数理解析研究所講究録
1198 巻 2001 年 108-116

周期的領域に対する数値等角写像
Numerical Conformal Mapping of Periodic Structure Domains

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Abstract

In this paper, we propose a numerical conformal mapping of periodic structure domains onto periodic parallel slit domains. The method presented here is obtained by extending Amano's method of numerical conformal mapping based on the charge simulation method. Some numerical examples show that the method presented here is efficient. We also apply our method to the analysis of potential flow past obstacles in a periodic array.

1 Introduction

Conformal mapping is a basic problem in complex analysis and is important in applications to science and engineering, for example, the analysis of two-dimensional potential flow, electromagnetic field, and so on. But the exact solution of comformal mapping is known for few cases. Therefore computational method of conformal mappings, that is numerical conformal mapping, has been an attractive problem in numerical analysis. See Henrici[5], Kythe[6], Nehari[7] and Trefethen[8] for surveys of numerical conformal mappings.

Amano et al. [1, 2, 3] proposed a numerical conformal mapping based on the charge simulation method, which is a fast solver for potential problems. In the method, the problem of numerical conformal mapping is reduced to the one of approximating the mapping function, which is expressed by using the charge simulation method, i.e., the approximate mapping function is expressed by using a linear combination of complex logarithmic potentials

$$\sum_{j=1}^{N} Q_j \log(z-\zeta_j), \quad Q_j \in \mathbb{R},\ \zeta_j \in \mathbb{C}\ (\, j=1,2,\ldots,N\,). \tag{1}$$

The method was applied to the conformal mappings of multiply connected domains onto various slit domains and shown to be very efficient from some numerical experiments. An application to the analysis of potential flows was also presented[4].

In this paper, by extending the above method, we propose a numerical conformal mapping of periodic structure domains onto periodic parallel slit domains (See Figure 1). In the method presented here, the problem is reduced to the one of approximating a periodic analytic function, which is approximated by a linear combination of periodic logarithmic potentials

$$\sum_{j=1}^{N} Q_j \log\sin\left[\frac{\pi}{a}(z-\zeta_j)\right], \quad a>0,\ Q_j \in \mathbb{R},\ \zeta_j \in \mathbb{C}\ (\, j=1,2,\ldots,N\,). \tag{2}$$

Some numerical experiments show that the method presented here is very efficient. We

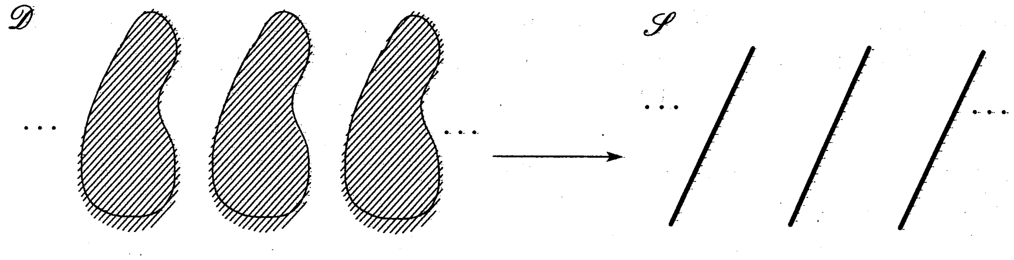


Figure 1: Conformal mapping of the periodic structure domain $\mathscr{D}$ onto the periodic parallel slit domain $\mathscr{S}$.

also apply our method to the analysis of potential flows past obstacles in a periodic array.

In section 2, we prepare some mathematical notations. In section 3, we propose our method for the numerical conformal mapping of periodic structure domains. In section 4, we show some numerical examples and applications to the analysis of potential flow. In section 5, we conclude this paper and refer to future problems.

2 Notations

First we define exactly the periodic structure domain and the periodic parallel slit domain. Let a be a positive constant.

