u)\dot{u}^2 + \dot{v}^2\,dx\right)^{1/2} d\theta\right\}$$

is a desired one. By an easy computation we find

$$\begin{aligned}
&\int_0^1 \left(\int_{-\infty}^{\infty} \varphi''(u)\dot{u}^2 + \dot{v}^2\,dx\right)^{1/2} d\theta \\
&= \int_0^1 \left(\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial\theta}\int_0^u \sqrt{\varphi''(r)}dr\right)^2 + \dot{v}^2 dx\right)^{1/2} d\theta \\
&\ge \left(\int_{-\infty}^{\infty} \left(\int_0^1 \frac{\partial}{\partial\theta}\left(\int_0^u \sqrt{\varphi''(r)}dr\right) d\theta\right)^2 + \left(\int_0^1 \dot{v}d\theta\right)^2 dx\right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty} \left(\int_{u(0)}^{u(1)} \sqrt{\varphi''(r)}dr\right)^2 + (v(1) - v(0))^2\,dx\right)^{1/2}.
\end{aligned}$$

On the other hand, the function $\psi(r) := \int_0^r \sqrt{\varphi''(s)}\, ds$ is strictly increasing and satisfies $\lim_{r\to\infty}\psi(r) = \infty$ and $\lim_{r\to-\infty}\psi(r) = -\infty$, because $\varphi''(r) \geq c_0 > 0$ for $r \in \mathbf{R}$. Now, let $u_0, u_1 \in L^2(\mathbf{R})$ and define $u(x,\theta)$ by

$$\psi(u(x,\theta)) = \psi(u_0(x)) + \theta(\psi(u_1(x)) - \psi(u_0(x))) \qquad \text{for } \theta \in [0,1].$$

Then we have $\psi'(u)\dot{u} = \psi(u_1) - \psi(u_0)$, namely

$$\sqrt{\varphi''(u)}\dot{u} = \int_{u_0}^{u_1} \sqrt{\varphi''(r)}\, dr.$$

For $v_0, v_1 \in L^2(\mathbf{R})$, the function v defined by $v(x,\theta) = v_0(x) + \theta(v_1(x) - v_0(x))$ satisfies $\dot{v} = v_1 - v_0$. Under an appropriate condition, the desired functional V is thus given by

$$V((u,v),(\hat{u},\hat{v})) = \left(\int_{-\infty}^{\infty} \left(\int_{\hat{u}}^{u} \sqrt{\varphi''(r)}dr\right)^2 + (v-\hat{v})^2 dx\right)^{1/2}.$$

2. Approximation of Semigroups of Lipschitz Operators

The space X is assumed to be approximated by a sequence $\{X_n\}$ of Banach spaces in the sense that for each n there exists $P_n \in B(X, X_n)$ such that

$$\lim_{n\to\infty} \|P_n x\|_n = \|x\| \qquad \text{for } x \in X. \tag{2.1}$$

This notion was introduced by Trotter [19] (see also [11]). Note that X is a function space and X_n is a space of sequences which is identified with a space of discrete functions defined only at certain grid points, in most applications.

Moreover, the set D is assumed to be approximated by a sequence $\{D_n\}$ of sets where D_n is closed in X_n for $n \geq 1$, in the following sense:
For any $x \in D$, there exist $x_n \in D_n$ such that $\lim_{n\to\infty} \|x_n - P_n x\|_n = 0$.

Note that there is no difficulty in verifying this assumption because the property that

$$P_n(D) \subset D_n$$

holds for $n \geq 1$, in most applications.

An approximation of a semigroup of Lipschitz operators by a sequence of discrete parameter semigroups is discussed. An approximation theorem of Lipschitz operators is given by the following theorem.

Theorem 2.1 ([9,Theorem 4.5]). *Let $\{h_n\}$ be a null sequence of positive numbers. For each $n \geq 1$, let C_n be a Lipschitz operator from D_n into itself satisfying the stability condition*

$$\|C_n^l x - C_n^l y\|_n \leq M e^{\omega h_n l}\|x - y\|_n$$

for $x, y \in D_n$ and $l = 1, 2, \ldots$, where $M \geq 1$ and $\omega \geq 0$ are independent of n.