Let D_0 be a domain surrounded by a closed Jordan curve in $z(= x + \mathrm{i}y)$-plane and D_m ($m \in \mathbb{Z}$) the domain defined by

$$D_m = \{ z + ma \mid z \in D_0 \} \tag{3}$$

which satisfy

$$\overline{D_m} \cap \overline{D_l} = \emptyset \quad (m \neq l). \tag{4}$$

The periodic structure domain $\mathscr{D}$ is defined as the exterior to the domains D_m ($m \in \mathbb{Z}$).

$$\mathscr{D} = \mathbb{C} \setminus \left\{ \bigcup_{m \in \mathbb{Z}} \overline{D_m} \right\}. \tag{5}$$

Let φ be an angle such that $-\pi/2 < \varphi \leq \pi/2$, w_0 a point in $w(= u + \mathrm{i}v)$-plane, d a positive constant and S_m ($m \in \mathbb{Z}$) the rectilinear slit defined by

$$S_m = \left\{ w_0 + ma + tde^{\mathrm{i}\varphi} \mid 0 \leq t \leq 1 \right\}. \tag{6}$$

The periodic parallel slit domain $\mathscr{S}$ is defined as the exterior to the slits S_m ($m \in \mathbb{Z}$).

$$\mathscr{S} = \mathbb{C} \setminus \left\{ \bigcup_{m \in \mathbb{Z}} S_m \right\}. \tag{7}$$

Our problem is to find numerically a conformal mapping $f : \mathscr{D} \to \mathscr{S}$. From the periodic structure of the domains, the mapping function is expected to satisfy the periodicity $f(z+a) = f(z) + a \quad (\, z \in \mathscr{D}\,)$. In fact, for a given periodic structure domain $\mathscr{D}$ and an angle $\varphi\,(-\pi/2 < \varphi \leq \pi/2)$, there exist a periodic parallel slit domain $\mathscr{S}$ and a conformal mapping $f : \mathscr{D} \to \mathscr{S}$ which satisfies the following properties.

- **(C1)** (boundary condition) $f(\partial D_m) = S_m \quad (\, m \in \mathbb{Z}\,)$
- **(C2)** (periodicity) $f(z+a) = f(z) + a \quad (\, z \in \mathscr{D}\,)$
- **(C3)** (asymptotic condition) $f(z) = z + \mathrm{O}(1) \quad (\,\mathrm{Re}\, z \text{ fixed, } \mathrm{Im}\, z \to \pm\infty\,)$

We are concerned with a conformal mapping $f : \mathscr{D} \to \mathscr{S}$ satisfying the properties **(C1)**, **(C2)** and **(C3)**.

3 Numerical Conformal Mapping

In this section, we propose a numerical conformal mapping of the periodic structure domain $\mathscr{D}$ onto the periodic parallel slit domain $\mathscr{S}$ by using the charge simulation method.

The mapping function $f(z)$ of the conformal mapping $\mathscr{D} \to \mathscr{S}$ is an analytic function in the domain $\mathscr{D}$. Thus the problem of the numerical conformal mapping is equivalent to the one of approximating the function $f(z)$ analytic in $\mathscr{D}$.

We write the mapping function as

$$f(z) = z + \mathrm{e}^{\mathrm{i}(\varphi\quad\pi/2)}\quad(z), \tag{8}$$

where $\quad(z)$ is a function analytic in $\mathscr{D}$ and periodic with the period a. We approximate $\quad(z)$ by the charge simulation method.

From the viewpoint of function approximation, we can say that the charge simulation method is the method of approximating an analytic function by a linear combination of complex logarithmic potentials

$$\sum_{j=1}^{N} Q_j \log(z - \zeta_j), \quad Q_j \in \mathbb{R},\ \zeta_j \in \mathbb{C}\ (\, j = 1, 2, \ldots, N\,). \tag{9}$$

In our case, $\quad(z)$ is a periodic function of the period a from the property **(C1)**.

$$\quad(z+a) = \quad(z) \quad (\, z \in \mathscr{D}\,).$$

The ordinary formula of the charge simulation method (9), however, is not suitable for approximating periodic functions. We approximate the function $\varphi(z)$ in the following way by modifying the formula (9).

$$\quad(z) \approx \sum_{j=1}^{N} Q_j \log\sin\left[\frac{\pi}{a}(z - \zeta_j)\right], \tag{10}$$

where Q_j $(j = 1, 2, \ldots, N)$ are real coefficients called the *charges* and ζ_j $(j = 1, 2, \ldots, N)$ fixed points in D_0 called the *charge points*. Thus we have the approximate mapping function

$$f(z) \approx F(z) = z + \mathrm{e}^{\mathrm{i}(\varphi\ \ \pi/2)} \sum_{j=1}^{N} Q_j \log \sin \left[\frac{\pi}{a}(z - \zeta_j) \right]. \tag{11}$$

The right hand side of (11) includes complex logarithmic functions. In order to make $F(z)$ single-valued, we pose on Q_j $(\, j = 1, 2, \ldots, N\,)$ the following constraint.

$$\sum_{j=1}^{N} Q_j = 0. \tag{12}$$

We can show that the function $F(z)$ is single-valued under the constraint (12) because, for an arbitrary closed curve $\widetilde{C}$ surrounding the domain D_m $(\, m \in \mathbb{Z}\,)$, we have

$$\begin{aligned} \int_{\widetilde{C}} \mathrm{d}F(z) &= \mathrm{e}^{\mathrm{i}(\varphi\ \ \pi/2)} \sum_{j=1}^{N} Q_j \int_{\widetilde{C}} \mathrm{d} \left(\log \sin \left[\frac{\pi}{a}(z - \zeta_j - ma) \right] \right) \\ &= 2\pi\mathrm{i}\, \mathrm{e}^{\mathrm{i}(\varphi\ \ \pi/2)} \sum_{j=1}^{N} Q_j \\ &= 0, \end{aligned}$$

which implies that $F(z)$ is single-valued in $\mathscr{D}$.

We can show easily that the approximate mapping function $F(z)$, which is defined by (11) and is subject to (12), satisfies the following properties.

(C2)$'$	(periodicity)	$F(z+a) = F(z) + a \quad (\, z \in \mathscr{D}\,)$
(C3)$'$	(asymptotic condition)	$F(z) = z + \mathrm{O}(1) \quad (\,\mathrm{Re}\, z \text{ fixed, } \mathrm{Im}\, z \to \pm\infty\,)$

which respectively correspond to the properties **(C2)** and **(C3)** of the exact mapping function $f(z)$.

We treat the boundary condition **(C1)** in the following way. The condition **(C1)** is rewritten as

$$\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}(\pi/2\ \ \varphi)} f(z) \right\} = u_0 \quad (\, z \in \partial D_0\,), \tag{13}$$

where u_0 is a real constant. Instead of (13), we impose on $F(z)$ the following condition.

$$\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}(\pi/2\ \ \varphi)} F(z_i) \right\} = U_0 \quad (\, i = 1, 2, \ldots, N\,), \tag{14}$$

where z_i $(\, i = 1, 2, \ldots, N\,)$ are fixed points on ∂D_0 called the *collocation points* and U_0 approximates the value u_0. We call (14) the *collocation condition.*

The condition (14) is rewritten as

$$\sum_{j=1}^{N} Q_j \log \left| \sin \left[\frac{\pi}{a}(z_i - \zeta_j) \right] \right| - U_0 = -\sin\varphi \cdot x_i + \cos\varphi \cdot y_i \quad (\, i = 1, 2, \ldots, N\,). \tag{15}$$

The equalities (12) and (15) form $(N+1)$ simultaneous linear equations with respect to Q_j ($j = 1, 2, \ldots, N$) and U_0. By solving (12) and (15), we determine the charges Q_j ($j = 1, 2, \ldots, N$) and obtain the approximate mapping function $F(z)$.

We must modify the expression of the approximate mapping function $F(z)$ (11) because the function $\log[(\pi/a)(z-\zeta_j)]$ has its discontinuity on the half-infinite line $(-\infty + \mathrm{i}(\operatorname{Im}\zeta_j), \zeta_j]$ parallel to the real axis, which makes the computation difficult.