Assume that A is a continuous operator from D into X satisfying the property

$$\text{if } x_n \in D_n,\ x \in D \text{ and } \lim_{n\to\infty}\|x_n - P_n x\|_n = 0 \text{ then } \lim_{n\to\infty}\|(C_n x_n - x_n)/h_n - P_n A x\|_n = 0,$$

and the subtangential condition

$$\liminf_{h\downarrow 0} d(x + hAx, D)/h = 0 \qquad \text{for } x \in D.$$

Then A is the infinitesimal generator of a semigroup $\{T(t); t \geq 0\}$ of Lipschitz operators on D. Moreover, for $x_n \in D_n$ and $x \in D$ with $\lim_{n\to\infty}\|x_n - P_n x\|_n = 0$, we have

$$\lim_{n\to\infty}\|C_n^{[t/h_n]}x_n - P_n T(t)x\|_n = 0,$$

and the convergence is uniform on every compact subinterval of $[0, \infty)$. Here $[r]$ denotes the integer part of $r \geq 0$.

The existence problem of the global solution of the following quasi-linear wave equation of Kirchhoff type is discussed by using a finite difference scheme of Lax-Friedrichs type, as an application of Theorem 2.1.

$$(2.2) \qquad \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \beta'(\|u\|_{L^2}^2)\partial_x u - \nu v \\ u|_{t=0} = u_0, \qquad v|_{t=0} = v_0. \end{cases}$$

Here $\nu > 0$ and $\beta \in C^2([0,\infty); [0,\infty))$ is assumed to be a *convex* function satisfying $\beta(0) = 0$ and $\beta'(s) \geq m_0 > 0$ for $s \geq 0$.

For this purpose, let $\{h_n\}$ and $\{k_n\}$ be two null sequences of positive numbers such that $h_n/k_n = r$, where r is an appropriate positive constant determined later.

Let us consider a difference scheme of Lax-Friedrichs type

$$(2.3) \qquad \begin{cases} (u_{l+1,i} - (u_{l,i+1} + u_{l,i-1})/2)/h_n = (v_{l,i+1} - v_{l,i-1})/2k_n, \\ (v_{l+1,i} - (v_{l,i+1} + v_{l,i-1})/2)/h_n \\ \quad = \beta'(\|\{u_{l+1,i}\}\|_n^2)(u_{l,i+1} - u_{l,i-1})/2k_n - \nu(v_{l,i+1} + v_{l,i-1})/2, \\ u_{0,i} = \frac{1}{k_n}\int_{(i-1/2)k_n}^{(i+1/2)k_n} u_0(x)\,dx, \qquad v_{0,i} = \frac{1}{k_n}\int_{(i-1/2)k_n}^{(i+1/2)k_n} v_0(x)\,dx \end{cases}$$

for $l = 0, 1, 2, \ldots$ and $i = 0, \pm 1, \pm 2, \ldots$, where the symbol $\|\cdot\|_n$ is the norm in l^2 defined by

$$\|u\|_n = \left(\sum_{i=-\infty}^{\infty} u_i^2 k_n\right)^{1/2}.$$

Let $R > 0$ and set $E_R = \beta(R) + \beta'(R)R$, and choose $r > 0$ such that

$$r \cdot \sup\{\sqrt{\beta'(\xi)}; \xi \in [0, E_R/m_0]\} < 1. \tag{2.4}$$

Let X_n be a real Banach space $l^2 \times l^2$ equipped with the norm defined by

$$\|(u, v)\|_n = (\|u\|_n^2 + \|v\|_n^2)^{1/2}.$$

It is well-known that $X = L^2(\mathbf{R}) \times L^2(\mathbf{R})$ is approximated by a sequence of $\{X_n\}$ in the sense of (2.1), by considering the operator P_n defined by

$$P_n(u, v) = \left\{\left(\frac{1}{k_n}\int_{(i-1/2)k_n}^{(i+1/2)k_n} u(x)\, dx, \frac{1}{k_n}\int_{(i-1/2)k_n}^{(i+1/2)k_n} v(x)\, dx\right)\right\}.$$

We begin by showing that a family of solution operators for the difference scheme of Lax-Friedrichs type forms a discrete parameter semigroup on a certain closed set satisfying the stability condition. Rearranging the equation (2.3) we have

$$\begin{cases} u_{l+1,i} = (u_{l,i+1} + u_{l,i-1})/2 + r(v_{l,i+1} - v_{l,i-1})/2, \\ v_{l+1,i} = (v_{l,i+1} + v_{l,i-1})/2 + r\beta'(\|\{u_{l+1,i}\}\|_n^2)(u_{l,i+1} - u_{l,i-1})/2 - \nu h_n(v_{l,i+1} + v_{l,i-1})/2. \end{cases}$$