In case that the boundary ∂D_0 is starlike with respect to the point ζ_0 in D_0, subtracting $0 = \sum_{j=1}^{N} Q_j \log\sin[(\pi/a)(z-\zeta_0)]$ from the both sides of (11), we have the expression of the approximate mapping function.

$$F(z) = z + \mathrm{e}^{\mathrm{i}(\varphi \quad \pi/2)} \sum_{j=1}^{N} Q_j \log\left(\frac{\sin[(\pi/a)(z-\zeta_j)]}{\sin[(\pi/a)(z-\zeta_0)]}\right). \tag{16}$$

The function $\log(\sin[(\pi/a)(z-\zeta_j)]/\sin[(\pi/a)(z-\zeta_0)])$ is continuous in the domain $\mathscr{D}$. In fact, the function $\sin[(\pi/a)(z-\zeta_j)]/\sin[(\pi/a)(z-\zeta_0)]$ has only one zero at the point $z = \zeta_j + ma$ and only one pole of the order one at the point $z = \zeta_0 + ma$ in each domain D_m ($m \in \mathbb{Z}$). From the argument principle, we have

$$\int_{\widetilde{C}} \mathrm{d}\log\left(\frac{\sin[(\pi/a)(z-\zeta_j)]}{\sin[(\pi/a)(z-\zeta_0)]}\right) \mathrm{d}z = 2\pi\mathrm{i}(1-1) = 0$$

for an arbitrary closed path $\widetilde{C}$ surrounding the domain D_m, which implies that the function $\log(\sin[(\pi/a)(z-\zeta_j)]/\sin[(\pi/a)(z-\zeta_0)])$ has no discontinuity on the path $\widetilde{C}$, i.e., the function $\log(\sin[(\pi/a)(z-\zeta_j)]/\sin[(\pi/a)(z-\zeta_0)])$ is continuous in the domain $\mathscr{D}$.

In case that the boundary ∂D_0 is not starlike, we can use the expression

$$F(z) = z + \mathrm{e}^{\mathrm{i}(\varphi \quad \pi/2)} \sum_{j=1}^{N \quad 1} \widetilde{Q}_j \log\left(\frac{\sin[(\pi/a)(z-\zeta_j)]}{\sin[(\pi/a)(z-\zeta_{j+1})]}\right), \tag{17}$$

where $\widetilde{Q}_j = Q_1 + Q_2 + \cdots + Q_j$ ($j = 1, 2, \ldots, N-1$). We can prove that the right hand side of (17) is continuous in the domain $\mathscr{D}$ in a similar way to the case that the boundary ∂D_0 is starlike. The expression (17) is obtained from (11) in the following way. Modifying the right hand side of (11), we have

$$\begin{aligned} F(z) &= z + \mathrm{e}^{\mathrm{i}(\varphi \quad \pi/2)} \left\{ Q_1 \log\sin\left[\frac{\pi}{a}(z-\zeta_1)\right] + \sum_{j=2}^{N} (\widetilde{Q}_j - \widetilde{Q}_{j \quad 1}) \log\sin\left[\frac{\pi}{a}(z-\zeta_j)\right] \right\} \\ &= z + \mathrm{e}^{\mathrm{i}(\varphi \quad \pi/2)} \left\{ \sum_{j=1}^{N \quad 1} \widetilde{Q}_j \log\sin\left(\frac{\sin[(\pi/a)(z-\zeta_j)]}{\sin[(\pi/a)(z-\zeta_{j+1}]}\right) + \widetilde{Q}_N \log\sin\left[\frac{\pi}{a}(z-\zeta_N)\right] \right\} \end{aligned}$$

and, using the equality $\widetilde{Q}_N = 0$ which is obtained from (12), we have the expression (17).

4 Numerical Examples

4.1 Examples of Numerical Conformal Mappings

We show the examples of the numerical conformal mappings for some typical domains. Computations were carried out on a SONY PCV-L330A/BP personal computer using programs coded in C with double precision working.

The first example is for the domain exterior to circles in a periodic array.

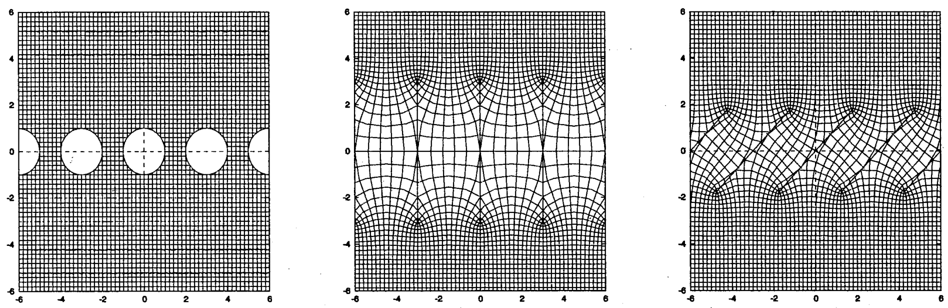
$$\mathscr{D}_1 = \{\, z \in \mathbb{C} \mid |z - 3m| > 1 \quad (\, m \in \mathbb{Z}\,)\,\}. \tag{18}$$

Figure 2 shows the result of the numerical conformal mapping of the domain $\mathscr{D}_1$. The collocation points z_i ($i = 1, 2, \ldots, N$) and the charge points ζ_j ($j = 1, 2, \ldots, N$) are respectively given by

$$z_i = \exp\left(\mathrm{i}\frac{2\pi(i-1)}{N}\right) \quad (\, i = 1, 2, \ldots, N\,), \tag{19}$$

$$\zeta_j = 0.5 \exp\left(\mathrm{i}\frac{2\pi(j-1)}{N}\right) \quad (\, j = 1, 2, \ldots, N\,), \tag{20}$$

where $N = 64$.



domain $\mathscr{D}_1$ — slit domain ($\varphi = \pi/2$) — slit domain ($\varphi = \pi/4$)

Figure 2: Numerical conformal mapping of the domain exterior to periodic disks $\mathscr{D}_1$.

The second example is the domain exterior to ellipses in a periodic array.

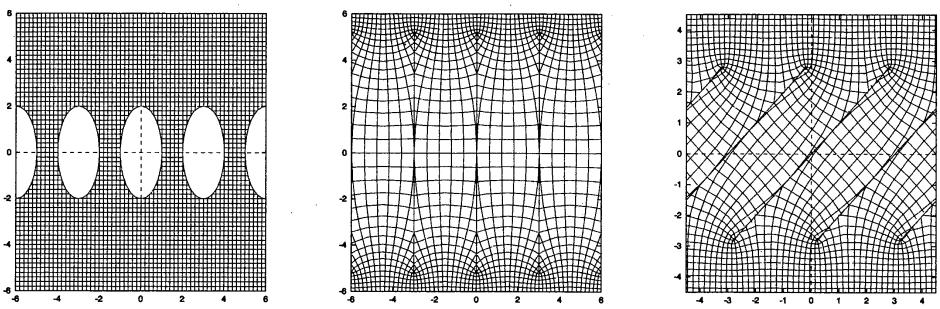
$$\mathscr{D}_2 = \left\{\, z = x + \mathrm{i}y \in \mathbb{C} \;\middle|\; x^2 + \frac{(y-3m)^2}{2^2} > 1 \quad (\, m \in \mathbb{Z}\,)\right\}. \tag{21}$$

Figure 3 shows the result of the numerical conformal mapping of the domain $\mathscr{D}_2$. The collocation points z_i ($i = 1, 2, \ldots, N$) and the charge points ζ_j ($j = 1, 2, \ldots, N$) are respectively given by

$$\left.\begin{aligned} z_i &= r_i \mathrm{e}^{\mathrm{i}\theta_i} \\ \theta_i &= \frac{2\pi(i-1)}{N}, \quad r_i = \left(\cos^2\theta_i + \frac{\sin^2\theta_i}{2^2}\right)^{1/2} \end{aligned}\right\} \quad (\, i = 1, 2, \ldots, N\,), \tag{22}$$

$$\zeta_j = z_j + 0.5\,|z_{j+1} - z_{j-1}|\exp\left(\mathrm{i}\arg(z_{j+1} - z_{j-1}) + \mathrm{i}\frac{\pi}{2}\right) \quad (j = 1, 2, \ldots, N), \tag{23}$$

where $z_0 = z_N$, $z_{N+1} = z_1$ and $N = 64$ (The choise of the points (22) and (23) was originally proposed by Amano). We observe from the figure that the boundary of each ellipse is mapped onto a rectilinear slit.



domain $\mathscr{D}_2$ | slit domain ($\varphi = \pi/2$) | slit domain ($\varphi = \pi/4$)

Figure 3: Numerical conformal mapping of the domain exterior to periodic ellipses $\mathscr{D}_2$.