We now consider a mapping C_n from D_n into X_n by the following relation: $(w, z) = C_n(u, v)$ if and only if

$$\begin{cases} w_i = (u_{i+1} + u_{i-1})/2 + r(v_{i+1} - v_{i-1})/2, \\ z_i = (v_{i+1} + v_{i-1})/2 + r\beta'(\|w\|_n^2)(u_{i+1} - u_{i-1})/2 - \nu h_n(v_{i+1} + v_{i-1})/2, \end{cases}$$

for $i = 0, \pm 1, \pm 2, \ldots$. Here D_n is a closed subset of X_n of the form

$$D_n = \{(u, v); H_n(u, v) \le s\},$$

where $H_n(u, v)$ is defined by

$$H_n(u, v) = \|u\|_n^2 + \|\delta_n^0 u\|_n^2 + \|\delta_n^0\delta_n^0 u\|_n^2 + \frac{1}{\beta'(\|u\|_n^2)}(\|v\|_n^2 + \|\delta_n^0 v\|_n^2 + \|\delta_n^0\delta_n^0 v\|_n^2)$$

where $\delta_n^0 u = \{(u_{i+1} - u_{i-1})/2k_n\}$. The number $s > 0$ can be chosen such that the operator C_n maps D_n into *itself* and that the discrete semigroup $\{C_n^l; l = 0, 1, 2, \ldots\}$ is stable in the following sense, by condition (2.4): There exist $M \geq 1$ and $\omega \geq 0$ independent of n such that

$$\|C_n^l(u,v) - C_n^l(\hat{u},\hat{v})\|_n \leq Me^{\omega h_n l}\|(u,v) - (\hat{u},\hat{v})\|_n$$

for $(u,v), (\hat{u},\hat{v}) \in X_n$, $n = 1,2,\ldots$ and $l = 1,2,\ldots$.

The operator A in X defined by

$$\begin{cases} A(u,v) = (\partial_x v, \beta'(\|u\|_{L^2}^2)\partial_x u - \nu v) & \text{for } (u,v) \in D \\ D = \{(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R}); H(u,v) \leq r_0\} \end{cases}$$

is continuous on D, by using the Landau inequality. Here $H(u,v)$ is defined by

$$H(u,v) = \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \frac{1}{\beta'(\|u\|_{L^2}^2)}(\|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2)$$

for $(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$. The number $r_0 > 0$ can be chosen such that $P_n(D) \subset D_n$ for all $n \geq 1$ and that the subtangential condition is satisfied.

Consequently, it is deduced from Theorem 2.1 that the solution which are computed recursively by Lax-Friedrichs scheme (2.3) converges to the smooth solution of Kirchhoff equation (2.2).

Theorem 2.2 ([9,Theorem 5.4]). *There exists an $r_0 > 0$ such that the following assertions hold for each $(u_0, v_0) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ satisfying $\|(u_0,v_0)\|_{H^2\times H^2} \leq r_0$:*

(i) *Problem (2.2) has a unique solution $(u(t), v(t))$ in the class $C^1([0,\infty); L^2(\mathbf{R})\times L^2(\mathbf{R}))$ satisfying the initial condition $(u(0), v(0)) = (u_0, v_0)$.*

(ii) *The solution $(u(t), v(t))$ is approximated by a sequence of solutions $\{(\{u_{l,i}\}, \{v_{l,i}\})\}$ of (2.3) in the sense that*

$$\lim_{n\to\infty} \|P_n(u(t,\cdot), v(t,\cdot)) - \{(u_{[t/h_n],i}, v_{[t/h_n],i})\}\|_n = 0$$

uniformly on every compact subinterval of $[0,\infty)$, where P_n is defined by

$$P_n(u,v) = \left\{\left(\tfrac{1}{k_n}\int_{(i-1/2)k_n}^{(i+1/2)k_n} u(x)\,dx, \tfrac{1}{k_n}\int_{(i-1/2)k_n}^{(i+1/2)k_n} v(x)\,dx\right)\right\}.$$

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