In order to estimate the error of the numerical conformal mapping, we computed the value

$$\varepsilon = \max_{z \in \partial D_0} \left|\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}(\varphi - \pi/2)}F(z)\right\} - U_0\right|, \tag{24}$$

which is the distance between the image of the boundary ∂D_0 by the numerical conformal mapping and the slit S_0. The value ε does not give the upper bound of the error but is expected to give a rough estimate of the error. Table 1 shows the error estimates ε for the numerical conformal mappings of the domains $\mathscr{D}_1$ and $\mathscr{D}_2$. From the table, we can say that the numerical conformal mapping presented here achieves high accuracy, especially the accuracy of double precision for the domain exterior to circles $\mathscr{D}_1$.

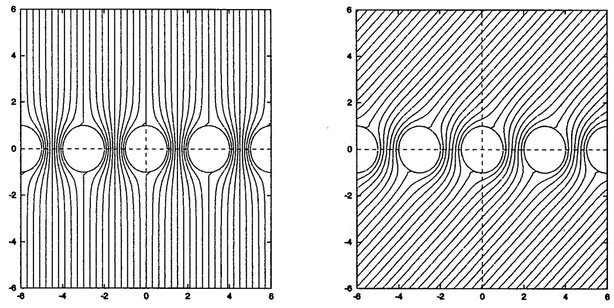
Table 1: Error estimate of the numerical conformal mapping.

domain	ε
$\mathscr{D}_1$	3.6×10^{-15}
$\mathscr{D}_2$	4.0×10^{-4}

4.2 Applications to Potential Flow Analysis

Our method can be applied to the analysis of potential flows past obstacles in a periodic array.

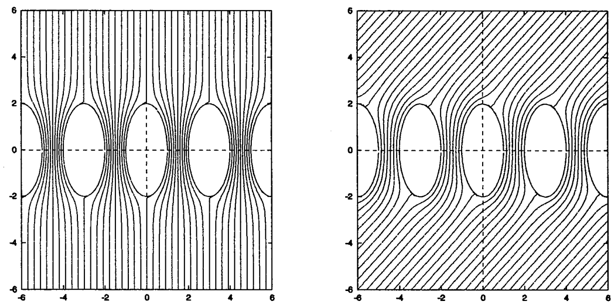
Figure 4 shows the contourlines of the function $\mathrm{Im}\{\mathrm{e}^{-\mathrm{i}\varphi}F(z)\}$, where $F(z)$ is the approximate mapping function of the domain $\mathscr{D}_1$ defined by (18). The figure illustrates the streamlines of potential flows past cylinders in a periodic array.



$(\varphi = \pi/2)$ $(\varphi = \pi/4)$

Figure 4: Potential flows past cylinders in a periodic array.

Figure 5 shows the contourlines of the function $\mathrm{Im}\{\mathrm{e}^{\mathrm{i}\varphi}F(z)\}$, where $F(z)$ is the approximate mapping function of the domain $\mathscr{D}_2$ defined by (21). The figure illustrates the streamlines of potential flows past ellipses in a periodic array.



$(\varphi = \pi/2)$ $(\varphi = \pi/4)$

Figure 5: Potential flows past ellipses in a periodic array.

5 Conclusions

We presented in this paper a numerical conformal mapping of periodic structure domains onto periodic parallel slit domains using the charge simulation method. Numerical examples in some typical cases show that the method presented here is very efficient, especially it achieves the accuracy of double precision for the case of the domain exterior to circles.

As future works, we are interested in the following problems.

1. Can we analyse the dynamics of more practical fluid, for example, Stokes flow, Oseen flow and so on, past obstacles in a periodic array?

2. Can we compute conformal mappings of two-dimensional periodic structure domains?

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